Chapter 19

Discrete factorization

> No endeavor that is worthwhile is simple in prospect; if it is right, it will be simple in retrospect.
>
> —Edward Teller

THE UTILITY of discrete symmetries in reducing spectrum calculations is familiar from quantum mechanics. Here we show that the classical spectral determinants factor in essentially the same way as the quantum ones. In the process we 1.) learn that the classical dynamics, once recast into the language of evolution operators, is much closer to quantum mechanics than is apparent in the Newtonian, ODE formulation (linear evolution operators/PDEs, group-theoretical spectral decompositions, . . .), 2.) that once the symmetry group is quotiented out, the dynamics simplifies, and 3.) it's a triple home run: simpler symbolic dynamics, fewer cycles needed, much better convergence of cycle expansions. Once you master this, going back is unthinkable.

The main result of this chapter can be stated as follows:

If the dynamics possesses a discrete symmetry, the contribution of a cycle p of multiplicity m_p to a dynamical zeta function factorizes into a product over the d_α-dimensional irreducible representations D_α of the symmetry group,

$$(1 - t_p)^{m_p} = \prod_\alpha \det\left(1 - D_\alpha(h_{\tilde{p}})t_{\tilde{p}}\right)^{d_\alpha}, \quad t_p = t_{\tilde{p}}^{g/m_p} ,$$

where $t_{\tilde{p}}$ is the cycle weight evaluated on the relative periodic orbit $\tilde{p}$, $g = |G|$ is the order of the group, $h_{\tilde{p}}$ is the group element relating the fundamental domain cycle $\tilde{p}$ to a segment of the full space cycle p, and m_p is the multiplicity of the p cycle. As dynamical zeta functions have particularly simple cycle expansions, a geometrical shadowing interpretation of their convergence, and suffice for determination of leading eigenvalues, we shall use them to explain the group-theoretic factorizations; the full spectral determinants can be factorized using the same techniques. p-cycle into a cycle weight t_p.

This chapter is meant to serve as a detailed guide to the computation of dynamical zeta functions and spectral determinants for systems with discrete symmetries. Familiarity with basic group-theoretic notions is assumed, with the definitions relegated to appendix H.1. We develop here the cycle expansions for factorized determinants, and exemplify them by working two cases of physical interest: $C_2 = D_1$, $C_{3v} = D_3$ symmetries. $C_{2v} = D_2 \times D_2$ and $C_{4v} = D_4$ symmetries are discussed in appendix H.

19.1 Preview

As we saw in chapter 9, discrete symmetries relate classes of periodic orbits and reduce dynamics to a fundamental domain. Such symmetries simplify and improve the cycle expansions in a rather beautiful way; in classical dynamics, just as in quantum mechanics, the symmetrized subspaces can be probed by linear operators of different symmetries. If a linear operator commutes with the symmetry, it can be block-diagonalized, and, as we shall now show, the associated spectral determinants and dynamical zeta functions factorize.

19.1.1 Reflection symmetric 1-d maps

Consider f, a map on the interval with reflection symmetry $f(-x) = -f(x)$. A simple example is the piecewise-linear sawtooth map of figure 9.1. Denote the reflection operation by $Rx = -x$. The symmetry of the map implies that if $\{x_n\}$ is a trajectory, than also $\{Rx_n\}$ is a trajectory because $Rx_{n+1} = Rf(x_n) = f(Rx_n)$. The dynamics can be restricted to a fundamental domain, in this case to one half of the original interval; every time a trajectory leaves this interval, it can be mapped back using R. Furthermore, the evolution operator commutes with R, $\mathcal{L}(y, x) = \mathcal{L}(Ry, Rx)$. R satisfies $R^2 = \mathbf{e}$ and can be used to decompose the state space into mutually orthogonal symmetric and antisymmetric subspaces by means of projection operators

$$
\begin{aligned}
P_{A_1} &= \frac{1}{2}(\mathbf{e} + R)\,, \qquad P_{A_2} = \frac{1}{2}(\mathbf{e} - R)\,, \\
\mathcal{L}_{A_1}(y, x) &= P_{A_1}\mathcal{L}(y, x) = \frac{1}{2}\left(\mathcal{L}(y, x) + \mathcal{L}(-y, x)\right)\,, \\
\mathcal{L}_{A_2}(y, x) &= P_{A_2}\mathcal{L}(y, x) = \frac{1}{2}\left(\mathcal{L}(y, x) - \mathcal{L}(-y, x)\right)\,.
\end{aligned}
\tag{19.1}
$$

To compute the traces of the symmetrization and antisymmetrization projection operators (19.1), we have to distinguish three kinds of cycles: asymmetric cycles a, symmetric cycles s built by repeats of irreducible segments $\tilde{s}$, and boundary cycles b. Now we show that the spectral determinant can be written as the product over the three kinds of cycles: $\det(1 - \mathcal{L}) = \det(1 - \mathcal{L})_a \det(1 - \mathcal{L})_{\tilde{s}} \det(1 - \mathcal{L})_b$.

Asymmetric cycles: A periodic orbits is not symmetric if $\{x_a\} \cap \{Rx_a\} = \emptyset$, where $\{x_a\}$ is the set of periodic points belonging to the cycle a. Thus R generates a second orbit with the same number of points and the same stability properties. Both orbits give the same contribution to the first term and no contribution to the second term in (19.1); as they are degenerate, the prefactor $1/2$ cancels. Resuming as in the derivation of (17.15) we find that asymmetric orbits yield the same contribution to the symmetric and the antisymmetric subspaces:

$$\det(1-\mathcal{L}_\pm)_a = \prod_a \prod_{k=0}^{\infty}\left(1-\frac{t_a}{\Lambda_a^k}\right), \qquad t_a = \frac{z^{n_a}}{|\Lambda_a|}.$$

Symmetric cycles: A cycle s is reflection symmetric if operating with R on the set of cycle points reproduces the set. The period of a symmetric cycle is always even ($n_s = 2n_{\tilde{s}}$) and the mirror image of the x_s cycle point is reached by traversing the irreducible segment $\tilde{s}$ of length $n_{\tilde{s}}$, $f^{n_{\tilde{s}}}(x_s) = Rx_s$. $\delta(x - f^n(x))$ picks up $2n_{\tilde{s}}$ contributions for every even traversal, $n = rn_{\tilde{s}}$, r even, and $\delta(x + f^n(x))$ for every odd traversal, $n = rn_{\tilde{s}}$, r odd. Absorb the group-theoretic prefactor in the stability eigenvalue by defining the stability computed for a segment of length $n_{\tilde{s}}$,

$$\Lambda_{\tilde{s}} = -\left.\frac{\partial f^{n_{\tilde{s}}}(x)}{\partial x}\right|_{x=x_s}.$$

Restricting the integration to the infinitesimal neighborhood $\mathcal{M}_s$ of the s cycle, we obtain the contribution to $\mathrm{tr}\,\mathcal{L}^n_\pm$:

$$\begin{aligned}
z^n \mathrm{tr}\,\mathcal{L}^n_\pm &\to \int_{\mathcal{M}_s} dx\, z^n \,\frac{1}{2}\,\left(\delta(x-f^n(x)) \pm \delta(x+f^n(x))\right) \\
&= n_{\tilde{s}}\left(\sum_{r=2}^{\text{even}} \delta_{n,rn_{\tilde{s}}} \frac{t^r_{\tilde{s}}}{1-1/\Lambda^r_{\tilde{s}}} \pm \sum_{r=1}^{\text{odd}} \delta_{n,rn_{\tilde{s}}} \frac{t^r_{\tilde{s}}}{1-1/\Lambda^r_{\tilde{s}}}\right) \\
&= n_{\tilde{s}} \sum_{r=1}^{\infty} \delta_{n,rn_{\tilde{s}}} \frac{(\pm t_{\tilde{s}})^r}{1-1/\Lambda^r_{\tilde{s}}}.
\end{aligned}$$

Substituting all symmetric cycles s into $\det(1-\mathcal{L}_\pm)$ and resuming we obtain:

$$\det(1-\mathcal{L}_\pm)_{\tilde{s}} = \prod_{\tilde{s}} \prod_{k=0}^{\infty}\left(1 \mp \frac{t_{\tilde{s}}}{\Lambda^k_{\tilde{s}}}\right)$$

Boundary cycles: In the example at hand there is only one cycle which is neither symmetric nor antisymmetric, but lies on the boundary of the fundamental domain, the fixed point at the origin. Such cycle contributes simultaneously to both $\delta(x - f^n(x))$

and $\delta(x+f^n(x))$:

$$
\begin{aligned}
z^n \mathrm{tr}\,\mathcal{L}_\pm^n &\to \int_{\mathcal{M}_b} dx\, z^n \frac{1}{2}\left(\delta(x-f^n(x)) \pm \delta(x+f^n(x))\right) \\
&= \sum_{r=1}^{\infty} \delta_{n,r}\, t_b^r \frac{1}{2}\left(\frac{1}{1-1/\Lambda_b^r} \pm \frac{1}{1+1/\Lambda_b^r}\right) \\
z^n\, \mathrm{tr}\,\mathcal{L}_+^n &\to \sum_{r=1}^{\infty} \delta_{n,r} \frac{t_b^r}{1-1/\Lambda_b^{2r}}\,; \qquad z^n\, \mathrm{tr}\,\mathcal{L}_-^n \to \sum_{r=1}^{\infty} \delta_{n,r} \frac{1}{\Lambda_b^r} \frac{t_b^r}{1-1/\Lambda_b^{2r}}\,.
\end{aligned}
$$

Boundary orbit contributions to the factorized spectral determinants follow by resummation:

$$
\det(1-\mathcal{L}_+)_b = \prod_{k=0}^{\infty}\left(1-\frac{t_b}{\Lambda_b^{2k}}\right), \qquad \det(1-\mathcal{L}_-)_b = \prod_{k=0}^{\infty}\left(1-\frac{t_b}{\Lambda_b^{2k+1}}\right)
$$

Only the even derivatives contribute to the symmetric subspace, and only the odd ones to the antisymmetric subspace, because the orbit lies on the boundary.

Finally, the symmetry reduced spectral determinants follow by collecting the above results:

$$
\begin{aligned}
F_+(z) &= \prod_a \prod_{k=0}^{\infty}\left(1-\frac{t_a}{\Lambda_a^k}\right) \prod_{\tilde{s}} \prod_{k=0}^{\infty}\left(1-\frac{t_{\tilde{s}}}{\Lambda_{\tilde{s}}^k}\right) \prod_{k=0}^{\infty}\left(1-\frac{t_b}{\Lambda_b^{2k}}\right) \\
F_-(z) &= \prod_a \prod_{k=0}^{\infty}\left(1-\frac{t_a}{\Lambda_a^k}\right) \prod_{\tilde{s}} \prod_{k=0}^{\infty}\left(1+\frac{t_{\tilde{s}}}{\Lambda_{\tilde{s}}^k}\right) \prod_{k=0}^{\infty}\left(1-\frac{t_b}{\Lambda_b^{2k+1}}\right) \qquad (19.2)
\end{aligned}
$$

We shall work out the symbolic dynamics of such reflection symmetric systems in some detail in sect. 19.5. As reflection symmetry is essentially the only discrete symmetry that a map of the interval can have, this example completes the group-theoretic factorization of determinants and zeta functions for 1-d maps. We now turn to discussion of the general case. [exercise 19.1]

19.2 Discrete symmetries

A dynamical system is invariant under a symmetry group $G = \{e, g_2, \ldots, g_{|G|}\}$ if the equations of motion are invariant under all symmetries $g \in G$. For a map $x_{n+1} = f(x_n)$ and the evolution operator $\mathcal{L}(y,x)$ defined by (15.23) this means

$$
\begin{aligned}
f(x) &= \mathbf{g}^{-1} f(\mathbf{g}x) \\
\mathcal{L}(y,x) &= \mathcal{L}(\mathbf{g}y, \mathbf{g}x)\,. \qquad (19.3)
\end{aligned}
$$

Bold face letters for group elements indicate a suitable representation on state space. For example, if a 2-dimensional map has the symmetry $x_1 \to -x_1$, $x_2 \to -x_2$, the symmetry group G consists of the identity and C, a rotation by π around the origin. The map f must then commute with rotations by π, $f(Rx) = \mathbf{C} f(x)$, with R given by the $[2 \times 2]$ matrix

$$R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{19.4}$$

R satisfies $R^2 = e$ and can be used to decompose the state space into mutually orthogonal symmetric and antisymmetric subspaces by means of projection operators (19.1). More generally the projection operator onto the α irreducible subspace of dimension d_α is given by $P_\alpha = (d_\alpha/|G|) \sum \chi_\alpha(h)\mathbf{h}^{-1}$, where $\chi_\alpha(h) = \operatorname{tr} D_\alpha(h)$ are the group characters, and the transfer operator $\mathcal{L}$ splits into a sum of inequivalent irreducible subspace contributions $\sum_\alpha \operatorname{tr} \mathcal{L}_\alpha$,

$$\mathcal{L}_\alpha(y, x) = \frac{d_\alpha}{|G|} \sum_{h \in G} \chi_\alpha(h) \mathcal{L}(\mathbf{h}^{-1} y, x) \ . \tag{19.5}$$

The prefactor d_α in the above reflects the fact that a d_α-dimensional representation occurs d_α times.

19.2.1 Cycle degeneracies

Taking into account these degeneracies, the Euler product (17.15) takes the form

$$\prod_p (1 - t_p) = \prod_{\hat{p}} (1 - t_{\hat{p}})^{m_{\hat{p}}} . \tag{19.6}$$

The Euler product (17.15) for the C_{3v} symmetric 3-disk problem is given in (18.36).

19.3 Dynamics in the fundamental domain

If the dynamics is invariant under a discrete symmetry, the state space M can be completely tiled by the fundamental domain $\tilde{M}$ and its images $a\tilde{M}$, $b\tilde{M}$, $\ldots$ under the action of the symmetry group $G = \{e, a, b, \ldots\}$,

$$M = \sum_{a \in G} M_a = \sum_{a \in G} a\tilde{M} \, .$$

In the above example (19.4) with symmetry group $G = \{e, C\}$, the state space $M = \{x_1\text{-}x_2 \text{ plane}\}$ can be tiled by a fundamental domain $\tilde{M} = \{\text{half-plane } x_1 \geq 0\}$, and $\mathbf{C}\tilde{M} = \{\text{half-plane } x_1 \leq 0\}$, its image under rotation by π.

If the space M is decomposed into g tiles, a function $\phi(x)$ over M splits into a g-dimensional vector $\phi_a(x)$ defined by $\phi_a(x) = \phi(x)$ if $x \in M_a$, $\phi_a(x) = 0$ otherwise. Let $h = ab^{-1}$ conflicts with be the symmetry operation that maps the endpoint domain M_b into the starting point domain M_a, and let $D(h)_{ba}$, the left regular representation, be the $[g \times g]$ matrix whose b, a-th entry equals unity if $a = hb$ and zero otherwise; $D(h)_{ba} = \delta_{bh,a}$. Since the symmetries act on state space as well, the operation h enters in two guises: as a $[g \times g]$ matrix $D(h)$ which simply permutes the domain labels, and as a $[d \times d]$ matrix representation $\mathbf{h}$ of a discrete symmetry operation on the d state space coordinates. For instance, in the above example (19.4) $h \in C_2$ and $D(h)$ can be either the identity or the interchange of the two domain labels,

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(C) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{19.7}$$

Note that $D(h)$ is a permutation matrix, mapping a tile M_a into a different tile $M_{ha} \neq M_a$ if $h \neq e$. Consequently only $D(e)$ has diagonal elements, and $\operatorname{tr} D(h) = g\delta_{h,e}$. However, the state space transformation $\mathbf{h} \neq \mathbf{e}$ leaves invariant sets of *boundary* points; for example, under reflection σ across a symmetry axis, the axis itself remains invariant. The boundary periodic orbits that belong to such pointwise invariant sets will require special care in $\operatorname{tr} \mathcal{L}$ evaluations.

One can associate to the evolution operator (15.23) a $[g \times g]$ matrix evolution operator defined by

$$\mathcal{L}_{ba}(y, x) = D(h)_{ba} \mathcal{L}(y, x) \,,$$

if $x \in M_a$ and $y \in M_b$, and zero otherwise. Now we can use the invariance condition (19.3) to move the starting point x into the fundamental domain $x = \mathbf{a}\tilde{x}$, $\mathcal{L}(y, x) = \mathcal{L}(\mathbf{a}^{-1} y, \tilde{x})$, and then use the relation $a^{-1}b = h^{-1}$ to also relate the endpoint y to its image in the fundamental domain, $\tilde{\mathcal{L}}(\tilde{y}, \tilde{x}) := \mathcal{L}(\mathbf{h}^{-1}\tilde{y}, \tilde{x})$. With this operator which is restricted to the fundamental domain, the global dynamics reduces to

$$\mathcal{L}_{ba}(y, x) = D(h)_{ba} \tilde{\mathcal{L}}(\tilde{y}, \tilde{x}) \,.$$

While the global trajectory runs over the full space M, the restricted trajectory is brought back into the fundamental domain $\tilde{M}$ any time it crosses into adjoining tiles; the two trajectories are related by the symmetry operation h which maps the global endpoint into its fundamental domain image.

Now the traces (17.3) required for the evaluation of the eigenvalues of the transfer operator can be evaluated on the fundamental domain alone

$$\operatorname{tr} \mathcal{L} = \int_M dx \mathcal{L}(x, x) = \int_{\tilde{M}} d\tilde{x} \sum_h \operatorname{tr} D(h)\ \mathcal{L}(\mathbf{h}^{-1}\tilde{x}, \tilde{x}) \tag{19.8}$$

The fundamental domain integral $\int d\tilde{x}\ \mathcal{L}(\mathbf{h}^{-1}\tilde{x}, \tilde{x})$ picks up a contribution from every global cycle (for which $h = e$), but it also picks up contributions from shorter segments of global cycles. The permutation matrix $D(h)$ guarantees by the identity $\mathrm{tr}\, D(h) = 0$, $h \neq e$, that only those repeats of the fundamental domain cycles $\tilde{p}$ that correspond to complete global cycles p contribute. Compare, for example, the contributions of the $\overline{12}$ and $\overline{0}$ cycles of figure 11.2. $\mathrm{tr}\, D(h)\tilde{\mathcal{L}}$ does not get a contribution from the $\overline{0}$ cycle, as the symmetry operation that maps the first half of the $\overline{12}$ into the fundamental domain is a reflection, and $\mathrm{tr}\, D(\sigma) = 0$. In contrast, $\sigma^2 = e$, $\mathrm{tr}\, D(\sigma^2) = 6$ insures that the repeat of the fundamental domain fixed point $\mathrm{tr}\,(D(h)\tilde{\mathcal{L}})^2 = 6t_0^2$, gives the correct contribution to the global trace $\mathrm{tr}\,\mathcal{L}^2 = 3 \cdot 2t_{12}$.

Let p be the full orbit, $\tilde{p}$ the orbit in the fundamental domain and $h_{\tilde{p}}$ an element of $\mathcal{H}_p$, the symmetry group of p. Restricting the volume integrations to the infinitesimal neighborhoods of the cycles p and $\tilde{p}$, respectively, and performing the standard resummations, we obtain the identity

$$(1 - t_p)^{m_p} = \det\left(1 - D(h_{\tilde{p}})t_{\tilde{p}}\right) , \tag{19.9}$$

valid cycle by cycle in the Euler products (17.15) for $\det\,(1 - \mathcal{L})$. Here “det” refers to the $[g \times g]$ matrix representation $D(h_{\tilde{p}})$; as we shall see, this determinant can be evaluated in terms of standard characters, and no explicit representation of $D(h_{\tilde{p}})$ is needed. Finally, if a cycle p is invariant under the symmetry subgroup $\mathcal{H}_p \subseteq G$ of order h_p, its weight can be written as a repetition of a fundamental domain cycle

$$t_p = t_{\tilde{p}}^{h_p} \tag{19.10}$$

computed on the irreducible segment that corresponds to a fundamental domain cycle. For example, in figure 11.2 we see by inspection that $t_{12} = t_0^2$ and $t_{123} = t_1^3$.

19.3.1 Boundary orbits

Before we can turn to a presentation of the factorizations of dynamical zeta functions for the different symmetries we have to discuss an effect that arises for orbits that run on a symmetry line that borders a fundamental domain. In our 3-disk example, no such orbits are possible, but they exist in other systems, such as in the bounded region of the Hénon-Heiles potential and in 1-d maps. For the symmetrical 4-disk billiard, there are in principle two kinds of such orbits, one kind bouncing back and forth between two diagonally opposed disks and the other kind moving along the other axis of reflection symmetry; the latter exists for bounded systems only. While there are typically very few boundary orbits, they tend to be among the shortest orbits, and their neglect can seriously degrade the convergence of cycle expansions, as those are dominated by the shortest cycles.

While such orbits are invariant under some symmetry operations, their neighborhoods are not. This affects the fundamental matrix M_p of the linearization perpendicular to the orbit and thus the eigenvalues. Typically, *e.g.* if the symmetry is a reflection, some eigenvalues of M_p change sign. This means that instead of a weight $1/\det(\mathbf{1} - M_p)$ as for a regular orbit, boundary cycles also pick up contributions of form $1/\det(\mathbf{1} - \mathbf{h}M_p)$, where $\mathbf{h}$ is a symmetry operation that leaves the orbit pointwise invariant; see for example sect. 19.1.1.

Consequences for the dynamical zeta function factorizations are that sometimes a boundary orbit does not contribute. A derivation of a dynamical zeta function (17.15) from a determinant like (17.9) usually starts with an expansion of the determinants of the Jacobian. The leading order terms just contain the product of the expanding eigenvalues and lead to the dynamical zeta function (17.15). Next to leading order terms contain products of expanding and contracting eigenvalues and are sensitive to their signs. Clearly, the weights t_p in the dynamical zeta function will then be affected by reflections in the Poincaré surface of section perpendicular to the orbit. In all our applications it was possible to implement these effects by the following simple prescription.

If an orbit is invariant under a little group $\mathcal{H}_p = \{e, b_2, \ldots, b_h\}$, then the corresponding group element in (19.9) will be replaced by a projector. If the weights are insensitive to the signs of the eigenvalues, then this projector is

$$g_p = \frac{1}{h}\sum_{i=1}^{h} b_i\,. \tag{19.11}$$

In the cases that we have considered, the change of sign may be taken into account by defining a sign function $\epsilon_p(g) = \pm 1$, with the "-" sign if the symmetry element g flips the neighborhood. Then (19.11) is replaced by

$$g_p = \frac{1}{h}\sum_{i=1}^{h} \epsilon(b_i)\, b_i\,. \tag{19.12}$$

We have illustrated the above in sect. 19.1.1 by working out the full factorization for the 1-dimensional reflection symmetric maps.

19.4 Factorizations of dynamical zeta functions

In chapter 9 we have shown that a discrete symmetry induces degeneracies among periodic orbits and decomposes periodic orbits into repetitions of irreducible segments; this reduction to a fundamental domain furthermore leads to a convenient symbolic dynamics compatible with the symmetry, and, most importantly, to a factorization of dynamical zeta functions. This we now develop, first in a general setting and then for specific examples.

19.4.1 Factorizations of dynamical dynamical zeta functions

According to (19.9) and (19.10), the contribution of a degenerate class of global cycles (cycle p with multiplicity $m_p = g/h_p$) to a dynamical zeta function is given by the corresponding fundamental domain cycle $\tilde{p}$:

$$(1 - t_{\tilde{p}}^{h_p})^{g/h_p} = \det\left(1 - D(h_{\tilde{p}})t_{\tilde{p}}\right) \tag{19.13}$$

Let $D(h) = \bigoplus_\alpha d_\alpha D_\alpha(h)$ be the decomposition of the matrix representation $D(h)$ into the d_α dimensional irreducible representations α of a finite group G. Such decompositions are block-diagonal, so the corresponding contribution to the Euler product (17.9) factorizes as

$$\det(1 - D(h)t) = \prod_\alpha \det(1 - D_\alpha(h)t)^{d_\alpha}\ , \tag{19.14}$$

where now the product extends over all distinct d_α-dimensional irreducible representations, each contributing d_α times. For the cycle expansion purposes, it has been convenient to emphasize that the group-theoretic factorization can be effected cycle by cycle, as in (19.13); but from the transfer operator point of view, the key observation is that the symmetry reduces the transfer operator to a block diagonal form; this block diagonalization implies that the dynamical zeta functions (17.15) factorize as

$$\frac{1}{\zeta} = \prod_\alpha \frac{1}{\zeta_\alpha^{d_\alpha}}\ , \qquad \frac{1}{\zeta_\alpha} = \prod_{\tilde{p}} \det\left(1 - D_\alpha(h_{\tilde{p}})t_{\tilde{p}}\right)\ . \tag{19.15}$$

Determinants of d-dimensional irreducible representations can be evaluated using the expansion of determinants in terms of traces,

$$\begin{aligned}\det(1 + M) &= 1 + \operatorname{tr} M + \frac{1}{2}\left((\operatorname{tr} M)^2 - \operatorname{tr} M^2\right)\\ &\quad + \frac{1}{6}\left((\operatorname{tr} M)^3 - 3\,(\operatorname{tr} M)(\operatorname{tr} M^2) + 2\operatorname{tr} M^3\right)\\ &\quad + \cdots + \frac{1}{d!}\left((\operatorname{tr} M)^d - \cdots\right)\ ,\end{aligned} \tag{19.16}$$

and each factor in (19.14) can be evaluated by looking up the characters $\chi_\alpha(h) = \operatorname{tr} D_\alpha(h)$ in standard tables [10]. In terms of characters, we have for the 1-dimensional representations

$$\det(1 - D_\alpha(h)t) = 1 - \chi_\alpha(h)t\ ,$$

for the 2-dimensional representations

$$\det(1 - D_\alpha(h)t) = 1 - \chi_\alpha(h)t + \frac{1}{2}\left(\chi_\alpha(h)^2 - \chi_\alpha(h^2)\right)t^2,$$

and so forth.

In the fully symmetric subspace $\mathrm{tr}\, D_{A_1}(h) = 1$ for all orbits; hence a straightforward fundamental domain computation (with no group theory weights) always yields a part of the full spectrum. In practice this is the most interesting subspectrum, as it contains the leading eigenvalue of the transfer operator.

[exercise 19.2]

19.4.2 Factorizations of spectral determinants

Factorization of the full spectral determinant (17.3) proceeds in essentially the same manner as the factorization of dynamical zeta functions outlined above. By (19.5) and (19.8) the trace of the transfer operator $\mathcal{L}$ splits into the sum of inequivalent irreducible subspace contributions $\sum_\alpha \mathrm{tr}\,\mathcal{L}_\alpha$, with

$$\mathrm{tr}\,\mathcal{L}_\alpha = d_\alpha \sum_{h\in G} \chi_\alpha(h) \int_{\tilde{M}} d\tilde{x}\,\mathcal{L}(\mathbf{h}^{-1}\tilde{x}, \tilde{x})\,.$$

This leads by standard manipulations to the factorization of (17.9) into

$$\begin{aligned} F(z) &= \prod_\alpha F_\alpha(z)^{d_\alpha} \\ F_\alpha(z) &= \exp\left(-\sum_{\tilde{p}}\sum_{r=1}^{\infty}\frac{1}{r}\frac{\chi_\alpha(h_{\tilde{p}}^r) z^{n_{\tilde{p}} r}}{|\det\left(\mathbf{1}-\tilde{M}_{\tilde{p}}^r\right)|}\right)\,, \end{aligned} \tag{19.17}$$

where $\tilde{M}_{\tilde{p}} = \mathbf{h}_{\tilde{p}} M_{\tilde{p}}$ is the fundamental domain Jacobian. Boundary orbits require special treatment, discussed in sect. 19.3.1, with examples given in the next section as well as in the specific factorizations discussed below.

The factorizations (19.15), (19.17) are the central formulas of this chapter. We now work out the group theory factorizations of cycle expansions of dynamical zeta functions for the cases of C_2 and C_{3v} symmetries. The cases of the C_{2v}, C_{4v} symmetries are worked out in appendix H below.

19.5 C_2 factorization

As the simplest example of implementing the above scheme consider the C_2 symmetry. For our purposes, all that we need to know here is that each orbit or configuration is uniquely labeled by an infinite string $\{s_i\}$, $s_i = +, -$ and that the dynamics is invariant under the $+ \leftrightarrow -$ interchange, i.e., it is C_2 symmetric. The C_2 symmetry cycles separate into two classes, the self-dual configurations $+-$, $++--$, $+++---$, $+--+-++-$, $\cdots$, with multiplicity $m_p = 1$, and the asymmetric configurations $+$, $-$, $++-$, $--+$, $\cdots$, with multiplicity $m_p = 2$.

For example, as there is no absolute distinction between the "up" and the "down" spins, or the "left" or the "right" lobe, $t_+ = t_-$, $t_{++-} = t_{+--}$, and so on.

[exercise 19.4]

The symmetry reduced labeling $\rho_i \in \{0, 1\}$ is related to the standard $s_i \in \{+, -\}$ Ising spin labeling by

$$\begin{array}{llcll} \text{If} & s_i & = & s_{i-1} & \text{then} \quad \rho_i = 1 \\ \text{If} & s_i & \neq & s_{i-1} & \text{then} \quad \rho_i = 0 \end{array} \tag{19.18}$$

For example, $\overline{+} = \cdots + + + + \cdots$ maps into $\cdots 111 \cdots = \overline{1}$ (and so does $\overline{-}$), $\overline{-+} = \cdots - + - + \cdots$ maps into $\cdots 000 \cdots = \overline{0}$, $\overline{- + +-} = \cdots - - + + - - + + \cdots$ maps into $\cdots 0101 \cdots = \overline{01}$, and so forth. A list of such reductions is given in table 11.2.

Depending on the maximal symmetry group $\mathcal{H}_p$ that leaves an orbit p invariant (see sects. 19.2 and 19.3 as well as sect. 19.1.1), the contributions to the dynamical zeta function factor as

$$\begin{array}{rccc} & & & A_1 \quad\quad A_2 \\ \mathcal{H}_p = \{e\}: & (1 - t_{\tilde{p}})^2 & = & (1 - t_{\tilde{p}})(1 - t_{\tilde{p}}) \\ \mathcal{H}_p = \{e, \sigma\}: & (1 - t_{\tilde{p}}^2) & = & (1 - t_{\tilde{p}})(1 + t_{\tilde{p}}) \,, \end{array} \tag{19.19}$$

For example:

$$\begin{array}{rccl} \mathcal{H}_{++-} = \{e\}: & (1 - t_{++-})^2 & = & (1 - t_{001})(1 - t_{001}) \\ \mathcal{H}_{+-} = \{e, \sigma\}: & (1 - t_{+-}) & = & (1 - t_0)\ (1 + t_0), \qquad t_{+-} = t_0^2 \end{array}$$

This yields two binary cycle expansions. The A_1 subspace dynamical zeta function is given by the standard binary expansion (18.7). The antisymmetric A_2 subspace dynamical zeta function ζ_{A_2} differs from ζ_{A_1} only by a minus sign for cycles with an odd number of 0's:

$$\begin{aligned} 1/\zeta_{A_2} &= (1 + t_0)(1 - t_1)(1 + t_{10})(1 - t_{100})(1 + t_{101})(1 + t_{1000}) \\ &\quad (1 - t_{1001})(1 + t_{1011})(1 - t_{10000})(1 + t_{10001}) \\ &\quad (1 + t_{10010})(1 - t_{10011})(1 - t_{10101})(1 + t_{10111}) \ldots \\ &= 1 + t_0 - t_1 + (t_{10} - t_1 t_0) - (t_{100} - t_{10} t_0) + (t_{101} - t_{10} t_1) \\ &\quad -(t_{1001} - t_1 t_{001} - t_{101} t_0 + t_{10} t_0 t_1) - \ldots\ldots \end{aligned} \tag{19.20}$$

Note that the group theory factors do not destroy the curvature corrections (the cycles and pseudo cycles are still arranged into shadowing combinations).

If the system under consideration has a boundary orbit (*cf.* sect. 19.3.1) with group-theoretic factor $\mathbf{h}_p = (\mathbf{e} + \sigma)/2$, the boundary orbit does not contribute to the antisymmetric subspace

$$\begin{array}{rccc} & & & A_1 \quad\quad A_2 \\ \text{boundary:} & (1 - t_p) & = & (1 - t_{\tilde{p}})(1 - 0 t_{\tilde{p}}) \end{array} \tag{19.21}$$

This is the $1/\zeta$ part of the boundary orbit factorization of sect. 19.1.1.

19.6 C_{3v} factorization: 3-disk game of pinball

The next example, the C_{3v} symmetry, can be worked out by a glance at figure 11.2 (a). For the symmetric 3-disk game of pinball the fundamental domain is bounded by a disk segment and the two adjacent sections of the symmetry axes that act as mirrors (see figure 11.2 (b)). The three symmetry axes divide the space into six copies of the fundamental domain. Any trajectory on the full space can be pieced together from bounces in the fundamental domain, with symmetry axes replaced by flat mirror reflections. The binary $\{0,1\}$ reduction of the ternary three disk $\{1,2,3\}$ labels has a simple geometric interpretation: a collision of type 0 reflects the projectile to the disk it comes from (back–scatter), whereas after a collision of type 1 projectile continues to the third disk. For example, $\overline{23} = \cdots 232323 \cdots$ maps into $\cdots 000 \cdots = \overline{0}$ (and so do $\overline{12}$ and $\overline{13}$), $\overline{123} = \cdots 12312 \cdots$ maps into $\cdots 111 \cdots = \overline{1}$ (and so does $\overline{132}$), and so forth. A list of such reductions for short cycles is given in table 11.1.

C_{3v} has two 1-dimensional irreducible representations, symmetric and antisymmetric under reflections, denoted A_1 and A_2, and a pair of degenerate 2-dimensional representations of mixed symmetry, denoted E. The contribution of an orbit with symmetry g to the $1/\zeta$ Euler product (19.14) factorizes according to

$$\det(1-D(h)t) = (1-\chi_{A_1}(h)t)\,(1-\chi_{A_2}(h)t)\left(1-\chi_E(h)t+\chi_{A_2}(h)t^2\right)^2 \quad (19.22)$$

with the three factors contributing to the C_{3v} irreducible representations A_1, A_2 and E, respectively, and the 3-disk dynamical zeta function factorizes into $\zeta = \zeta_{A_1}\zeta_{A_2}\zeta_E^2$. Substituting the C_{3v} characters [10]

C_{3v}	A_1	A_2	E
e	1	1	2
C, C^2	1	1	-1
σ_v	1	-1	0

into (19.22), we obtain for the three classes of possible orbit symmetries (indicated in the first column)

$$\begin{array}{rlcl}
 & \mathbf{h}_{\tilde{p}} & & \quad A_1 \qquad A_2 \qquad E \\
e: & (1-t_{\tilde{p}})^6 & = & (1-t_{\tilde{p}})(1-t_{\tilde{p}})(1-2t_{\tilde{p}}+t_{\tilde{p}}^2)^2 \\
C, C^2: & (1-t_{\tilde{p}}^3)^2 & = & (1-t_{\tilde{p}})(1-t_{\tilde{p}})(1+\ t_{\tilde{p}}+t_{\tilde{p}}^2)^2 \\
\sigma_v: & (1-t_{\tilde{p}}^2)^3 & = & (1-t_{\tilde{p}})(1+t_{\tilde{p}})(1+0t_{\tilde{p}}-t_{\tilde{p}}^2)^2 .
\end{array} \quad (19.23)$$

where σ_v stands for any one of the three reflections.

The Euler product (17.15) on each irreducible subspace follows from the factorization (19.23). On the symmetric A_1 subspace the ζ_{A_1} is given by the standard binary curvature expansion (18.7). The antisymmetric A_2 subspace ζ_{A_2} differs from ζ_{A_1} only by a minus sign for cycles with an odd number of 0's, and is given in (19.20). For the mixed-symmetry subspace E the curvature expansion is given by

$$\begin{aligned}
1/\zeta_E &= (1 + zt_1 + z^2t_1^2)(1 - z^2t_0^2)(1 + z^3t_{100} + z^6t_{100}^2)(1 - z^4t_{10}^2) \\
&\quad (1 + z^4t_{1001} + z^8t_{1001}^2)(1 + z^5t_{10000} + z^{10}t_{10000}^2) \\
&\quad (1 + z^5t_{10101} + z^{10}t_{10101}^2)(1 - z^5t_{10011})^2 \ldots \\
&= 1 + zt_1 + z^2(t_1^2 - t_0^2) + z^3(t_{001} - t_1t_0^2) \\
&\quad + z^4\left[t_{0011} + (t_{001} - t_1t_0^2)t_1 - t_{01}^2\right] \\
&\quad + z^5\left[t_{00001} + t_{01011} - 2t_{00111} + (t_{0011} - t_{01}^2)t_1 + (t_1^2 - t_0^2)t_{100}\right] + \cdots
\end{aligned} \tag{19.24}$$

We have reinserted the powers of z in order to group together cycles and pseudocycles of the same length. Note that the factorized cycle expansions retain the curvature form; long cycles are still shadowed by (somewhat less obvious) combinations of pseudocycles.

Referring back to the topological polynomial (13.31) obtained by setting $t_p = 1$, we see that its factorization is a consequence of the C_{3v} factorization of the ζ function:

$$1/\zeta_{A_1} = 1 - 2z\,, \quad 1/\zeta_{A_2} = 1\,, \quad 1/\zeta_E = 1 + z\,, \tag{19.25}$$

as obtained from (18.7), (19.20) and (19.24) for $t_p = 1$.

Their symmetry is $K = \{\mathbf{e}, \sigma\}$, so according to (19.11), they pick up the group-theoretic factor $\mathbf{h}_p = (\mathbf{e} + \sigma)/2$. If there is no sign change in t_p, then evaluation of $\det(1 - \frac{\mathbf{e}+\sigma}{2}t_{\tilde{p}})$ yields

$$\begin{array}{lcccl}
 & & A_1 \quad\; A_2 \quad\; E & & \\
\text{boundary:} \quad (1 - t_p)^3 & = & (1 - t_{\tilde{p}})(1 - 0t_{\tilde{p}})(1 - t_{\tilde{p}})^2\,, & & t_p = t_{\tilde{p}}\,.
\end{array} \tag{19.26}$$

However, if the cycle weight changes sign under reflection, $t_{\sigma\tilde{p}} = -t_{\tilde{p}}$, the boundary orbit does not contribute to the subspace symmetric under reflection across the orbit;

$$\begin{array}{lcccl}
 & & A_1 \quad\; A_2 \quad\; E & & \\
\text{boundary:} \quad (1 - t_p)^3 & = & (1 - 0t_{\tilde{p}})(1 - t_{\tilde{p}})(1 - t_{\tilde{p}})^2\,, & & t_p = t_{\tilde{p}}\,.
\end{array} \tag{19.27}$$

Résumé

If a dynamical system has a discrete symmetry, the symmetry should be exploited; much is gained, both in understanding of the spectra and ease of their evaluation.

Once this is appreciated, it is hard to conceive of a calculation without factorization; it would correspond to quantum mechanical calculations without wave–function symmetrizations.

While the reformulation of the chaotic spectroscopy from the trace sums to the cycle expansions does not reduce the exponential growth in number of cycles with the cycle length, in practice only the short orbits are used, and for them the labor saving is dramatic. For example, for the 3-disk game of pinball there are 256 periodic points of length 8, but reduction to the fundamental domain non-degenerate prime cycles reduces the number of the distinct cycles of length 8 to 30.

In addition, cycle expansions of the symmetry reduced dynamical zeta functions converge dramatically faster than the unfactorized dynamical zeta functions. One reason is that the unfactorized dynamical zeta function has many closely spaced zeros and zeros of multiplicity higher than one; since the cycle expansion is a polynomial expansion in topological cycle length, accommodating such behavior requires many terms. The dynamical zeta functions on separate subspaces have more evenly and widely spaced zeros, are smoother, do not have symmetry-induced multiple zeros, and fewer cycle expansion terms (short cycle truncations) suffice to determine them. Furthermore, the cycles in the fundamental domain sample state space more densely than in the full space. For example, for the 3-disk problem, there are 9 distinct (symmetry unrelated) cycles of length 7 or less in full space, corresponding to 47 distinct periodic points. In the fundamental domain, we have 8 (distinct) periodic orbits up to length 4 and thus 22 different periodic points in 1/6-th the state space, i.e., an increase in density by a factor 3 with the same numerical effort.

We emphasize that the symmetry factorization (19.23) of the dynamical zeta function is *intrinsic* to the classical dynamics, and not a special property of quantal spectra. The factorization is not restricted to the Hamiltonian systems, or only to the configuration space symmetries; for example, the discrete symmetry can be a symmetry of the Hamiltonian phase space [2]. In conclusion, the manifold advantages of the symmetry reduced dynamics should thus be obvious; full state space cycle expansions, such as those of exercise 18.8, are useful only for cross checking purposes.

Commentary

Remark 19.1 Symmetry reductions in periodic orbit theory. This chapter is based on long collaborative effort with B. Eckhardt, ref. [1]. The group-theoretic factorizations of dynamical zeta functions that we develop here were first introduced and applied in ref. [4]. They are closely related to the symmetrizations introduced by Gutzwiller [4] in the context of the semiclassical periodic orbit trace formulas, put into more general group-theoretic context by Robbins [2], whose exposition, together with Lauritzen's [3] treatment of the boundary orbits, has influenced the presentation given here. The symmetry reduced trace formula for a finite symmetry group $G = \{e, g_2, \ldots, g_{|G|}\}$ with $|G|$ group elements, where the integral over Haar measure is replaced by a finite group discrete sum $|G|^{-1} \sum_{g\in G} = 1$, was derived in ref. [1]. A related group-theoretic decomposition in context of hyperbolic billiards was utilized in ref. [10], and for the Selberg's zeta function in ref. [11]. One of its loftier antecedents is the Artin factorization formula of algebraic number theory, which expresses the zeta-function of a finite extension of a given field as a product of L-functions over all irreducible representations of the corresponding Galois group.

Remark 19.2 Computations. The techniques of this chapter have been applied to computations of the 3-disk classical and quantum spectra in refs. [7, 13], and to a "Zeeman effect" pinball and the x^2y^2 potentials in ref. [12]. In a larger perspective, the factorizations developed above are special cases of a general approach to exploiting the group-theoretic invariances in spectra computations, such as those used in enumeration of periodic geodesics [10, 3, 13] for hyperbolic billiards [12] and Selberg zeta functions [18].

Remark 19.3 Other symmetries. In addition to the symmetries exploited here, time reversal symmetry and a variety of other non-trivial discrete symmetries can induce further relations among orbits; we shall point out several of examples of cycle degeneracies under time reversal. We do not know whether such symmetries can be exploited for further improvements of cycle expansions.

Exercises

19.1. **Sawtooth map desymmetrization.** Work out the some of the shortest global cycles of different symmetries and fundamental domain cycles for the sawtooth map of figure 9.1. Compute the dynamical zeta function and the spectral determinant of the Perron-Frobenius operator for this map; check explicitly the factorization (19.2).

19.2. **2-d asymmetric representation.** The above expressions can sometimes be simplified further using standard group-theoretical methods. For example, the $\frac{1}{2}\left((\mathrm{tr}\, M)^2 - \mathrm{tr}\, M^2\right)$ term in (19.16) is the trace of the antisymmetric part of the $M \times M$ Kronecker product. Show that if α is a 2-dimensional representation, this is the A_2 antisymmetric representation, and

$$\text{2-dim:} \quad \det(1-D_\alpha(h)t) = 1-\chi_\alpha(h)t+\chi_{A_2}(h)t^2. \tag{19.28}$$

19.3. **3-disk desymmetrization.**

a) Work out the 3-disk symmetry factorization for the 0 and 1 cycles, i.e. which symmetry do they have, what is the degeneracy in full space and how do they factorize (how do they look in the A_1, A_2 and the E representations).

b) Find the shortest cycle with no symmetries and factorize it as in a)

c) Find the shortest cycle that has the property that its time reversal is not described by the same symbolic dynamics.

d) Compute the dynamical zeta functions and the spectral determinants (symbolically) in the three representations; check the factorizations (19.15) and (19.17).

(Per Rosenqvist)

19.4. **C_2 factorizations: the Lorenz and Ising systems.** In the Lorenz system [1, 3] the labels $+$ and $-$ stand for the left or the right lobe of the attractor and the symmetry is a rotation by π around the z-axis. Similarly, the Ising Hamiltonian (in the absence of an external magnetic field) is invariant under spin flip. Work out the factorizations for some of the short cycles in either system.

19.5. **Ising model.** The Ising model with two states $\epsilon_i = \{+, -\}$ per site, periodic boundary condition, and