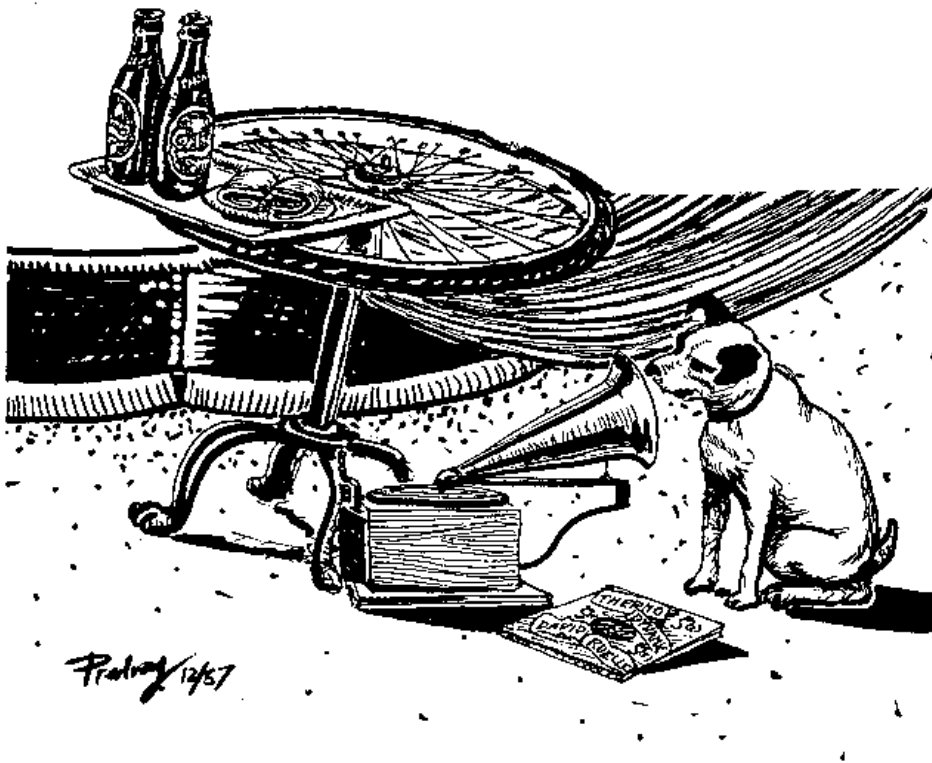


Chaos: Classical and Quantum

I: Deterministic Chaos



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Gábor Vattay

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Contributors

No man but a blockhead ever wrote except for money
—Samuel Johnson

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Daniel Borrero Oct 23 2008, `soluCycles.tex`

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I feel I never want to write another book. What's the good!
I can eke living on stories and little articles, that don't cost
a tithe of the output a book costs. Why write novels any
more!

—D.H. Lawrence

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F. Haake's heartfelt lament on page 377 was uttered at the end of the first conference presentation of cycle expansions, in 1988. G.P. Morriss advice to students as how to read the introduction to this book, page 6, was offered during a 2002 graduate course in Dresden. K. Huang's C.N. Yang interview quoted on page 337 is available on ChaosBook.org/extras. T.D. Lee remarks on as to who is to blame, page 37 and page 269, as well as M. Shub's helpful technical remark on page 476 came during the Rockefeller University December 2004 "Feigenbaum Fest." Quotes on pages 37, 127, and 334 are taken from a book review by J. Guckenheimer [1.1].

Who is the 3-legged dog reappearing throughout the book? Long ago, when we were innocent and knew not Borel measurable α to Ω sets, P. Cvitanović asked V. Baladi a question about dynamical zeta functions, who then asked J.-P. Eckmann, who then asked D. Ruelle. The answer was transmitted back: “The master says: ‘It is holomorphic in a strip’.” Hence His Master’s Voice logo, and the 3-legged dog is us, still eager to fetch the bone. The answer has made it to the book, though not precisely in His Master’s voice. As a matter of fact, the answer *is* the book. We are still chewing on it.

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