Appendix E

Counting itineraries

E.1 Counting curvatures

One consequence of the finiteness of topological polynomials is that the contributions to curvatures at every order are even in number, half with positive and half with negative sign. For instance, for complete binary labeling (20.7),

$$
\begin{aligned}
c_4 &= -t_{0001} - t_{0011} - t_{0111} - t_0 t_{01} t_1 \\
&\quad + t_0 t_{001} + t_0 t_{011} + t_{001} t_1 + t_{011} t_1 \,.
\end{aligned} \tag{E.1}
$$

We see that 2^3 terms contribute to c_4, and exactly half of them appear with a negative sign - hence if all binary strings are admissible, this term vanishes in the counting expression.

exercise E.2

Such counting rules arise from the identity

$$\prod_p \left(1 + t_p\right) = \prod_p \frac{1 - {t_p}^2}{1 - t_p} \,. \tag{E.2}$$

Substituting $t_p = z^{n_p}$ and using (15.18) we obtain for unrestricted symbol dynamics with N letters

$$\prod_p^{\infty} (1 + z^{n_p}) = \frac{1 - Nz^2}{1 - Nz} = 1 + Nz + \sum_{k=2}^{\infty} z^k \left(N^k - N^{k-1}\right)$$

The z^n coefficient in the above expansion is the number of terms contributing to c_n curvature, so we find that for a complete symbolic dynamics of N symbols and

$n > 1$, the number of terms contributing to c_n is $(N-1)N^{k-1}$ (of which half carry a minus sign).

exercise E.4

We find that for complete symbolic dynamics of N symbols and $n > 1$, the number of terms contributing to c_n is $(N-1)N^{n-1}$. So, superficially, not much is gained by going from periodic orbits trace sums which get N^n contributions of n to the curvature expansions with $N^n(1 - 1/N)$. However, the point is not the number of the terms, but the cancelations between them.

Exercises

E.1. **Lefschetz zeta function.** Elucidate the relation between the topological zeta function and the Lefschetz zeta function.

E.2. **Counting the 3-disk pinball counterterms.** Verify that the number of terms in the 3-disk pinball curvature expansion (20.35) is given by

$$
\begin{aligned}
\prod_p \left(1 + t_p\right) &= \frac{1 - 3z^4 - 2z^6}{1 - 3z^2 - 2z^3} = 1 + 3z^2 + 2z^3 + \frac{z^4(6 + 12z + 2z^2)}{1 - 3z^2 - 2z^3} \\
&= 1 + 3z^2 + 2z^3 + 6z^4 + 12z^5 + 20z^6 + 48z^7 + 84z^8 + 184z^9 + \cdots
\end{aligned} \tag{E.3}
$$

This means that, for example, c_6 has a total of 20 terms, in agreement with the explicit 3-disk cycle expansion (20.36).

E.3. **Cycle expansion denominators**.** Prove that the denominator of c_k is indeed D_k, as asserted (D.14).

E.4. **Counting subsets of cycles.** The techniques developed above can be generalized to counting subsets of cycles. Consider the simplest example of a dynamical system with a complete binary tree, a repeller map (11.4) with two straight branches, which we label 0 and 1. Every cycle weight for such map factorizes, with a factor t_0 for each 0, and factor t_1 for each 1 in its symbol string. The transition matrix traces (15.7) collapse to $tr(T^k) = (t_0 + t_1)^k$, and $1/\zeta$ is simply

$$\prod_p \left(1 - t_p\right) = 1 - t_0 - t_1 \tag{E.4}$$

Substituting into the identity

$$\prod_p \left(1 + t_p\right) = \prod_p \frac{1 - {t_p}^2}{1 - t_p}$$

we obtain

$$
\begin{aligned}
\prod_p \left(1 + t_p\right) &= \frac{1 - t_0^2 - t_1^2}{1 - t_0 - t_1} = 1 + t_0 + t_1 + \frac{2 \ldots}{1 - \ldots} \\
&= 1 + t_0 + t_1 + \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} 2 \binom{n-2}{k-1} t_0^k \ldots
\end{aligned}
$$

Hence for $n \geq 2$ the number of terms in the expansion ?! with k 0's and $n-k$ 1's in their symbol sequences is $2\binom{n-2}{k-1}$. This is the degeneracy of distinct cycle eigenvalues in fig. ?!; for systems with non-uniform hyperbolicity this degeneracy is lifted (see fig. ?!).

In order to count the number of prime cycles in each such subset we denote with $M_{n,k}$ $(n = 1, 2, \ldots; k = \{0, 1\}$ for $n = 1$; $k = 1, \ldots, n-1$ for $n \geq 2)$ the number of prime n-cycles whose labels contain k zeros, use binomial string counting and Möbius inversion and obtain

$$
\begin{aligned}
M_{1,0} &= M_{1,1} = 1 \\
nM_{n,k} &= \sum_{m \mid \frac{n}{k}} \mu(m) \binom{n/m}{k/m}, \qquad n \geq 2, k = 1, \ldots
\end{aligned}
$$

where the sum is over all m which divide both n and k.