Chapter 9

World in a mirror

A detour of a thousand pages starts with a single misstep.
—Chairman Miaw

DYNAMICAL SYSTEMS often come equipped with symmetries, such as the reflection and rotation symmetries of various potentials. In this chapter we study quotienting of discrete symmetries, and in the next chapter we study symmetry reduction for continuous symmetries. We look at individual orbits, and the ways they are interrelated by symmetries. This sets the stage for a discussion of how symmetries affect global densities of trajectories, and the factorization of spectral determinants to be undertaken in chapter 21.

As we shall show here and in chapter 21, discrete symmetries simplify the dynamics in a rather beautiful way: If dynamics is invariant under a set of discrete symmetries G, the state space $\mathcal{M}$ is tiled by a set of symmetry-related tiles, and the dynamics can be reduced to dynamics within one such tile, the *fundamental domain* $\mathcal{M}/G$. In presence of a symmetry the notion of a prime periodic orbit has to be reexamined: a set of symmetry-related full state space cycles is replaced by often much shorter *relative periodic orbit*, the shortest segment of the full state space cycle which tiles the cycle and all of its copies under the action of the group. Furthermore, the group operations that relate distinct tiles do double duty as letters of an alphabet which assigns symbolic itineraries to trajectories.

section 11.1

Familiarity with basic group-theoretic notions is assumed, with details relegated to appendix H.1. We find the abstract notions easier to digest by working out the examples interspersed throughout this chapter.The erudite reader might prefer to skip the lengthy group-theoretic overture and go directly to $C_2 = D_1$ example 9.12, example 9.14, and $C_{3v} = D_3$ example 9.1, backtrack as needed.

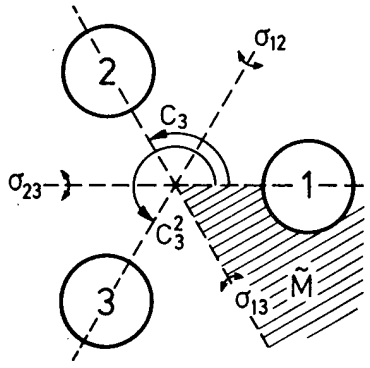


Figure 9.1: The symmetries of three disks on an equilateral triangle. A fundamental domain is indicated by the shaded wedge.

9.1 Discrete symmetries

Normal is just a setting on a washing machine.
—Borgette, Borgo's daughter

We show that a symmetry equates multiplets of equivalent orbits, or 'stratifies' the state space into equivalence classes, each class a 'group orbit'. We start by defining a finite (discrete) group, its state space representations, and what we mean by a *symmetry* (*invariance* or *equivariance*) of a dynamical system. As is always the problem with 'gruppenpest' (read appendix A.2.3) way too many abstract notions have to be defined before an intelligent conversation can take place. Perhaps best to skim through this section on the first reading, then return to it later as needed.

Definition: A **group** consists of a set of elements

$$G = \{e, g_2, \ldots, g_n, \ldots\} \tag{9.1}$$

and a group multiplication rule $g_j \circ g_i$ (often abbreviated as $g_j g_i$), satisfying

1. Closure: If $g_i, g_j \in G$, then $g_j \circ g_i \in G$
2. Associativity: $g_k \circ (g_j \circ g_i) = (g_k \circ g_j) \circ g_i$
3. Identity e: $g \circ e = e \circ g = g$ for all $g \in G$
4. Inverse g^{-1}: For every $g \in G$, there exists a unique element $h = g^{-1} \in G$ such that
   $h \circ g = g \circ h = e$.

If the group is finite, the number of elements, $|G| = n$, is called the *order* of the group.

example H.1
example H.2
example H.3

Example 9.1 $C_{3v} = D_3$ ***symmetry of the 3-disk game of pinball:*** *If the three unit-radius disks in figure 9.1 are equidistantly spaced, our game of pinball has a sixfold*

symmetry. The symmetry group of relabeling the 3 disks is the permutation group S_3; however, it is more instructive to think of this group geometrically, as C_{3v}, also known as the dihedral group

$$D_3 = \{e, \sigma_{12}, \sigma_{13}, \sigma_{23}, C^{1/3}, C^{2/3}\}\,, \tag{9.2}$$

the group of order $|G| = 6$ consisting of the identity element e, three reflections across symmetry axes $\{\sigma_{12}, \sigma_{23}, \sigma_{13}\}$, and two rotations by $2\pi/3$ and $4\pi/3$ denoted $\{C^{1/3}, C^{2/3}\}$. (continued in example 9.6)

Definition: Coordinate transformations. Consider a map $x' = f(x)$, $x, x' \in \mathcal{M}$. An *active* coordinate transformation Mx corresponds to a non-singular $[d\times d]$ matrix M that maps the vector $x \in \mathcal{M}$ onto another vector $Mx \in \mathcal{M}$. The corresponding *passive* coordinate transformation $f(x) \to M^{-1} f(x)$ changes the coordinate system with respect to which the vector $f(x) \in \mathcal{M}$ is measured. Together, a passive and active coordinate transformations yield the map in the transformed coordinates:

$$\hat{f}(x) = M^{-1} f(Mx)\,. \tag{9.3}$$

Example 9.2 ***Discrete groups of order 2 on $\mathbb{R}^3$.*** *Three types of discrete group of order 2 can arise by linear action on our 3-dimensional Euclidian space $\mathbb{R}^3$:*

$$\begin{aligned} \textit{reflections: } \sigma(x,y,z) &= (x,y,-z) \\ \textit{rotations: } C^{1/2}(x,y,z) &= (-x,-y,z) \\ \textit{inversions: } P(x,y,z) &= (-x,-y,-z)\,. \end{aligned} \tag{9.4}$$

σ is a reflection *(or an inversion) through the $[x,y]$ plane. $C^{1/2}$ is $[x,y]$-plane, constant z* rotation *by π about the z-axis (or an inversion thorough the z-axis). P is an* inversion *(or parity operation) through the point $(0,0,0)$. Singly, each operation generates a group of order 2: $D_1 = \{e, \sigma\}$, $C_2 = \{e, C^{1/2}\}$, and $D_1 = \{e, P\}$. Together, they form the dihedral group $D_2 = \{e, \sigma, C^{1/2}, P\}$ of order 4. (continued in example 9.3)*

Definition: Matrix group. The set of $[d\times d]$-dimensional real non-singular matrices $A, B, C, \ldots \in GL(d)$ acting in a d-dimensional vector space $V \in \mathbb{R}^d$ forms the general linear group $GL(d)$ under matrix multiplication. The product of matrices A and B gives the matrix C, $Cx = B(Ax) = (BA)x \in V$, for all $x \in V$. The unit matrix $\mathbf{1}$ is the identity element which leaves all vectors in V unchanged. Every matrix in the group has a unique inverse.

Definition: Matrix representation. Linear action of a group element g on states $x \in \mathcal{M}$ is given by a finite non-singular $[d \times d]$ matrix g, the *matrix representation* of element $g \in G$. We shall denote by 'g' both the abstract group element and its matrix representation.

However, when dealing simultaneously with several representations of the same group action, notation $D_j(g)$, j a representation label, is preferable (see appendix H.1). A linear or matrix representation $\mathsf{D}(G)$ of the abstract group G acting on a *representation space* V is a group of matrices $\mathsf{D}(G)$ such that

1. Any $g \in G$ is mapped to a matrix $D(g) \in \mathsf{D}(G)$.
2. The group product $g_2 \circ g_1$ is mapped onto the matrix product $D(g_2 \circ g_1) = D(g_2)D(g_1)$.
3. The associativity follows from the associativity of matrix multiplication, $D(g_3 \circ (g_2 \circ g_1)) = D(g_3)(D(g_2)D(g_1)) = (D(g_3)(D(g_2))D(g_1)$.
4. The identity element $e \in G$ is mapped onto the unit matrix $D(e) = 1$ and the inverse element $g^{-1} \in G$ is mapped onto the inverse matrix $D(g^{-1}) = [D(g)]^{-1} \equiv D^{-1}(g)$.

Example 9.3 ***Discrete operations on*** $\mathbb{R}^3$***.*** *(continued from example 9.2) The matrix representation of reflections, rotations and inversions defined by (9.4) is*

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad C^{1/2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{9.5}$$

with $\det C^{1/2} = 1$, $\det \sigma = \det P = -1$*; that is why we refer to* $C^{1/2}$ *as a rotation, and* σ, P *as inversions. As* $g^2 = e$ *in all three cases, these are groups of order 2. (continued in example 9.5)*

If the coordinate transformation g belongs to a linear non-singular representation of a discrete finite group G, for any element $g \in G$ there exists a number $m \leq |G|$ such that

$$g^m \equiv \underbrace{g \circ g \circ \ldots \circ g}_{m \text{ times}} = e \quad \rightarrow \quad |\det g| = 1\,. \tag{9.6}$$

As the modulus of its determinant is unity, $\det g$ is an mth root of 1. Hence all finite groups have unitary representations.

Definition: Symmetry of a dynamical system. A group G is a *symmetry* of the dynamics if for every solution $f(x) \in \mathcal{M}$ and $g \in G$, $gf(x)$ is also a solution.

Another way to state this: A dynamical system $(\mathcal{M}, f)$ is *invariant* (or *G-equivariant*) under a symmetry group G if the time evolution $f : \mathcal{M} \to \mathcal{M}$ (a

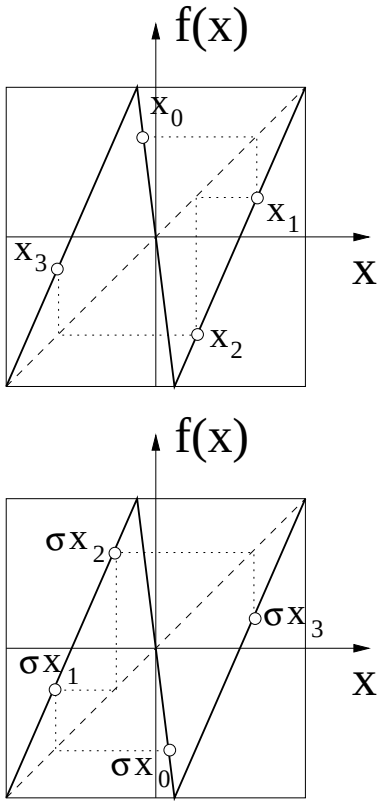


Figure 9.2: The bimodal Ulam sawtooth map with the D_1 symmetry $f(-x) = -f(x)$. If the trajectory $x_0 \to x_1 \to x_2 \to \cdots$ is a solution, so is its reflection $\sigma x_0 \to \sigma x_1 \to \sigma x_2 \to \cdots$. (continued in figure 9.4)

discrete time map f, or the continuous flow f^t map from the d-dimensional manifold $\mathcal{M}$ into itself) commutes with all actions of G,

$$f(gx) = gf(x)\,. \tag{9.7}$$

In the language of physicists: The 'law of motion' is invariant, i.e., retains its form in any symmetry-group related coordinate frame (9.3),

$$f(x) = g^{-1} f(gx)\,, \tag{9.8}$$

for $x \in \mathcal{M}$ and *any* finite non-singular $[d\times d]$ matrix representation g of element $g \in G$. As these are true *any* state x, one can state this more compactly as $f \circ g = g \circ f$, or $f = g^{-1} \circ f \circ g$.

Why 'equivariant?' A scalar function $h(x)$ is said to be *G-invariant* if $h(x) = h(gx)$ for all $g \in G$. The group actions map the solution $f : \mathcal{M} \to \mathcal{M}$ into different (but equivalent) solutions $gf(x)$, hence the invariance condition $f(x) = g^{-1} f(gx)$ appropriate to vectors (and, more generally, tensors). The full set of such solutions is *G-invariant*, but the flow that generates them is said to be *G*-equivariant. It is obvious from the context, but for verbal emphasis applied mathematicians like to distinguish the two cases by *in/equi*-variant. The distinction is helpful in distinguishing the dynamics written in the original, equivariant coordinates from the dynamics rewritten in terms of *invariant* coordinates, see sects. 9.5 and 10.4. exercise 9.7

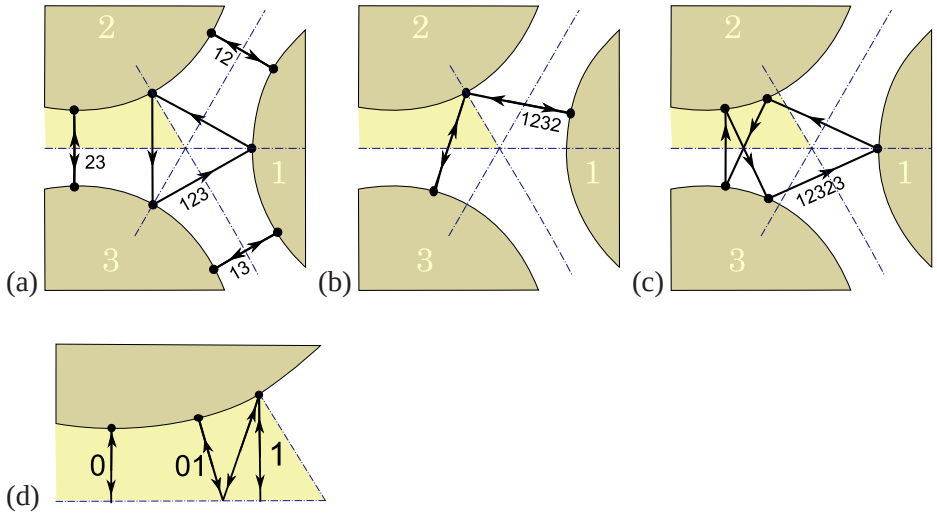


Figure 9.3: The 3-disk pinball cycles: (a) $\overline{12}$, $\overline{13}$, $\overline{23}$, $\overline{123}$; the clockwise $\overline{132}$ not drawn. (b) Cycle $\overline{1232}$; the symmetry related $\overline{1213}$ and $\overline{1323}$ not drawn. (c) $\overline{12323}$; $\overline{12123}$, $\overline{12132}$, $\overline{12313}$, $\overline{13131}$ and $\overline{13232}$ not drawn. (d) The fundamental domain, i.e., the 1/6th wedge indicated in (a), consisting of a section of a disk, two segments of symmetry axes acting as straight mirror walls, and the escape gap to the left. The above 14 full-space cycles restricted to the fundamental domain and recoded in binary reduce to the two fixed points $\overline{0}$, $\overline{1}$, 2-cycle $\overline{10}$, and 5-cycle $\overline{00111}$ (not drawn). See figure 9.9 for the $\overline{001}$ cycle.

Example 9.4 ***A reflection symmetric*** $1d$ ***map.*** *Consider a* $1d$ *map* f *with reflection symmetry* $f(-x) = -f(x)$*, such as the bimodal 'sawtooth' map of figure 9.2, piecewise-linear on the state space* $\mathcal{M} = [-1, 1]$*, a compact 1-dimensional line interval, split into three regions* $\mathcal{M} = \mathcal{M}_L \cup \mathcal{M}_C \cup \mathcal{M}_R$*. Denote the reflection operation by* $\sigma x = -x$*. The 2-element group* $G = \{e, \sigma\}$ *goes by many names, such as* Z_2 *or* C_2*. Here we shall refer to it as* D_1*, dihedral group generated by a single reflection. The* G*-equivariance of the map implies that if* $\{x_n\}$ *is a trajectory, than also* $\{\sigma x_n\}$ *is a symmetry-equivalent trajectory because* $\sigma x_{n+1} = \sigma f(x_n) = f(\sigma x_n)$ *(continued in example 9.12)*

Example 9.5 ***Equivariance of the Lorenz flow.*** *(continued from example 9.3) The velocity field in Lorenz equations (2.12)*

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma(y - x) \\ \rho x - y - xz \\ xy - bz \end{bmatrix}$$

is equivariant under the action of cyclic group $C_2 = \{e, C^{1/2}\}$ *acting on* $\mathbb{R}^3$ *by a* π *rotation about the* z *axis,*

$$C^{1/2}(x, y, z) = (-x, -y, z)\,. \tag{9.9}$$

(continued in example 9.14)

Example 9.6 ***3-disk game of pinball - symmetry-related orbits:*** *(continued from example 9.1) Applying an element (identity, rotation by* $\pm 2\pi/3$*, or one of the three possible reflections) of this symmetry group to a trajectory yields another trajectory. For instance,* σ_{23}*, the flip across the symmetry axis going through disk* 1 *interchanges the symbols* 2 *and* 3*; it maps the cycle* $\overline{12123}$ *into* $\overline{13132}$*, figure 9.3 (c). Cycles* $\overline{12}$*,* $\overline{23}$*, and* $\overline{13}$ *in figure 9.3 (a) are related to each other by rotation by* $\pm 2\pi/3$*, or, equivalently, by a relabeling of the disks. (continued in example 9.8)*

Example 9.7 ***Discrete symmetries of the plane Couette flow.*** *The plane Couette flow is a fluid flow bounded by two countermoving planes, in a cell periodic in streamwise and spanwise directions. The Navier-Stokes equations for the plane Couette flow have two discrete symmetries: reflection through the (streamwise , wall-normal) plane, and rotation by π in the (streamwise , wall-normal) plane. That is why the system has equilibrium and periodic orbit solutions, (as opposed to relative equilibrium and relative periodic orbit solutions discussed in chapter 10). They belong to discrete symmetry subspaces. (continued in example 10.4)*

9.1.1 Subgroups, cosets, classes

Inspection of figure 9.3 indicates that various 3-disk orbits are the same up to a symmetry transformation. Here we set up some abstract group-theoretic notions needed to describe such relations. The reader might prefer to skip to sect. 9.2, backtrack as needed.

Definition: Subgroup. A set of group elements $H = \{e, b_2, b_3, \ldots, b_h\} \subseteq G$ closed under group multiplication forms a subgroup.

Definition: Coset. Let $H = \{e, b_2, b_3, \ldots, b_h\} \subseteq G$ be a subgroup of order $h = |H|$. The set of h elements $\{c, cb_2, cb_3, \ldots, cb_h\}$, $c \in G$ but not in H, is called left *coset* cH. For a given subgroup H the group elements are partitioned into H and $m-1$ cosets, where $m = |G|/|H|$. The cosets *cannot be* subgroups, since they do not include the identity element. We learn that a nontrival subgroup can exist only if $|G|$, the order of the group, is divisible by $|H|$, the order of the subgroup, i.e., only if $|G|$ is not a prime number.

Example 9.8 ***Subgroups, cosets of D_3:*** *(continued from example 9.6) The 3-disks symmetry group, the D_3 dihedral group (9.2) has six subgroups*

$$\{e\}, \quad \{e, \sigma_{12}\}, \ \{e, \sigma_{13}\}, \ \{e, \sigma_{23}\}, \quad \{e, C^{1/3}, C^{2/3}\}, \quad D_3\,. \tag{9.10}$$

The left cosets of subgroup $D_1 = \{e, \sigma_{12}\}$ are $\{\sigma_{13}, C^{1/3}\}$, $\{\sigma_{23}, C^{2/3}\}$. The coset of subgroup $C_3 = \{e, C^{1/3}, C^{2/3}\}$ is $\{\sigma_{12}, \sigma_{13}, \sigma_{23}\}$. The significance of the coset is that if a solution has a symmetry H, for example the symmetry of a 3-cycle $\overline{123}$ is C_3, then all elements in a coset act on it the same way, for example $\{\sigma_{12}, \sigma_{13}, \sigma_{23}\}\overline{123} = \overline{132}$.

The nontrivial subgroups of D_3 are $D_1 = \{e, \sigma\}$, consisting of the identity and any one of the reflections, of order 2, and $C_3 = \{e, C^{1/3}, C^{2/3}\}$, of order 3, so possible cycle multiplicities are $|G|/|G_p| = 1, 2, 3$ or 6. Only the fixed point at the origin has full symmetry $G_p = G$. Such equilibria exist for smooth potentials, but not for the 3-disk billiard. Examples of other multiplicities are given in figure 9.3 and figure 9.7. (continued in example 9.9)

Next we need a notion that will, for example, identify the three 3-disk 2-cycles in figure 9.3 as belonging to the same class.

Definition: Class. An element $b \in G$ is *conjugate* to a if $b = c\,a\,c^{-1}$ where c is some other group element. If b and c are both conjugate to a, they are conjugate to each other. Application of all conjugations separates the set of group elements into mutually not-conjugate subsets called *classes*, *types* or *conjugacy classes*. The identity e is always in the class $\{e\}$ of its own. This is the only class which is a subgroup, all other classes lack the identity element.

exercise 9.3

exercise 9.5

Example 9.9 ***D_3 symmetry - classes:*** *(continued from example 9.8) The three classes of the 3-disk symmetry group $D_3 = \{e, C^{1/3}, C^{2/3}, \sigma, \sigma C^{1/3}, \sigma C^{2/3}\}$, are the identity, any one of the reflections, and the two rotations,*

$$\{e\}\,, \quad \left\{ \begin{array}{c} \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{array} \right\}\,, \quad \left\{ \begin{array}{c} C^{1/3} \\ C^{2/3} \end{array} \right\}\,. \tag{9.11}$$

In other words, the group actions either flip or rotate. (continued in example 9.13)

Physical importance of classes is clear from (9.8), the way coordinate transformations act on mappings: action of elements of a class (say reflections, or rotations) is equivalent up to a redefinition of the coordinate frame.

Definition: Invariant subgroup. A subgroup $H \subseteq G$ is an *invariant* subgroup or *normal divisor* if it consists of complete classes. Class is complete if no conjugation takes an element of the class out of H.

Think of action of H within each coset as identifying its $|H|$ elements as equivalent. This leads to the notion of the *factor group* or *quotient group* G/H of G, with respect to the invariant subgroup H. H thus divides G into H and $m-1$ cosets, each of order $|H|$. The order of G/H is $m = |G|/|H|$, and its multiplication table can be worked out from the G multiplication table class by class, with the subgroup H playing the role of identity. G/H is *homeomorphic* to G, with $|H|$ elements in a class of G represented by a single element in G/H.

9.1.2 Orbits, quotient space

So far we have discussed the structure of a group as an abstract entity. Now we switch gears and describe the action of the group on the state space. This is the key step; if a set of solutions is equivalent by symmetry (a circle, let's say), we would like to represent it by a single solution (cut the circle at a point, or rewrite the dynamics in a 'reduced state space,' where the circle of solutions is represented by a single point).

section 2.1

Definition: Orbit. The subset $\mathcal{M}_{x_0} \subset \mathcal{M}$ traversed by the infinite-time trajectory of a given point x_0 is called the *orbit* (or *time orbit*, or *solution*) $x(t) = f^t(x_0)$. An orbit is a *dynamically invariant* notion: it refers to the set of all states that can

be reached in time from x_0, thus as a set it is invariant under time evolution. The full state space $\mathcal{M}$ is foliated into a union of such orbits. We label a generic orbit $\mathcal{M}_{x_0}$ by any point belonging to it, $x_0 = x(0)$ for example.

A generic orbit might be ergodic, unstable and essentially uncontrollable. The ChaosBook strategy is to populate the state space by a hierarchy of orbits which are *compact invariant sets* (equilibria, periodic orbits, invariant tori, ...), each computable in a finite time. They are a set of zero Lebesgue measure, but dense on the non–wandering set, and are to a generic orbit what fractions are to normal numbers on the unit interval. We label orbits confined to compact invariant sets by whatever alphabet we find convenient in a given context: point $EQ = x_{EQ} = \mathcal{M}_{EQ}$ for an equilibrium, 1-dimensional loop $p = \mathcal{M}_p$ for a prime periodic orbit p, etc. (note also discussion on page 205, and the distinction between trajectory and orbit made in sect. 2.1; a trajectory is a finite-time segment of an orbit).

Definition: Group orbit or the *G-orbit* of the point $x \in \mathcal{M}$ is the set

$$\mathcal{M}_x = \{g\, x \mid g \in G\} \tag{9.12}$$

of all state space points into which x is mapped under the action of G. If G is a symmetry, intrinsic properties of an equilibrium (such as stability eigenvalues) or a cycle p (period, Floquet multipliers) evaluated anywhere along its G-orbit are the same.

A symmetry thus reduces the number of inequivalent solutions $\mathcal{M}_p$. So we also need to describe the symmetry of a *solution*, as opposed to (9.8), the symmetry of the *system*. We start by defining the notions of *reduced state space*, of *isotropy* of a state space point, and of the *symmetry of an orbit*.

Definition: Reduced state space. The action of group G partitions the state space $\mathcal{M}$ into a union of group orbits. This set of group orbits, denoted $\mathcal{M}/G$, has many names: *reduced state space*, *quotient space* or any of the names listed on page 195.

Reduction of the dynamical state space is discussed in sect. 9.4 for discrete symmetries, and in sect. 10.4 for continuous symmetries.

Definition: Fixed-point subspace. $\mathcal{M}_H$ is the set of all state space points left *H-fixed*, *point-wise* invariant under subgroup or 'centralizer' $H \subset G$ action

$$\mathcal{M}_H = \mathrm{Fix}\,(H) = \{x \in \mathcal{M} : h\, x = x \text{ for all } h \in H\}\,. \tag{9.13}$$

Points in state space subspace $\mathcal{M}_G$ which are fixed points of the full group action are called *invariant points*,

$$\mathcal{M}_G = \mathrm{Fix}\,(G) = \{x \in \mathcal{M} : g\, x = x \text{ for all } g \in G\}\,. \tag{9.14}$$

Definition: Flow invariant subspace. A typical point in fixed-point subspace $\mathcal{M}_H$ moves with time, but, due to equivariance (9.7), its trajectory $x(t) = f^t(x)$ remains within $f(\mathcal{M}_H) \subseteq \mathcal{M}_H$ for all times,

$$h f^t(x) = f^t(hx) = f^t(x)\,, \quad h \in H\,, \tag{9.15}$$

i.e., it belongs to a *flow invariant subspace*. This suggests a systematic approach to seeking compact invariant solutions. The larger the symmetry subgroup, the smaller $\mathcal{M}_H$, easing the numerical searches, so start with the largest subgroups H first.

We can often decompose the state space into smaller subspaces, with group acting within each 'chunk' separately:

Definition: Invariant subspace. $\mathcal{M}_\alpha \subset \mathcal{M}$ is an *invariant* subspace if

$$\{\mathcal{M}_\alpha : gx \in \mathcal{M}_\alpha \text{ for all } g \in G \text{ and } x \in \mathcal{M}_\alpha\}\,. \tag{9.16}$$

$\{0\}$ and $\mathcal{M}$ are always invariant subspaces. So is any Fix(H) which is point-wise invariant under action of G.

Definition: Irreducible subspace. A space $\mathcal{M}_\alpha$ whose only invariant subspaces are $\{0\}$ and $\mathcal{M}_\alpha$ is called *irreducible*.

9.2 Symmetries of solutions

The solutions of an equivariant system can satisfy all of the system's symmetries, a subgroup of them, or have no symmetry at all. For a generic ergodic orbit $f^t(x)$ the trajectory and any of its images under action of $g \in G$ are distinct with probability one, $f^t(x) \cap g f^{t'}(x) = \emptyset$ for all t, t'. For example, a typical turbulent trajectory of pipe flow has no symmetry beyond the identity, so its symmetry group is the trivial $\{e\}$. For compact invariant sets, such as fixed points and periodic orbits the situation is very different. For example, the symmetry of the laminar solution of the plane Couette flow is the full symmetry of its Navier-Stokes equations. In between we find solutions whose symmetries are subgroups of the full symmetry of dynamics.

Definition: Isotropy subgroup. The maximal set of group actions which maps a state space point x into itself,

$$G_x = \{g \in G : gx = x\}\,, \tag{9.17}$$

is called the *isotropy group* or *little group* of x.

A solution usually exhibits less symmetry than the equations of motion. The symmetry of a solution is thus a subgroup of the symmetry group of dynamics. We thus also need a notion of *set-wise* invariance, as opposed to the *point-wise* invariance under G_x. exercise 9.2

Definition: Symmetry of a solution, G_p-symmetric cycle. We shall refer to the subset of nontrivial group actions $G_p \subseteq G$ on state space points within a compact set $\mathcal{M}_p$, which leave no point fixed but leave the set invariant, as the symmetry G_p of the solution $\mathcal{M}_p$,

$$G_p = \{g \in G_p : gx \in \mathcal{M}_p,\ gx \neq x \text{ for } g \neq e\}\,, \tag{9.18}$$

and reserve the notion of 'isotropy' of a set $\mathcal{M}_p$ for the subgroup G_p that leaves each point in it fixed.

A cycle p is G_p*-symmetric* (*set-wise symmetric*, *self-dual*) if the action of elements of G_p on the set of periodic points $\mathcal{M}_p$ reproduces the set. $g \in G_p$ acts as a shift in time, mapping the periodic point $x \in \mathcal{M}_p$ into another periodic point.

Example 9.10 ***D_1-symmetric cycles:*** *For D_1 the period of a set-wise symmetric cycle is even ($n_s = 2n_{\tilde{s}}$), and the mirror image of the x_s periodic point is reached by traversing the relative periodic orbit segment $\tilde{s}$ of length $n_{\tilde{s}}$, $f^{n_{\tilde{s}}}(x_s) = \sigma x_s$, see figure 9.4 (b).*

Definition: Conjugate symmetry subgroups. The splitting of a group G into a symmetry group G_p of orbit $\mathcal{M}_p$ and $m-1$ cosets cG_p relates the orbit $\mathcal{M}_p$ to $m-1$ other distinct orbits $c\mathcal{M}_p$. All of them have equivalent symmetry subgroups, exercise 9.4
or, more precisely, the points on the same group orbit have *conjugate symmetry subgroups* (or *conjugate stabilizers*):

$$G_{c\,p} = c\,G_p\,c^{-1}\,, \tag{9.19}$$

i.e., if G_p is the symmetry of orbit $\mathcal{M}_p$, elements of the coset space $g \in G/G_p$ generate the $m_p - 1$ distinct copies of $\mathcal{M}_p$, so for discrete groups the multiplicity of orbit p is $m_p = |G|/|G_p|$.

Definition: G_p-fixed orbits: An equilibrium x_q or a compact solution p is point-wise or G_p*-fixed* if it lies in the invariant points subspace $\mathrm{Fix}\left(G_p\right)$, $gx = x$ for all $g \in G_p$, and $x = x_q$ or $x \in \mathcal{M}_p$. A solution that is G-invariant under all group G operations has multiplicity 1. Stability of such solutions will have to be examined with care, as they lie on the boundaries of domains related by the action of the symmetry group.

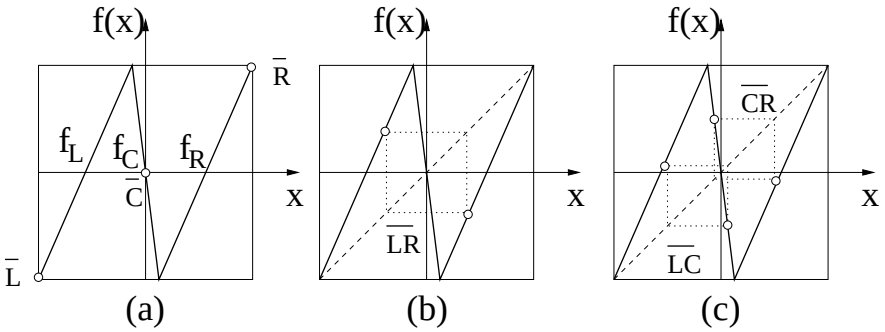


Figure 9.4: The D_1-equivariant bimodal sawtooth map of figure 9.2 has three types of periodic orbits: (a) D_1-fixed fixed point $\overline{C}$, asymmetric fixed points pair $\{\overline{L},\overline{R}\}$. (b) D_1-symmetric (setwise invariant) 2-cycle $\overline{LR}$. (c) Asymmetric 2-cycles pair $\{\overline{LC},\overline{CR}\}$. (continued in figure 9.8) (Y. Lan)

Example 9.11 ***D_1-invariant cycles:*** *In the example at hand there is only one G-invariant (point-wise invariant) orbit, the fixed point $\overline{C}$ at the origin, see figure 9.4 (a). As reflection symmetry is the only discrete symmetry that a map of the interval can have, this example completes the group-theoretic analysis of 1-dimensional maps. We shall continue analysis of this system in example 9.16, and work out the symbolic dynamics of such reflection symmetric systems in example 12.5.*

In the literature the symmetry group of a solution is often called *stabilizer* or *isotropy subgroup*. Saying that G_p is the symmetry of the solution p, or that the orbit $\mathcal{M}_p$ is 'G_p-invariant,' accomplishes as much without confusing you with all these names (see remark 9.1). In what follows we say "the symmetry of the periodic orbit p is $\mathrm{C}_2 = \{e, R\}$," rather than bandy about 'stabilizers' and such.

The key concept in the classification of dynamical orbits is their symmetry. We note three types of solutions: (i) fully asymmetric solutions a, (ii) subgroup $G_{\tilde{s}}$ set-wise invariant cycles s built by repeats of relative cycle segments $\tilde{s}$, and (iii) isotropy subgroup G_{EQ}-invariant equilibria or point-wise G_p-fixed cycles b.

Definition: Asymmetric orbits. An equilibrium or periodic orbit is not symmetric if $\{x_a\} \cap \{gx_a\} = \emptyset$ for any $g \in G$, where $\{x_a\}$ is the set of periodic points belonging to the cycle a. Thus $g \in G$ generate $|G|$ distinct orbits with the same number of points and the same stability properties.

Example 9.12 ***Group D_1 - a reflection symmetric $1d$ map:*** *Consider the bimodal 'sawtooth' map of example 9.4, with the state space $\mathcal{M} = [-1, 1]$ split into three regions $\mathcal{M} = \{\mathcal{M}_L, \mathcal{M}_C, \mathcal{M}_R\}$ which we label with a 3-letter alphabet L(eft), C(enter), and R(ight). The symbolic dynamics is complete ternary dynamics, with any sequence of letters $\mathcal{A} = \{L, C, R\}$ corresponding to an admissible trajectory ('complete' means no additional grammar rules required, see example 11.6 below). The D_1-equivariance of the map, $D_1 = \{e, \sigma\}$, implies that if $\{x_n\}$ is a trajectory, so is $\{\sigma x_n\}$.*

$\mathrm{Fix}\,(G)$, the set of points invariant under group action of D_1, $\tilde{\mathcal{M}} \cap \sigma\tilde{\mathcal{M}}$, is just this fixed point $x = 0$, the reflection symmetry point. If a is an asymmetric cycle, σ maps it into the reflected cycle σa, with the same period and the same stability properties, see the fixed points pair $\{\overline{L},\overline{R}\}$ and the 2-cycles pair $\{\overline{LC},\overline{CR}\}$ in figure 9.4 (c).

The next illustration brings in the non-abelian, noncommutative group structure: for the 3-disk game of pinball of sect. 1.3, example 9.1 and example 9.17, the symmetry group has elements that do not commute. exercise 9.5

Example 9.13 ***3-disk game of pinball - cycle symmetries:*** *(continued from example 9.9) The C_3 subgroup $G_p = \{e, C^{1/3}, C^{2/3}\}$ invariance is exemplified by the two cycles $\overline{123}$ and $\overline{132}$ which are invariant under rotations by $2\pi/3$ and $4\pi/3$, but are mapped into each other by any reflection, figure 9.7 (a), and have multiplicity $|G|/|G_p| = 2$.*

The C_v type of a subgroup is exemplified by the symmetries of $\hat{p} = 1213$. This cycle is invariant under reflection $\sigma_{23}\{\overline{1213}\} = \overline{1312} = \overline{1213}$, so the invariant subgroup is $G_{\hat{p}} = \{e, \sigma_{23}\}$, with multiplicity is $m_{\hat{p}} = |G|/|G_p| = 3$; the cycles in this class, $\overline{1213}$, $\overline{1232}$ and $\overline{1323}$, are related by $2\pi/3$ rotations, figure 9.7 (b).

A cycle of no symmetry, such as $\overline{12123}$, has $G_p = \{e\}$ and contributes in all six copies (the remaining cycles in the class are $\overline{12132}$, $\overline{12313}$, $\overline{12323}$, $\overline{13132}$ and $\overline{13232}$), figure 9.7 (c).

Besides the above spatial symmetries, for Hamiltonian systems cycles may be related by time reversal symmetry. An example are the cycles $\overline{121212313}$ and $\overline{313212121} = \overline{121213132}$ which have the same periods and stabilities, but are related by no space symmetry, see figure 9.7. (continued in example 9.17)

Consider next perhaps the simplest 3-dimensional flow with a symmetry, the iconic flow of Lorenz. The example is long but worth working throug: the symmetry-reduced dynamics is much simpler than the original Lorenz flow.

exercise 9.7
exercise 9.8
exercise 9.9

Example 9.14 ***Desymmetrization of Lorenz flow:*** *(continuation of example 9.5) Lorenz equation (2.12) is equivariant under (9.9), the action of order-2 group $C_2 = \{e, C^{1/2}\}$, where $C^{1/2}$ is $[x, y]$-plane, half-cycle rotation by π about the z-axis:*

$$(x, y, z) \to C^{1/2}(x, y, z) = (-x, -y, z)\,. \tag{9.20}$$

*$(C^{1/2})^2 = 1$ condition decomposes the state space into two linearly irreducible subspaces $\mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^-$, the z-axis $\mathcal{M}^+$ and the $[x, y]$ plane $\mathcal{M}^-$, with projection operators onto the two subspaces given by (see sect. **??**)*

$$\mathbf{P}^+ = \frac{1}{2}(1 + C^{1/2}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P}^- = \frac{1}{2}(1 - C^{1/2}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{9.21}$$

As the flow is C_2-invariant, so is its linearization $\dot{x} = Ax$. Evaluated at EQ_0, A commutes with $C^{1/2}$, and, as we have already seen in example 4.7, the EQ_0 stability matrix decomposes into $[x, y]$ and z blocks.

The 1-dimensional $\mathcal{M}^+$ subspace is the fixed-point subspace, with the z-axis points left point-wise invariant *under the group action*

$$\mathcal{M}^+ = \mathrm{Fix}\,(C_2) = \{x \in \mathcal{M} : g\,x = x \text{ for } g \in \{e, C^{1/2}\}\} \tag{9.22}$$

(here $x = (x, y, z)$ is a 3-dimensional vector, not the coordinate x). A C_2-fixed point $x(t)$ in $\mathrm{Fix}\,(C_2)$ moves with time, but according to (9.15) remains within $x(t) \in \mathrm{Fix}\,(C_2)$ for all times; the subspace $\mathcal{M}^+ = \mathrm{Fix}\,(C_2)$ is flow invariant. *In case at hand this jargon is a bit of an overkill: clearly for $(x, y, z) = (0, 0, z)$ the full state space Lorenz equation (2.12) is reduced to the exponential contraction to the EQ_0 equilibrium,*

$$\dot{z} = -b\,z\,. \tag{9.23}$$

However, for higher-dimensional flows the flow-invariant subspaces can be high-dimensional, with interesting dynamics of their own. Even in this simple case this subspace

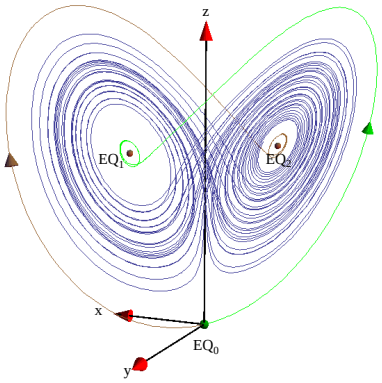


Figure 9.5: Lorenz attractor of figure 3.4, the full state space coordinates $[x, y, z]$, with the unstable manifold orbits $W^u(EQ_0)$. (Green) is a continuation of the unstable $\mathbf{e}^{(1)}$ of EQ_0, and (brown) is its π-rotated symmetric partner. Compare with figure 9.6. (E. Siminos)

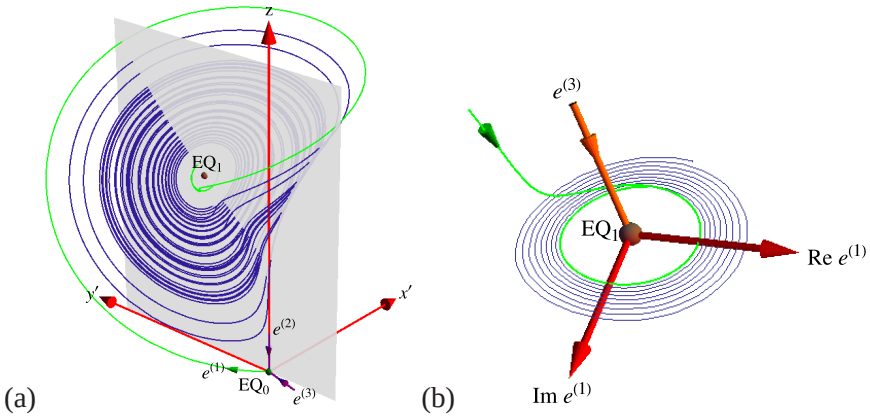


Figure 9.6: (a) Lorenz attractor plotted in $[\hat{x}, \hat{y}, z]$, the doubled-polar angle coordinates (9.24), with points related by π-rotation in the $[x, y]$ plane identified. Stable eigenvectors of EQ_0: $\mathbf{e}^{(3)}$ and $\mathbf{e}^{(2)}$, along the z axis (9.23). Unstable manifold orbit $W^u(EQ_0)$ (green) is a continuation of the unstable $\mathbf{e}^{(1)}$ of EQ_0. (b) Blow-up of the region near EQ_1: The unstable eigenplane of EQ_1 defined by Re $\mathbf{e}^{(2)}$ and Im $\mathbf{e}^{(2)}$, the stable eigenvector $\mathbf{e}^{(3)}$. The descent of the EQ_0 unstable manifold (green) defines the innermost edge of the strange attractor. As it is clear from (a), it also defines its outermost edge. (E. Siminos)

plays an important role as a topological obstruction: the orbits can neither enter it nor exit it, so the number of windings of a trajectory around it provides a natural, topological symbolic dynamics.

The $\mathcal{M}^-$ subspace is, however, not *flow-invariant, as the nonlinear terms $\dot{z} = xy - bz$ in the Lorenz equation (2.12) send all initial conditions within $\mathcal{M}^- = (x(0), y(0), 0)$ into the full, $z(t) \neq 0$ state space $\mathcal{M}/\mathcal{M}^+$.*

By taking as a Poincaré section any $C^{1/2}$-equivariant, non-self-intersecting surface that contains the z axis, the state space is divided into a half-space fundamental domain $\tilde{\mathcal{M}} = \mathcal{M}/C_2$ and its 180^o rotation $C^{1/2}\tilde{\mathcal{M}}$. An example is afforded by the $\mathcal{P}$ plane section of the Lorenz flow in figure 3.4. Take the fundamental domain $\tilde{\mathcal{M}}$ to be the half-space between the viewer and $\mathcal{P}$. Then the full Lorenz flow is captured by re-injecting back into $\tilde{\mathcal{M}}$ any trajectory that exits it, by a rotation of π around the z axis.

As any such $C^{1/2}$-invariant section does the job, a choice of a 'fundamental domain' is here largely mater of taste. For purposes of visualization it is convenient to make the double-cover nature of the full state space by $\tilde{\mathcal{M}}$ explicit, through any state space redefinition that maps a pair of points related by symmetry into a single point. In case at hand, this can be easily accomplished by expressing (x, y) in polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, and then plotting the flow in the 'doubled-polar angle representation:'

section 9.5
exercise 9.8

$$(\hat{x}, \hat{y}, z) = (r\cos 2\theta, r\sin 2\theta, z) = ((x^2 - y^2)/r, 2xy/r, z)\,, \tag{9.24}$$

*as in figure 9.6 (a). In contrast to the original G-equivariant coordinates $[x, y, z]$, the Lorenz flow expressed in the new coordinates $[\hat{x}, \hat{y}, z]$ is G-*invariant, *see example 9.18. In this representation the $\tilde{\mathcal{M}} = \mathcal{M}/C_2$ fundamental domain flow is a smooth, continuous flow, with (any choice of) the fundamental domain stretched out to seamlessly cover the entire $[\hat{x}, \hat{y}]$ plane. (continued in example 11.4)*

(E. Siminos and J. Halcrow)

Note: nonlinear coordinate transformations such as the doubled-polar angle representation (9.24) are *not* required to implement the symmetry quotienting $\mathcal{M}/G$.We deploy them only as a visualization aid that might help the reader disentangle 2-dimensional projections of higher-dimensional flows. All numerical calculations can still be carried in the initial, full state space formulation of a flow, with symmetry-related points identified by *linear* symmetry transformations.

in depth:
appendix H, p. 841

9.3 Relative periodic orbits

So far we have demonstrated that symmetry relates classes of orbits. Now we show that a symmetry reduces computation of periodic orbits to repeats of shorter, 'relative periodic orbit' segments.

Equivariance of a flow under a symmetry means that the symmetry image of a cycle is again a cycle, with the same period and stability. The new orbit may be topologically distinct (in which case it contributes to the multiplicity of the cycle) or it may be the same cycle.

A cycle p is G_p-*symmetric* under symmetry operation $g \in G_p$ if the operation acts on it as a shift in time, advancing a cycle point to a cycle point on the symmetry related segment. The cycle p can thus be subdivided into m_p repeats of a *relative periodic orbit segment*, 'prime' in the sense that the full state space cycle is built from its repeats. Thus in presence of a symmetry the notion of a periodic orbit is replaced by the notion of the shortest segment of the full state space cycle which tiles the cycle under the action of the group. In what follows we refer to this segment as a *relative periodic orbit*. In the literature this is sometimes referred to as a *short periodic orbit*, or, for finite symmetry groups, as a *pre-periodic* orbit.

Relative periodic orbits (or *equivariant periodic orbits*) are orbits $x(t)$ in state space $\mathcal{M}$ which exactly recur

$$x(t) = g\, x(t+T) \tag{9.25}$$

for the shortest fixed *relative period* T and a fixed group action $g \in G_p$. Parameters of this group action are referred to as 'phases' or 'shifts.' For a discrete group $g^m = e$ for some finite m, by (9.6), so the corresponding full state space orbit is periodic with period mT.

The period of the full orbit is given by the $m_p \times$(period of the relative periodic orbit), $T_p = |G_p| T_{\tilde{p}}$, and the ith Floquet multiplier $\Lambda_{p,i}$ is given by $\Lambda_{\tilde{p},i}^{m_p}$ of the

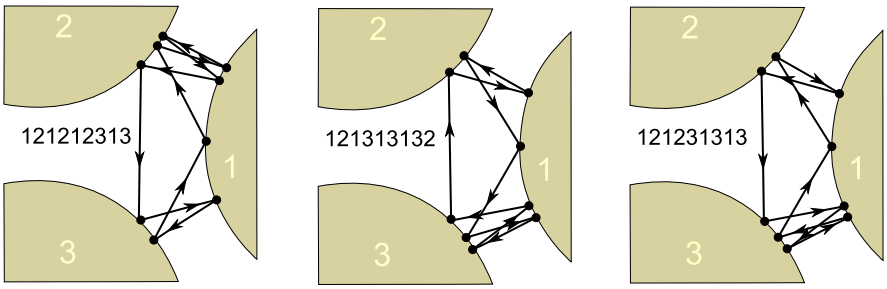


Figure 9.7: Cycle $\overline{121212313}$ has multiplicity 6; shown here is $\overline{121313132} = \sigma_{23}\overline{121212313}$. However, $\overline{121231313}$ which has the same stability and period is related to $\overline{121313132}$ by time reversal, but not by any C_{3v} symmetry.

relative periodic orbit. The elements of the quotient space $b \in G/G_p$ generate the copies bp, so the multiplicity of the full state space cycle p is $m_p = |G|/|G_p|$.

Example 9.15 ***Relative periodic orbits of Lorenz flow:*** *(continuation of example 9.14) The relation between the full state space periodic orbits, and the fundamental domain (9.24) reduced relative periodic orbits of the Lorenz flow: an asymmetric full state space cycle pair p, Rp maps into a single cycle $\tilde{p}$ in the fundamental domain, and any self-dual cycle $p = Rp = \tilde{p}R\tilde{p}$ is a repeat of a relative periodic orbit $\tilde{p}$.*

9.4 Dynamics reduced to fundamental domain

I submit my total lack of apprehension of fundamental concepts.

—John F. Gibson

So far we have used symmetry to effect a reduction in the number of independent cycles, by separating them into classes, and slicing them into 'prime' relative orbit segments. The next step achieves much more: it replaces each class by a single (typically shorter) prime cycle segment.

1. Discrete symmetry tessellates the state space into dynamically equivalent domains, and thus induces a natural partition of state space: If the dynamics is invariant under a discrete symmetry, the state space $\mathcal{M}$ can be completely tiled by a *fundamental domain* $\tilde{\mathcal{M}}$ and its symmetry images $\tilde{\mathcal{M}}_a = a\tilde{\mathcal{M}}$, $\tilde{\mathcal{M}}_b = b\tilde{\mathcal{M}}$, ... under the action of the symmetry group $G = \{e, a, b, \ldots\}$,

$$\mathcal{M} = \tilde{\mathcal{M}} \cup \tilde{\mathcal{M}}_a \cup \tilde{\mathcal{M}}_b \cdots \cup \tilde{\mathcal{M}}_{|G|} \,. \tag{9.26}$$

2. Discrete symmetriy can be used to restrict all computations to the fundamental domain $\tilde{\mathcal{M}} = \mathcal{M}/G$, the reduced state space quotient of the full state space $\mathcal{M}$ by the group actions of G.

   We can use the invariance condition (9.7) to move the starting point x into the fundamental domain $x = a\tilde{x}$, and then use the relation $a^{-1}b = h^{-1}$ to also relate the endpoint $y \in \tilde{\mathcal{M}}_b$ to its image in the fundamental domain $\tilde{\mathcal{M}}$. While the global trajectory runs over the full space $\mathcal{M}$, the restricted trajectory is brought back into the fundamental domain $\tilde{\mathcal{M}}$ any time it exits into

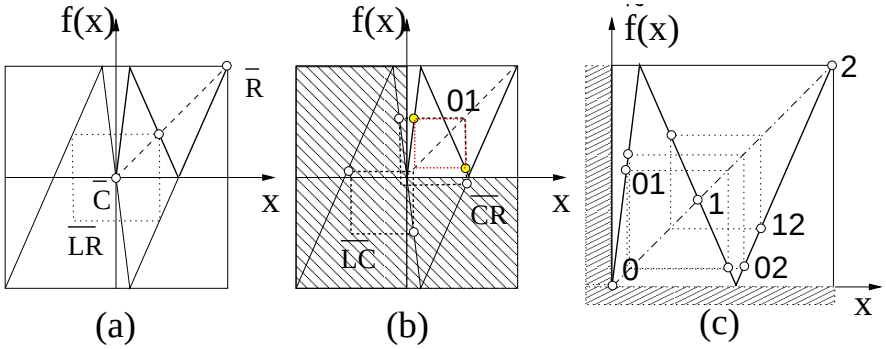


Figure 9.8: The bimodal Ulam sawtooth map of figure 9.4 with the D_1 symmetry $f(-x) = -f(x)$ restricted to the fundamental domain. $f(x)$ is indicated by the thin line, and fundamental domain map $\tilde{f}(\tilde{x})$ by the thick line. (a) Boundary fixed point $\overline{C}$ is the fixed point $\overline{0}$. The asymmetric fixed point pair $\{\overline{L},\overline{R}\}$ is reduced to the fixed point $\overline{2}$, and the full state space symmetric 2-cycle $\overline{LR}$ is reduced to the fixed point $\overline{1}$. (b) The asymmetric 2-cycle pair $\{\overline{LC},\overline{CR}\}$ is reduced to 2-cycle $\overline{01}$. (c) All fundamental domain fixed points and 2-cycles. (Y. Lan)

an adjoining tile; the two trajectories are related by the symmetry operation h which maps the global endpoint into its fundamental domain image.

3. Cycle multiplicities induced by the symmetry are removed by reduction of the full dynamics to the dynamics on a fundamental domain. Each symmetry-related set of global cycles p corresponds to precisely one fundamental domain (or relative) cycle $\tilde{p}$.

4. Conversely, each fundamental domain cycle $\tilde{p}$ traces out a segment of the global cycle p, with the end point of the cycle $\tilde{p}$ mapped into the irreducible segment of p with the group element $h_{\tilde{p}}$. A relative periodic orbit segment in the full state space is thus a periodic orbit in the fundamental domain.

5. The group elements $G = \{e, g_2, \ldots, g_{|G|}\}$ which map the fundamental domain $\tilde{\mathcal{M}}$ into its copies $g\tilde{\mathcal{M}}$, serve also as letters of a symbolic dynamics alphabet.

For a symmetry reduction in presence of continuous symmetries, see sect. 10.4.

exercise 9.6

Example 9.16 ***Group D_1 and reduction to the fundamental domain.*** *Consider again the reflection-symmetric bimodal Ulam sawtooth map $f(-x) = -f(x)$ of example 9.12, with symmetry group $D_1 = \{e, \sigma\}$. The state space $\mathcal{M} = [-1, 1]$ can be tiled by half-line $\tilde{\mathcal{M}} = [0, 1]$, and $\sigma\tilde{\mathcal{M}} = [-1, 0]$, its image under a reflection across $x = 0$ point. The dynamics can then be restricted to the* fundamental domain *$\tilde{x}_k \in \tilde{\mathcal{M}} = [0, 1]$; every time a trajectory leaves this interval, it is mapped back using σ.*

In figure 9.8 the fundamental domain map $\tilde{f}(\tilde{x})$ is obtained by reflecting $x < 0$ segments of the global map $f(x)$ into the upper right quadrant. $\tilde{f}$ is also bimodal and piecewise-linear, with $\tilde{\mathcal{M}} = [0, 1]$ split into three regions $\tilde{\mathcal{M}} = \{\tilde{\mathcal{M}}_0, \tilde{\mathcal{M}}_1, \tilde{\mathcal{M}}_2\}$ which we label with a 3-letter alphabet $\tilde{\mathcal{A}} = \{0, 1, 2\}$. The symbolic dynamics is again complete ternary dynamics, with any sequence of letters $\{0, 1, 2\}$ admissible.

However, the interpretation of the 'desymmetrized' dynamics is quite different - the multiplicity of every periodic orbit is now 1, and relative periodic segments of the full state space dynamics are all periodic orbits in the fundamental domain. Consider figure 9.8:

In (a) the boundary fixed point $\overline{C}$ is also the fixed point $\overline{0}$.

The asymmetric fixed point pair $\{\overline{L},\overline{R}\}$ is reduced to the fixed point $\overline{2}$, and the full state space symmetric 2-cycle $\overline{LR}$ is reduced to the fixed point $\overline{1}$. The asymmetric

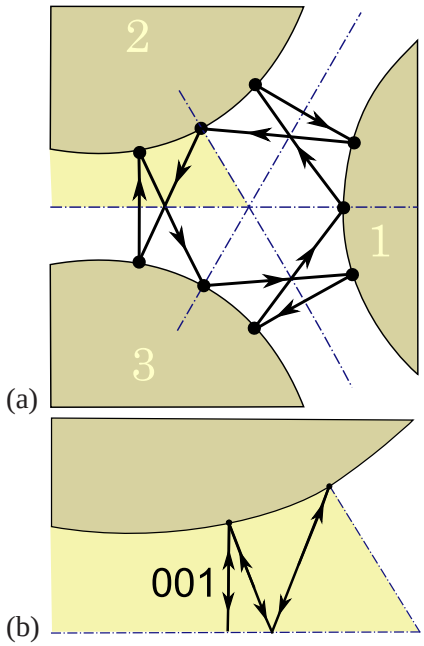


Figure 9.9: (a) The pair of full-space 9-cycles, the counter-clockwise $\overline{121232313}$ and the clockwise $\overline{131323212}$ correspond to (b) one fundamental domain 3-cycle $\overline{001}$.

2-cycle pair $\{\overline{LC},\overline{CR}\}$ *is reduced to the 2-cycle* $\overline{01}$. *Finally, the symmetric 4-cycle* $\overline{LCRC}$ *is reduced to the 2-cycle* $\overline{02}$. *This completes the conversion from the full state space for all fundamental domain fixed points and 2-cycles, figure 9.8 (c).*

Example 9.17 ***3-disk game of pinball in the fundamental domain***

If the dynamics is equivariant under interchanges of disks, the absolute disk labels $\epsilon_i = 1,2,\cdots,N$ *can be replaced by the symmetry-invariant relative disk→disk increments* g_i, *where* g_i *is the discrete group element that maps disk* $i-1$ *into disk* i. *For 3-disk system* g_i *is either reflection* σ *back to initial disk (symbol '0') or* $2\pi/3$ *rotation by* C *to the next disk (symbol '1'). An immediate gain arising from symmetry invariant relabeling is that* N*-disk symbolic dynamics becomes* $(N-1)$*-nary, with no restrictions on the admissible sequences.*

An irreducible segment corresponds to a periodic orbit in the fundamental domain, *a one-sixth slice of the full 3-disk system, with the symmetry axes acting as reflecting mirrors (see figure 9.3(d)). A set of orbits related in the full space by discrete symmetries maps onto a single fundamental domain orbit. The reduction to the fundamental domain desymmetrizes the dynamics and removes all global discrete symmetry-induced degeneracies: rotationally symmetric global orbits (such as the 3-cycles* $\overline{123}$ *and* $\overline{132}$*) have multiplicity 2, reflection symmetric ones (such as the 2-cycles* $\overline{12}$, $\overline{13}$ *and* $\overline{23}$*) have multiplicity 3, and global orbits with no symmetry are 6-fold degenerate. Table 12.2 lists some of the shortest binary symbols strings, together with the corresponding full 3-disk symbol sequences and orbit symmetries. Some examples of such orbits are shown in figures 9.7 and 9.9.* *(continued in example 12.7)*

9.5 Invariant polynomials

Physical laws should have the same form in symmetry-equivalent coordinate frames, so they are often formulated in terms of functions (Hamiltonians, Lagrangians,

$\cdots$) invariant under a given set of symmetries. The key result of the representation theory of invariant functions is:

Hilbert-Weyl theorem. For a compact group G there exists a finite G-invariant homogenous polynomial basis $\{u_1, u_2, \ldots, u_m\}$, $m \geq d$, such that any G-invariant polynomial can be written as a multinomial

$$h(x) = p(u_1(x), u_2(x), \ldots, u_m(x)), \qquad x \in \mathcal{M}. \tag{9.27}$$

These polynomials are linearly independent, but can be functionally dependent through nonlinear relations called *syzygies*.

Example 9.18 ***Polynomials invariant under discrete operations on*** $\mathbb{R}^3$***.*** *(continued from example 9.2)* *σ is a* reflection *through the $[x,y]$ plane. Any $\{e,\sigma\}$-invariant function can be expressed in the polynomial basis $\{u_1,u_2,u_3\} = \{x,y,z^2\}$.*

$C^{1/2}$ is a $[x,y]$-plane rotation *by π about the z-axis. Any $\{e,C^{1/2}\}$-invariant function can be expressed in the polynomial basis $\{u_1,u_2,u_3,u_4\} = \{x^2, xy, y^2, z\}$, with one syzygy between the basis polynomials, $(x^2)(y^2) - (xy)^2 = 0$.*

P is an inversion *through the point $(0,0,0)$. Any $\{e,P\}$-invariant function can be expressed in the polynomial basis $\{u_1,\cdots,u_6\} = \{x^2,y^2,z^2,xy,xz,yz\}$, with three syzygies between the basis polynomials, $(x^2)(y^2) - (xy)^2 = 0$, and its 2 permutations.*

For the D_2 dihedral group $G = \{e,\sigma,C^{1/2},P\}$ the G-invariant polynomial basis is $\{u_1,u_2,u_3,u_4\} = \{x^2,y^2,z^2,xy\}$, with one syzygy, $(x^2)(y^2) - (xy)^2 = 0$. *(continued in example 10.13)*

In practice, explicit construction of G-invariant basis can be a laborious undertaking, and we will not take this path except for a few simple low-dimensional cases, such as the 5-dimensional example of sect. 10.5. We prefer to apply the symmetry to the system as given, rather than undertake a series of nonlinear coordinate transformations that the theorem suggests. (What 'compact' in the above refers to will become clearer after we have discussed continuous symmetries. For now, it suffices to know that any finite discrete group is compact.) exercise 9.1

Résumé

A group G is a *symmetry* of the dynamical system $(\mathcal{M}, f)$ if its 'law of motion' retains its form under all symmetry-group actions, $f(x) = g^{-1} f(gx)$. A mapping u is said to be *invariant* if $gu = u$, where g is any element of G. If the mapping and the group actions commute, $gu = ug$, u is said to be *equivariant*. The governing dynamical equations are equivariant with respect to G.

We have shown here that if a dynamical system $(\mathcal{M}, f)$ has a symmetry G, the symmetry should be deployed to 'quotient' the state space to $\hat{\mathcal{M}} = \mathcal{M}/G$, i.e.,

identify all symmetry-equivalent $x \in \mathcal{M}$ on each group orbit, thus replacing the full state space dynamical system $(\mathcal{M}, f)$ by the symmetry-reduced $(\hat{\mathcal{M}}, \hat{f})$. The main result of this chapter can be stated as follows:

In presence of a discrete symmetry G, associated with each full state space solution p is the group of its symmetries $G_p \subseteq G$ of order $1 \leq |G_p| \leq |G|$, whose elements leave the orbit $\mathcal{M}_p$ invariant. The elements of G_p act on p as shifts, tiling it with $|G_p|$ copies of its shortest invariant segment, the relative periodic orbit $\tilde{p}$. The elements of the coset $b \in G/G_p$ generate $m_p = |G|/|G_p|$ equivalent copies of p.

Once you grasp the relation between the full state space $\mathcal{M}$ and the desymmetrized, G-quotiented reduced state space $\mathcal{M}/G$, you will find the life as a fundamentalist so much simpler that you will never return to your full state space ways of yesteryear. The reduction to the fundamental domain $\tilde{\mathcal{M}} = \mathcal{M}/G$ simplifies symbolic dynamics and eliminates symmetry-induced degeneracies. For the short orbits the labor saving is dramatic. For example, for the 3-disk game of pinball there are 256 periodic points of length 8, but reduction to the fundamental domain non-degenerate prime cycles reduces this number to 30. In the next chapter continuous symmetries will induce relative periodic orbits that never close a periodic orbit, and in the chapter 25 they will tile the infinite periodic state space, and reduce calculation of diffusion constant in an infinite domain to a calculation on a compact torus.

Commentary

Remark 9.1 Literature. We found Tinkham [9.1] the most enjoyable as a no-nonsense, the user friendliest introduction to the basic concepts. Byron and Fuller [9.2], the last chapter of volume two, offers an introduction even more compact than Tinkham's. For a summary of the theory of discrete groups see, for example, ref. [9.3]. Chapter 3 of Rebecca Hoyle [9.4] is a very student-friendly overview of the group theory a nonlinear dynamicist might need, with exception of the quotienting, reduction of dynamics to a fundamental domain, which is not discussed at all. We found sites such as en.wikipedia.org/wiki/Quotient_group helpful. Curiously, we have not read any of the group theory books that Hoyle recommends as background reading, which just confirms that there are way too many group theory books out there. For example, one that you will not find useful at all is ref. [9.5]. The reason is presumably that in the 20th century physics (which motivated much of the work on the modern group theory) the focus is on the linear representations used in quantum mechanics, crystallography and quantum field theory. We shall need these techniques in Chapter 21, where we reduce the linear action of evolution operators to irreducible subspaces. However, here we are looking at nonlinear dynamics, and the emphasis is on the symmetries of orbits, their reduced state space sisters, and the isotypic decomposition of their linear stability matrices.

appendix A.2.3

In ChaosBook we focus on chaotic dynamics, and skirt the theory of bifurcations, the landscape between the boredom of regular motions and the thrills of chaos. Chapter 4 of Rebecca Hoyle [9.4] is a student-friendly introduction to the treatment of bifurcations in presence of symmetries, worked out in full detail and generality in monographs by Golubitsky, Stewart and Schaeffer [9.6], Golubitsky and Stewart [9.7] and Chossat and Lauterbach [9.8]. Term 'stabilizer' is used, for example, by Broer *et al.* [9.9] to refer to a periodic orbit with Z_2 symmetry; they say that the relative or pre-periodic segment is in this case called a 'short periodic orbit.' In Efstathiou [9.10] a subgroup of 'short periodic orbit' symmetries is referred to as a 'nontrivial isotropy group or stabilizer.' Chap. 8 of Govaerts [9.11] offers a review of numerical methods that employ equivariance with respect to compact, and mostly discrete groups. (continued in remark 10.1)

Remark 9.2 Symmetries of the Lorenz equation: (continued from remark 2.3) After having studied example 9.14 you will appreciate why `ChaosBook.org` starts out with the symmetry-less Rössler flow (2.17), instead of the better known Lorenz flow (2.12). Indeed, getting rid of symmetry was one of Rössler's motivations. He threw the baby out with the water; for Lorenz flow dimensionalities of stable/unstable manifolds make possible a robust heteroclinic connection absent from Rössler flow, with unstable manifold of an equilibrium flowing into the stable manifold of another equilibrium. How such connections are forced upon us is best grasped by perusing the chapter 13 'Heteroclinic tangles' of the inimitable Abraham and Shaw illustrated classic [9.12]. Their beautiful hand-drawn sketches elucidate the origin of heteroclinic connections in the Lorenz flow (and its high-dimensional Navier-Stokes relatives) better than any computer simulation. Miranda and Stone [9.13] were first to quotient the C_2 symmetry and explicitly construct the desymmetrized, 'proto-Lorenz system,' by a nonlinear coordinate transformation into the Hilbert-Weyl polynomial basis invariant under the action of the symmetry group [9.14]. For in-depth discussion of symmetry-reduced ('images') and symmetry-extended ('covers') topology, symbolic dynamics, periodic orbits, invariant polynomial bases etc., of Lorenz, Rössler and many other low-dimensional systems there is no better reference

than the Gilmore and Letellier monograph [9.15]. They interpret [9.16] the proto-Lorenz and its 'double cover' Lorenz as 'intensities' being the squares of 'amplitudes,' and call quotiented flows such as (Lorenz)/C_2 'images.' Our 'doubled-polar angle' visualization figure 11.8 is a proto-Lorenz in disguise; we, however, integrate the flow and construct Poincaré sections and return maps in the original Lorenz $[x, y, z]$ coordinates, without any nonlinear coordinate transformations. The Poincaré return map figure 11.9 is reminiscent in shape both of the one given by Lorenz in his original paper, and the one plotted in a radial coordinate by Gilmore and Letellier. Nevertheless, it is profoundly different: our return maps are from unstable manifold $\to$ itself, and thus intrinsic and coordinate independent. In this we follow ref. [9.17]. This construction is necessary for high-dimensional flows in order to avoid problems such as double-valuedness of return map projections on arbitrary 1-dimensional coordinates encountered already in the Rössler example of figure 3.3. More importantly, as we know the embedding of the unstable manifold into the full state space, a periodic point of our return map *is* - regardless of the length of the cycle - the periodic point in the full state space, so no additional Newton searches are needed. In homage to Lorenz, we note that his return map was already symmetry-reduced: as z belongs to the symmetry invariant Fix (G) subspace, one can replace dynamics in the full space by $\dot{z}, \ddot{z}, \cdots$. That is G-invariant by construction [9.15].

Remark 9.3 Examples of systems with discrete symmetries. Almost any flow of interest is symmetric in some way or other: the list of examples is endless, we list here a handful that we found interesting. One has a C_2 symmetry in the Lorenz system (remark 2.3), the Ising model, and in the 3-dimensional anisotropic Kepler potential [9.18, 9.19, 9.20], a $D_4 = C_{4v}$ symmetry in quartic oscillators [9.21, 9.22], in the pure x^2y^2 potential [9.23, 9.24] and in hydrogen in a magnetic field [9.25], and a $D_2 = C_{2v} = V_4 = C_2 \times C_2$ symmetry in the stadium billiard [9.26]. A very nice nontrivial desymmetrization is carried out in ref. [9.27]. An example of a system with $D_3 = C_{3v}$ symmetry is provided by the motion of a particle in the Hénon-Heiles potential [9.28, 9.29, 9.30, 9.31]

$$V(r,\theta) = \frac{1}{2}r^2 + \frac{1}{3}r^3 \sin(3\theta) \ .$$

Our 3-disk coding is insufficient for this system because of the existence of elliptic islands and because the three orbits that run along the symmetry axis cannot be labeled in our code. As these orbits run along the boundary of the fundamental domain, they require the special treatment. A partial classification of the 67 possible symmetries of solutions of the plane Couette flow of example 9.7, and their reduction 5 conjugate classes is given in ref. [9.32].

Exercises

9.1. **Polynomials invariant under discrete operations on $\mathbb{R}^3$.** Prove that the $\{e, \sigma\}$, $\{e, C^{1/2}\}$, $\{e, P\}$ and $\{e, \sigma, C^{1/2}, P\}$-invariant polynomial basis and syzygies are those listed in example 9.18.

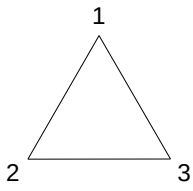
9.2. **$G_x \subset G$.** Prove that the set G_x as defined in (9.17) is a subgroup of G.

9.3. **Transitivity of conjugation.** Assume that g_1, g_2, $g_3 \in G$ and both g_1 and g_2 are conjugate to g_3. Prove that g_1 is conjugate to g_2.

9.4. **Isotropy subgroup of gx.** Prove that for $g \in G$, x and gx have conjugate isotropy subgroups:

$$G_{gx} = g\, G_x\, g^{-1}$$

9.5. **D_3: symmetries of an equilateral triangle.** Consider group $D_3 \cong C_{3v}$, the symmetry group of an equilateral triangle:



(a) List the group elements and the corresponding geometric operations

(b) Find the subgroups of the group D_3.

(c) Find the classes of D_3 and the number of elements in them, guided by the geometric interpretation of group elements. Verify your answer using the definition of a class.

(d) List the conjugacy classes of subgroups of D_3. (continued as exer:FractRot)

9.6. **Reduction of 3-disk symbolic dynamics to binary.** (continued from exercise 1.1)

(a) Verify that the 3-disk cycles
$\{\overline{1\,2},\ \overline{1\,3},\ \overline{2\,3}\}$, $\{\overline{1\,2\,3},\ \overline{1\,3\,2}\}$, $\{\overline{12\,13}$ + 2 perms.$\}$, $\{\overline{121\,232\,313}$ + 5 perms.$\}$, $\{\overline{121\,323}$+ 2 perms.$\}$, $\cdots$,
correspond to the fundamental domain cycles $\overline{0}$, $\overline{1}$, $\overline{01}$, $\overline{001}$, $\overline{011}$, $\cdots$ respectively.

(b) Check the reduction for short cycles in table 12.2 by drawing them both in the full 3-disk system and in the fundamental domain, as in figure 9.9.

(c) Optional: Can you see how the group elements listed in table 12.2 relate irreducible segments to the fundamental domain periodic orbits?

(continued in exercise 12.6)

9.7. **C_2-equivariance of Lorenz system.** Verify that the vector field in Lorenz equations (2.12)

$$\dot{x} = v(x) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma(y-x) \\ \rho x - y - xz \\ xy - bz \end{bmatrix} \tag{9.28}$$

is equivariant under the action of cyclic group $C_2 = \{e, C^{1/2}\}$ acting on $\mathbb{R}^3$ by a π rotation about the z axis,

$$C^{1/2}(x, y, z) = (-x, -y, z)\,,$$

as claimed in example 9.5. (continued in exercise 9.8)

9.8. **Lorenz system in polar coordinates: group theory.** Use (6.7), (6.8) to rewrite the Lorenz equation (9.28) in polar coordinates (r, θ, z), where $(x, y) = (r\cos\theta, r\sin\theta)$.

1. Show that in the polar coordinates Lorenz flow takes form

$$\begin{aligned} \dot{r} &= \frac{r}{2}\left(-\sigma - 1 + (\sigma + \rho - z)\sin 2\theta \right. \\ &\quad \left. +(1-\sigma)\cos 2\theta\right) \\ \dot{\theta} &= \frac{1}{2}\left(-\sigma + \rho - z + (\sigma - 1)\sin 2\theta \right. \\ &\quad \left. +(\sigma + \rho - z)\cos 2\theta\right) \\ \dot{z} &= -bz + \frac{r^2}{2}\sin 2\theta\,. \end{aligned} \tag{9.29}$$

2. Argue that the transformation to polar coordinates is invertible almost everywhere. Where does the inverse not exist? What is group-theoretically special about the subspace on which the inverse not exist?

3. Show that this is the (Lorenz)/C_2 quotient map for the Lorenz flow, i.e., that it identifies points related by the π rotation in the $[x, y]$ plane.

4. Rewrite (9.28) in the invariant polynomial basis of example 9.18 and exercise 9.29.

5. Show that a periodic orbit of the Lorenz flow in polar representation (9.29) is either a periodic orbit or a relative periodic orbit (9.25) of the Lorenz flow in the (x, y, z) representation.

By going to polar coordinates we have quotiented out the π-rotation $(x, y, z) \to (-x, -y, z)$ symmetry of the Lorenz equations, and constructed an explicit representation of the desymmetrized Lorenz flow.

9.9. **Proto-Lorenz system.** Here we quotient out the C_2 symmetry by constructing an explicit "intensity" representation of the desymmetrized Lorenz flow, following Miranda and Stone [9.13].

1. Rewrite the Lorenz equation (2.12) in terms of variables

$$(u, v, z) = (x^2 - y^2, 2xy, z)\,, \tag{9.30}$$

show that it takes form

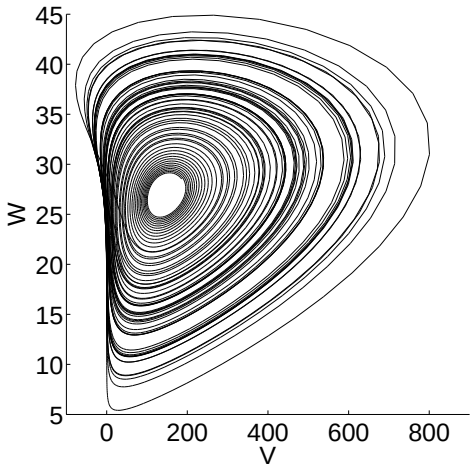
$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -(\sigma+1)u + (\sigma - r)v + (1-\sigma)N + vz \\ (r-\sigma)u - (\sigma+1)v + (r+\sigma)N - uz - uN \\ v/2 - bz \end{bmatrix} \tag{9.31}$$
$$N = \sqrt{u^2 + v^2}\,.$$

2. Show that this is the (Lorenz)/C_2 quotient map for the Lorenz flow, i.e., that it identifies points related by the π rotation (9.20).

3. Show that (9.30) is invertible. Where does the inverse not exist?

4. Compute the equilibria of proto-Lorenz and their stabilities. Compare with the equilibria of the Lorenz flow.

5. Plot the strange attractor both in the original form (2.12) and in the proto-Lorenz form (9.31)



for the Lorenz parameter values $\sigma = 10$, $b = 8/3$, $\rho = 28$. Topologically, does it resemble more the Lorenz, or the Rössler attractor, or neither? (plot by J. Halcrow)

7. Show that a periodic orbit of the proto-Lorenz is either a periodic orbit or a relative periodic orbit of the Lorenz flow.

8. Show that if a periodic orbit of the proto-Lorenz is also periodic orbit of the Lorenz flow, their Floquet multipliers are the same. How do the Floquet multipliers of relative periodic orbits of the Lorenz flow relate to the Floquet multipliers of the proto-Lorenz?

9 What does the volume contraction formula (4.43) look like now? Interpret.

10. Show that the coordinate change (9.30) is the same as rewriting (9.29) in variables

$$(u, v) = (r^2 \cos 2\theta, r^2 \sin 2\theta)\,,$$

i.e., squaring a complex number $z = x + iy$, $z^2 = u + iv$.

11. How is (9.31) related to the invariant polynomial basis of example 9.18 and exercise 9.29?

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