Chapter 8

Billiards

The dynamics that we have the best intuitive grasp on, and find easiest to grasp, with both numerically and conceptually, is the dynamics of billiards.

For billiards, discrete time is altogether natural; a particle moving through a billiard suffers a sequence of instantaneous kicks, and executes simple motion in between, so there is no need to contrive a Poincaré section. We have already used this system in sect. 1.3 as the intuitively most accessible example of chaos. Here we define billiard dynamics more precisely, anticipating the applications to come.

8.1 Billiard dynamics

A billiard is defined by a connected region \( Q \subset \mathbb{R}^D \), with boundary \( \partial Q \subset \mathbb{R}^{D-1} \) separating \( Q \) from its complement \( \mathbb{R}^D \setminus Q \). The region \( Q \) can consist of one compact, finite volume component (in which case the billiard phase space is bounded), as for the stadium billiard of figure 8.1, or can be infinite in extent, with its complement \( \mathbb{R}^D \setminus Q \) consisting of one or several finite or infinite volume components (in which case the phase space is open, as for the 3-disk pinball game in figure 1.1). In what follows we shall most often restrict our attention to planar billiards.

A point particle of mass \( m \) and momentum \( p_0 = mv_0 \) moves freely within the billiard, along a straight line, until it encounters the boundary. There it reflects specularly (specular = mirrorlike), with no change in the tangential component of momentum, and instantaneous reversal of the momentum component normal to the boundary,

\[
p' = p - 2(p \cdot \hat{n})\hat{n},
\]

with \( \hat{n} \) the unit vector normal to the boundary \( \partial Q \) at the collision point. The angle of incidence equals the angle of reflection, as illustrated in figure 8.2. A billiard is a Hamiltonian system with a 2D-dimensional phase space \( x = (q, p) \) and potential \( V(q) = 0 \) for \( q \in Q \), \( V(q) = \infty \) for \( q \in \partial Q \).

A billiard flow has a natural Poincaré section defined by Birkhoff coordinates \( s_n \), the arc length position of the \( n \)th bounce measured along the billiard boundary, and \( p_n = |p| \sin \phi_n \), the momentum component parallel to the boundary, where \( \phi_n \) is the angle between the outgoing trajectory and the normal to the boundary. We measure both the arc length \( s \) and the parallel momentum \( p \) counterclockwise relative to the outward normal (see figure 8.2 as well as figure 3.3). In \( D = 2 \), the Poincaré section is a cylinder (topologically an annulus), figure 8.3, where the parallel momentum \( p \) ranges for \(-|p| \) to \( |p| \), and the \( s \) coordinate is cyclic along each connected component of \( \partial Q \). The volume in the full phase space is preserved by the Liouville theorem (7.31). The Birkhoff coordinates \( x = (s, p) \in \mathcal{P} \), are the natural choice, because with them the Poincaré return map preserves the phase space volume of the \((s, p)\) parameterized Poincaré section (a perfectly good coordinate set \((s, \phi)\) does not do that).

Without loss of generality we set \( m = |v| = |p| = 1 \). Poincaré section condition eliminates one dimension, and the energy conservation \(|p| = 1\) eliminates another, so the Poincaré section return map \( P \) is \((2D - 2)\)-dimensional.

The dynamics is given by the Poincaré return map

\[
P: (s_n, p_n) \mapsto (s_{n+1}, p_{n+1})
\]
from the $n$th collision to the $(n+1)$st collision. The discrete time dynamics map $P$ is equivalent to the Hamiltonian flow (7.1) in the sense that both describe the same full trajectory. Let $t_n$ denote the instant of $n$th collision. Then the position of the pinball $Q$ at time $t_n + \tau \leq t_{n+1}$ is given by $2D - 2$ Poincaré section coordinates $(q_n, p_n) \in \mathcal{P}$ together with $\tau$, the distance reached by the pinball along the $n$th section of its trajectory (as we have set the pinball velocity to 1, the time of flight equals the distance traversed).

**Example 8.1 3-disk game of pinball:** In case of bounces off a circular disk, the position coordinate $s = r \theta$ is given by angle $\theta \in [0, 2\pi]$. For example, for the 3-disk game of pinball of figure 1.6 and figure 3.3 we have two types of collisions: 

$$P_0: \begin{cases} \dot{\phi} = -\phi + 2 \arcsin p \\ \dot{p}' = -p + \frac{\phi}{2} \sin \phi' \end{cases} \quad \text{back-reflection} \quad (8.3)$$

$$P_1: \begin{cases} \dot{\phi} = \phi - 2 \arcsin p + 2\pi/3 \\ \dot{p}' = p - \frac{\phi}{2} \sin \phi' \end{cases} \quad \text{reflect to 3rd disk.} \quad (8.4)$$

Here $a = \text{radius of a disk}$, and $R = \text{center-to-center separation}$. Actually, as in this example we are computing intersections of circles and straight lines, nothing more than high-school geometry is required. There is no need to compute $\arcsin$ - one only needs to compute one square root per each reflection, and the simulations can be very fast. 

Trajectory of the pinball in the 3-disk billiard is generated by a series of $P_0$'s and $P_1$'s. At each step one has to check whether the trajectory intersects the desired disk (and no disk in-between). With minor modifications, the above formulas are valid for any smooth billiard as long as we replace $a$ by the local curvature of the boundary at the point of collision.

### 8.2 Stability of billiards

We turn next to the question of local stability of discrete time billiard systems. Infinitesimal equations of variations (4.2) do not apply, but the multiplicative structure (4.44) of the finite-time Jacobian matrix does. As they are more physical than most maps studied by dynamacists, let us work out the billiard stability in some detail.

On the face of it, a plane billiard phase space is 4-dimensional. However, one dimension can be eliminated by energy conservation, and the other by the fact that the magnitude of the velocity is constant. We shall now show how going to a local frame of motion leads to a $[2 \times 2]$ Jacobian matrix.

Consider a 2-dimensional billiard with phase space coordinates $s = (q_1, q_2, p_1, p_2)$. Let $t_k$ be the instant of the $k$th collision of the pinball with the billiard boundary, and $\delta t_k = t_k - \tau$ positive and infinitesimal. With the mass and the velocity equal to 1, the momentum direction can be specified by angle $\theta$: $\theta = (q_1, q_2, \sin \theta, \cos \theta)$. Now parametrize the $2 \times 2$ dimensional neighborhood of a trajectory segment by $\delta \alpha = (\delta q_1, \delta \theta)$, where

$$\delta q_1 \cos \theta - \delta q_2 \sin \theta, \quad (8.5)$$

$\delta \theta$ is the variation in the direction of the pinball motion. Due to energy conservation, there is no need to keep track of $\delta q_1$, variation along the flow, as that remains constant. $(\delta q_1, \delta q_2)$ is the coordinate variation transverse to the $k$th segment of the flow. From the Hamilton’s equations of motion for a free particle, $dq_i/dt = p_i$, $dp_i/dt = 0$, we obtain the equations of motion (4.1) for the linearized neighborhood

$$\frac{d}{dt} \delta \theta = 0, \quad \frac{d}{dt} \delta q_1 = \delta \theta. \quad (8.6)$$

Let $\delta q_k = \delta q(t_k^*)$ and $\delta q_k^- = \delta q(t_k^-)$ be the local coordinates immediately after the $k$th collision, and $\delta q_k^+ = \delta q(t_k^+)$ immediately before. Integrating the free flight from $t_k^-$ to $t_k$ we obtain

$$\delta q_k^- = \delta q_{k-1}^- + \tau_k \delta \theta_{k-1}, \quad \tau_k = t_k - t_{k-1}$$

$$\delta q_k^+ = \delta q_{k-1}^+. \quad (8.7)$$

and the Jacobian matrix (4.43) for the $k$th free flight segment is

$$M_T(\mathbf{z}_k) = \begin{pmatrix} 1 & \tau_k \\ 0 & 1 \end{pmatrix}. \quad (8.8)$$

At incidence angle $\phi_k$ (the angle between the outgoing particle and the outgoing normal to the billiard edge), the incoming transverse variation $\delta q_k^-$ projects onto an arc on the billiard boundary of length $\delta q_k^- / \cos \phi_k$. The corresponding incidence angle variation $\delta \theta_k = \delta q_k^- / \rho_k \cos \phi_k$, $\rho_k =$ local radius of curvature, increases the angular spread to

$$\delta q_k = -\delta q_k^-$$

$$\delta \theta_k = -\delta \theta_k^- - \frac{2}{\rho_k \cos \phi_k} \delta q_k^- . \quad (8.9)$$
Figure 8.4: Defocusing of a beam of nearby trajectories at a billiard collision. (A. Wirzba)

so the Jacobian matrix associated with the reflection is

\[ M_p(x_k) = - \begin{pmatrix} 1 & 0 \\ r_k & 1 \end{pmatrix}, \quad r_k = \frac{2}{\rho_k \cos \phi_k}, \] (8.10)

The full Jacobian matrix for \( n_p \) consecutive flight bounces describes a beam of trajectories defocused by \( M_p \) along the free flight (the \( r_k \) terms below) and defocused/refocused at reflections by \( M_k \) (the \( r_k \) terms below)

\[ M_p = (-1)^{n_p} \prod_{k=0}^{1} \begin{pmatrix} 1 & 0 \\ r_k & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \] (8.11)

where \( r_k \) is the flight time of the \( k \)th free-flight segment of the orbit, \( r_k = 2/\rho_k \cos \phi_k \) is the defocusing due to the \( k \)th reflection, and \( \rho_k \) is the radius of curvature of the billiard boundary at the \( k \)th scattering point (for our 3-disk game of pinball, \( \rho = 1 \)). As the billiard dynamics is phase space volume preserving, \( \det M = 1 \), and the eigenvalues are given by (7.22).

This is an example of the Jacobian matrix chain rule (4.52) for discrete time systems (the Hénon map stability (4.53) is another example). Stability of every flight segment or reflection taken alone is a shear with two unit eigenvalues,

\[ \det M_T = \det \begin{pmatrix} 1 & \tau_k \\ 0 & 1 \end{pmatrix}, \quad \det M_R = \det \begin{pmatrix} 1 & 0 \\ \rho_k & 1 \end{pmatrix}, \] (8.12)

but acting in concert in the interwoven sequence (8.11) they can lead to a hyperbolic deformation of the infinitesimal neighborhood of a billiard trajectory.

As a concrete application, consider the 3-disk pinball system of sect. 1.3. Analytic expressions for the lengths and eigenvalues of \( \delta, \tilde{1} \) and \( \tilde{10} \) cycles follow from elementary geometrical considerations. Longer cycles require numerical evaluation by methods such as those described in chapter 13.
Here we take $p_k = -1$ for the semicircle sections of the boundary, and $\cos \phi_k$ remains constant for all bounces in a rotation sequence. The time of flight between two semicircle bounces is $\tau_k = 2 \cos \phi_k$. The Jacobian matrix of one semicircle reflection followed by the flight to the next bounce is
\[
J = (-1)^k \begin{pmatrix}
1 & 0 & 3 & 0 \\
-2 \cos \phi_k & 2 & -1 & 0 \\
0 & 0 & 1 & 1 \\
-1 & -2 \cos \phi_k & 0 & 1 \\
\end{pmatrix}
\]
A free flight must always be followed by $k = 1, 2, 3, \ldots$ bounces along a semicircle, hence the natural symbolic dynamics for this problem is nary, with the corresponding Jacobian matrix given by shear (i.e., the eigenvalues remain equal to 1 throughout the whole rotation), and $k$ bounces inside a circle lead to
\[
J^k = (-1)^k \begin{pmatrix}
-2k - 1 & 2k \cos \phi & 2k - 1 \\
2k \cos \phi & 2 & -1 \\
0 & 0 & 1 \\
-1 & -2k \cos \phi & 0 & 1 \\
\end{pmatrix}
\] (8.13)

The Jacobian matrix of a cycle $p$ of length $n_p$ is given by
\[
J_p = (-1)^{n_p} \prod_{k=1}^{n_p} \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 \\
-1 & -2 \cos \phi & 0 & 1 \\
\end{pmatrix}
\] (8.14)

Adopt your pinball simulator to the stadium billiard.

8.5 A test of your pinball simulator. Test your exercise 8.3 pinball simulator by computing numerically cycle stabilities by tracking distances to nearby orbits. Compare your result with the exact analytic formulas of exercise 13.7 and 13.8.

8.6 Birkhoff coordinates. Prove that the Birkhoff coordinates are phase space volume preserving.

References


