Chapter 5

Cycle stability

TOPOLOGICAL FEATURES of a dynamical system–singularities, periodic orbits, and the ways in which the orbits intertwine–are invariant under a general continuous change of coordinates. Surprisingly, there exist quantities that depend on the notion of metric distance between points, but nevertheless do not change value under a smooth change of coordinates. Local quantities such as the eigenvalues of equilibria and periodic orbits, and global quantities such as Lyapunov exponents, metric entropy, and fractal dimensions are examples of properties of dynamical systems independent of coordinate choice.

We now turn to the first, local class of such invariants, linear stability of periodic orbits of flows and maps. This will give us metric information about local dynamics. If you already know that the eigenvalues of periodic orbits are invariants of a flow, skip this chapter.

fast track:
chapter 7, p. 108

5.1 Stability of periodic orbits

As noted on page 35, a trajectory can be stationary, periodic or aperiodic. For chaotic systems almost all trajectories are aperiodic–nevertheless, equilibria and periodic orbits will turn out to be the key to unraveling chaotic dynamics. Here we note a few of the properties that makes them so precious to a theorist.

An obvious virtue of periodic orbits is that they are *topological* invariants: a fixed point remains a fixed point for any choice of coordinates, and similarly a periodic orbit remains periodic in any representation of the dynamics. Any reparametrization of a dynamical system that preserves its topology has to preserve topological relations between periodic orbits, such as their relative inter-windings and knots. So the mere existence of periodic orbits suffices to partially organize the spatial layout of a non–wandering set. No less important, as we shall now

show, is the fact that cycle eigenvalues are *metric* invariants: they determine the relative sizes of neighborhoods in a non–wandering set.

To prove this, we start by noting that due to the multiplicative structure (4.44) of fundamental matrices, the fundamental matrix for the rth repeat of a prime cycle p of period T_p is

$$J^{rT_p}(x) = J^{T_p}(f^{(r-1)T_p}(x)) \cdots J^{T_p}(f^{T_p}(x))J^{T_p}(x) = \left(J_p(x)\right)^r , \tag{5.1}$$

where $J_p(x) = J^{T_p}(x)$ is the fundamental matrix for a single traversal of the prime cycle p, $x \in p$ is any point on the cycle, and $f^{rT_p}(x) = x$ as $f^t(x)$ returns to x every multiple of the period T_p. Hence, it suffices to restrict our considerations to the stability of prime cycles.

fast track:
sect. 5.2, p. 87

5.1.1 Nomenclature, again

When dealing with periodic orbits, some of the quantities introduced above inherit terminology from the theory of differential equations with periodic coefficients.

For instance, if we consider the equation of variations (4.2) evaluated on a periodic orbit p,

$$\dot{\delta x} = A(t)\delta x , \qquad A(t) = A(x(t)) = A(t + T_p) , \tag{5.2}$$

the T_p periodicity of the stability matrix implies that if $\delta x(t)$ is a solution of (5.2) then also $\delta x(t + T_p)$ satisfies the same equation: moreover the two solutions are related by (see (4.6))

$$\delta x(t + T_p) = J_p(x)\delta x(t) . \tag{5.3}$$

Even though the fundamental matrix $J_p(x)$ depends upon x (the "starting" point of the periodic orbit), its eigenvalues do not, so we may write for its eigenvectors $\mathbf{e}^{(j)}$

$$J_p(x)\mathbf{e}^{(j)}(x) = \Lambda_{p,j}\mathbf{e}^{(j)}(x) = e^{T_p(\mu_p^{(j)} + i\omega_p^{(j)})}\mathbf{e}^{(j)}(x) ,$$

where $\mu_p^{(j)}$ and $\omega_p^{(j)}$ are independent of x, and expand

$$\delta x(t) = \sum_j u_j(t)\mathbf{e}^{(j)} .$$

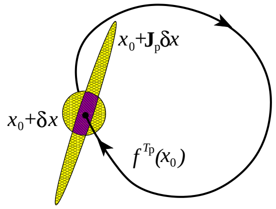

Figure 5.1: For a prime cycle p, fundamental matrix J_p returns an infinitesimal spherical neighborhood of $x_0 \in p$ stretched into an ellipsoid, with overlap ratio along the expanding eigdirection $\mathbf{e}^{(i)}$ of $J_p(x)$ given by the the expanding eigenvalue $1/|\Lambda_{p,i}|$. These ratios are invariant under smooth nonlinear reparametrizations of state space coordinates, and are intrinsic property of cycle p.

If we take (5.3) into account, we get

$$\delta x(t+T_p) = \sum_j u_j(t+T_p)\mathbf{e}^{(j)} = \sum_j u_j(t) e^{T_p(\mu_p^{(j)}+i\omega_p^{(j)})}\mathbf{e}^{(j)}$$

which shows that the coefficients $u_j(t)$ may be written as

$$u_j(t) = e^{t(\mu_p^{(j)}+i\omega_p^{(j)})} v_j(t)$$

where $v_j(t)$ is *periodic* with period T_p. Thus each solution of the equation of variations may be expressed as

$$\delta x(t) = \sum_j v_j(t) e^{t(\mu_p^{(j)}+i\omega_p^{(j)})}\mathbf{e}^{(j)} \qquad v_j(t+T_p) = v_j(t)\,, \tag{5.4}$$

the form predicted by Floquet theorem for differential equations with periodic coefficients.

The continuous time t appearing in (5.4) does not imply that eigenvalues of the fundamental matrix enjoy any multiplicative property: $\mu_p^{(j)}$ and $\omega_p^{(j)}$ refer to a full evolution over the complete periodic orbit. $\Lambda_{p,j}$ is called the Floquet multiplier, and $\mu_p^{(j)} + i\omega_p^{(j)}$ the Floquet or characteristic exponent, where $\Lambda_{p,j} = e^{T_p(\mu_p^{(j)}+i\omega_p^{(j)})}$.

5.1.2 Fundamental matrix eigenvalues and exponents

We sort the *Floquet multipliers* $\Lambda_{p,1}, \Lambda_{p,2}, \ldots, \Lambda_{p,d}$ of the $[d\times d]$ fundamental matrix J_p evaluated on the p-cycle into sets $\{e, m, c\}$

$$\begin{aligned}
\text{expanding:} \quad & \{\Lambda\}_e && = \{\Lambda_{p,j} : \left|\Lambda_{p,j}\right| > 1\} \\
\text{marginal:} \quad & \{\Lambda\}_m && = \{\Lambda_{p,j} : \left|\Lambda_{p,j}\right| = 1\} \\
\text{contracting:} \quad & \{\Lambda\}_c && = \{\Lambda_{p,j} : \left|\Lambda_{p,j}\right| < 1\}\,.
\end{aligned} \tag{5.5}$$

and denote by Λ_p (no jth eigenvalue index) the product of *expanding* Floquet multipliers

$$\Lambda_p = \prod_e \Lambda_{p,e}\,. \tag{5.6}$$

As J_p is a real matrix, complex eigenvalues always come in complex conjugate pairs, $\Lambda_{p,i+1} = \Lambda^*_{p,i}$, so the product of expanding eigenvalues Λ_p is always real.

The stretching/contraction rates per unit time are are given by the real parts of Floquet exponents

$$\mu_p^{(i)} = \frac{1}{T_p}\ln\left|\Lambda_{p,i}\right|\,. \tag{5.7}$$

The factor $\frac{1}{T_p}$ in the definition of the Floquet exponents is motivated by its form for the linear dynamical systems, for example (4.16), as well as the fact that exponents so defined can be interpreted as Lyapunov exponents (15.33) evaluated on the prime cycle p. As in the three cases of (5.5), we sort the Floquet exponents $\lambda = \mu \pm i\omega$ into three sets [section 15.3]

$$\begin{aligned}
\text{expanding:} \quad & \{\lambda\}_e && = \{\lambda_p^{(i)} : \mu_p^{(i)} > 0\} \\
\text{marginal:} \quad & \{\lambda\}_m && = \{\lambda_p^{(i)} : \mu_p^{(i)} = 0\} \\
\text{contracting:} \quad & \{\lambda\}_c && = \{\lambda_p^{(i)} : \mu_p^{(i)} < 0\}\,.
\end{aligned} \tag{5.8}$$

A periodic orbit p of a d-dimensional flow or a map is *stable* if real parts of all of its Floquet exponents (other than the vanishing longitudinal exponent, to be explained in sect. 5.2.1) are strictly negative, $\mu_p^{(i)} < 0$. The region of system parameter values for which a periodic orbit p is stable is called the *stability window* of p. The set $\mathcal{M}_p$ of initial points that are asymptotically attracted to p as $t \to +\infty$ (for a fixed set of system parameter values) is called the *basin of attraction* of p.

If *all* Floquet exponents (other than the vanishing longitudinal exponent) of *all* periodic orbits of a flow are strictly bounded away from zero, $|\mu^{(i)}| \geq \mu_{min} > 0$, the flow is said to be *hyperbolic*. Otherwise the flow is said to be *nonhyperbolic*. In particular, if all $\mu^{(i)} = 0$, the orbit is said to be *elliptic*. Such orbits proliferate in Hamiltonian flows. [section 7.3]

We often do care about $\sigma_p^{(j)} = \Lambda_{p,j}/|\Lambda_{p,j}|$, the sign of the jth eigenvalue, and, if $\Lambda_{p,j}$ is complex, its phase

$$\Lambda_{p,j} = \sigma_p^{(j)} e^{\lambda_p^{(j)} T_p} = \sigma_p^{(j)} e^{(\mu_p^{(j)} \pm i\omega_p^{(j)})T_p}\,. \tag{5.9}$$

[section 7.2]

Keeping track of this by case-by-case enumeration is an unnecessary nuisance, followed in much of the literature. To avoid this, almost all of our formulas will be stated in terms of the Floquet multipliers Λ_j rather than in the terms of the overall signs, Floquet exponents $\lambda^{(j)}$ and phases $\omega^{(j)}$.

Example 5.1 ***Stability of 1-d map cycles:*** *The simplest example of cycle stability is afforded by 1-dimensional maps. The stability of a prime cycle p follows from the chain rule (4.50) for stability of the n_pth iterate of the map*

$$\Lambda_p = \frac{d}{dx_0} f^{n_p}(x_0) = \prod_{m=0}^{n_p-1} f'(x_m), \quad x_m = f^m(x_0). \tag{5.10}$$

Λ_p is a property of the cycle, not the initial point, as taking any periodic point in the p cycle as the initial point yields the same result.

A critical point *x_c is a value of x for which the mapping $f(x)$ has vanishing derivative, $f'(x_c) = 0$. For future reference we note that a periodic orbit of a 1-dimensional map is* stable *if*

$$\left|\Lambda_p\right| = \left|f'(x_{n_p}) f'(x_{n_p-1}) \cdots f'(x_2) f'(x_1)\right| < 1,$$

and superstable *if the orbit includes a critical point, so that the above product vanishes. For a stable periodic orbit of period n the slope of the nth iterate $f^n(x)$ evaluated on a periodic point x (fixed point of the nth iterate) lies between -1 and 1. If $\left|\Lambda_p\right| > 1$, p-cycle is* unstable.

Example 5.2 ***Stability of cycles for maps:*** *No matter what method we use to determine the unstable cycles, the theory to be developed here requires that their Floquet multipliers be evaluated as well. For maps a fundamental matrix is easily evaluated by picking any cycle point as a starting point, running once around a prime cycle, and multiplying the individual cycle point fundamental matrices according to (4.51). For example, the fundamental matrix M_p for a Hénon map (3.18) prime cycle p of length n_p is given by (4.52),*

$$M_p(x_0) = \prod_{k=n_p}^{1} \begin{pmatrix} -2ax_k & b \\ 1 & 0 \end{pmatrix}, \qquad x_k \in p,$$

and the fundamental matrix M_p for a 2-dimensional billiard prime cycle p of length n_p

$$M_p = (-1)^{n_p} \prod_{k=n_p}^{1} \begin{pmatrix} 1 & \tau_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r_k & 1 \end{pmatrix}$$

follows from (8.11) of chapter 8. We shall compute Floquet multipliers of Hénon map cycles once we learn how to find their periodic orbits, see exercise 12.10.

5.2 Cycle Floquet multipliers are cycle invariants

The 1-dimensional map cycle Floquet multiplier Λ_p is a product of derivatives over all points around the cycle, and is therefore independent of which periodic point is chosen as the initial one. In higher dimensions the form of the fundamental matrix $J_p(x_0)$ in (5.1) does depend on the choice of coordinates and the initial point $x_0 \in p$. Nevertheless, as we shall now show, the cycle *Floquet multipliers* are intrinsic property of a cycle also for multi-dimensional flows. Consider the

ith eigenvalue, eigenvector pair $(\Lambda_{p,i}, \mathbf{e}^{(i)})$ computed from J_p evaluated at a cycle point,

$$J_p(x)\mathbf{e}^{(i)}(x) = \Lambda_{p,i}\mathbf{e}^{(i)}(x), \quad x \in p. \tag{5.11}$$

Consider another point on the cycle at time t later, $x' = f^t(x)$ whose fundamental matrix is $J_p(x')$. By the group property (4.44), $J^{T_p+t} = J^{t+T_p}$, and the fundamental matrix at x' can be written either as

$$J^{T_p+t}(x) = J^{T_p}(x')J^t(x) = J_p(x')J^t(x), \quad \text{or} \quad J_p(x')J^t(x) = J^t(x)J_p(x).$$

Multiplying (5.11) by $J^t(x)$, we find that the fundamental matrix evaluated at x' has the same eigenvalue,

$$J_p(x')\mathbf{e}^{(i)}(x') = \Lambda_{p,i}\mathbf{e}^{(i)}(x'), \quad \mathbf{e}^{(i)}(x') = J^t(x)\mathbf{e}^{(i)}(x), \tag{5.12}$$

but with the eigenvector $\mathbf{e}^{(i)}$ transported along the flow $x \to x'$ to $\mathbf{e}^{(i)}(x') = J^t(x)\mathbf{e}^{(i)}(x)$. Hence, J_p evaluated anywhere along the cycle has the same set of Floquet multipliers $\{\Lambda_{p,1}, \Lambda_{p,2}, \cdots \Lambda_{p,d-1}, 1\}$. As quantities such as $\mathrm{tr}\, J_p(x)$, $\det J_p(x)$ depend only on the eigenvalues of $J_p(x)$ and not on the starting point x, in expressions such as $\det\left(\mathbf{1} - J^r_p(x)\right)$ we may omit reference to x:

$$\det\left(\mathbf{1} - J^r_p\right) = \det\left(\mathbf{1} - J^r_p(x)\right) \quad \text{for any } x \in p. \tag{5.13}$$

We postpone the proof that the cycle Floquet multipliers are smooth conjugacy invariants of the flow to sect. 6.6.

5.2.1 Marginal eigenvalues

The presence of marginal eigenvalues signals either a continuous symmetry of the flow (which one should immediately exploit to simplify the problem), or a non-hyperbolicity of a flow (a source of much pain, hard to avoid). In that case (typical of parameter values for which bifurcations occur) one has to go beyond linear stability, deal with Jordan type subspaces (see example 4.3), and sub-exponential growth rates, such as t^α. [chapter 23]

[exercise 5.1]

For flow-invariant solutions such as periodic orbits, the time evolution is itself a continuous symmetry, hence a periodic orbit of a flow always has a *marginal eigenvalue*:

As $J^t(x)$ transports the velocity field $v(x)$ by (4.7), after a complete period

$$J_p(x)v(x) = v(x), \tag{5.14}$$

so a periodic orbit of a *flow* always has an eigenvector $\mathbf{e}^{(d)}(x) = v(x)$ parallel to the local velocity field with the unit eigenvalue

$$\Lambda_{p,d} = 1\,, \qquad \lambda_p^{(d)} = 0\,. \tag{5.15}$$

[exercise 6.2]

The continuous invariance that gives rise to this marginal eigenvalue is the invariance of a cycle under a translation of its points along the cycle: two points on the cycle (see figure 4.3) initially distance δx apart, $x'(0) - x(0) = \delta x(0)$, are separated by the exactly same δx after a full period T_p. As we shall see in sect. 5.3, this marginal stability direction can be eliminated by cutting the cycle by a Poincaré section and eliminating the continuous flow fundamental matrix in favor of the fundamental matrix of the Poincaré return map.

If the flow is governed by a time-independent Hamiltonian, the energy is conserved, and that leads to an additional marginal eigenvalue (remember, by symplectic invariance (7.19) real eigenvalues come in pairs).

5.3 Stability of Poincaré map cycles

(R. Paškauskas and P. Cvitanović)

If a continuous flow periodic orbit p pierces the Poincaré section $\mathcal{P}$ once, the section point is a fixed point of the Poincaré return map P with stability (4.56)

$$\hat{J}_{ij} = \left(\delta_{ik} - \frac{v_i\,U_k}{(v\cdot U)}\right) J_{kj}\,, \tag{5.16}$$

with all primes dropped, as the initial and the final points coincide, $x' = f^{T_p}(x) = x$. If the periodic orbit p pierces the Poincaré section n times, the same observation applies to the nth iterate of P.

We have already established in (4.57) that the velocity $v(x)$ is a zero-eigenvector of the Poincaré section fundamental matrix, $\hat{J}v = 0$. Consider next $(\Lambda_{p,\alpha}, \mathbf{e}^{(\alpha)})$, the full state space αth (eigenvalue, eigenvector) pair (5.11), evaluated at a cycle point on a Poincaré section,

$$J(x)\mathbf{e}^{(\alpha)}(x) = \Lambda_\alpha \mathbf{e}^{(\alpha)}(x)\,, \quad x \in \mathcal{P}\,. \tag{5.17}$$

Multiplying (5.16) by $\mathbf{e}^{(\alpha)}$ and inserting (5.17), we find that the full state space fundamental matrix and the Poincaré section fundamental matrix $\hat{J}$ has the same eigenvalue

$$\hat{J}(x)\hat{\mathbf{e}}^{(\alpha)}(x) = \Lambda_\alpha \hat{\mathbf{e}}^{(\alpha)}(x)\,, \quad x \in \mathcal{P}\,, \tag{5.18}$$

where $\hat{\mathbf{e}}^{(\alpha)}$ is a projection of the full state space eigenvector onto the Poincaré section:

$$(\hat{\mathbf{e}}^{(\alpha)})_i = \left(\delta_{ik} - \frac{v_i\,U_k}{(v\cdot U)}\right)(\mathbf{e}^{(\alpha)})_k\,. \tag{5.19}$$

Hence, $\hat{J}_p$ evaluated on any Poincaré section point along the cycle p has the same set of Floquet multipliers $\{\Lambda_{p,1}, \Lambda_{p,2}, \cdots \Lambda_{p,d}\}$ as the full state space fundamental matrix J_p.

As established in (4.57), due to the continuous symmetry (time invariance) $\hat{J}_p$ is a rank $d-1$ matrix. We shall refer to any such full rank $[(d-N)\times(d-N)]$ submatrix with N continuous symmetries quotiented out as the *monodromy matrix* M_p (from Greek *mono-* = alone, single, and *dromo* = run, racecourse, meaning a single run around the stadium).

5.4 There goes the neighborhood

In what follows, our task will be to determine the size of a *neighborhood* of $x(t)$, and that is why we care about the Floquet multipliers, and especially the unstable (expanding) ones. Nearby points aligned along the stable (contracting) directions remain in the neighborhood of the trajectory $x(t) = f^t(x_0)$; the ones to keep an eye on are the points which leave the neighborhood along the unstable directions. The sub-volume $|\mathcal{M}_i| = \prod_i^e \Delta x_i$ of the set of points which get no further away from $f^t(x_0)$ than L, the typical size of the system, is fixed by the condition that $\Delta x_i \Lambda_i = O(L)$ in each expanding direction i. Hence the neighborhood size scales as $\propto 1/|\Lambda_p|$ where Λ_p is the product of expanding eigenvalues (5.6) only; contracting ones play a secondary role. So secondary that even infinitely many of them will not matter.

So the physically important information is carried by the expanding sub-volume, not the total volume computed so easily in (4.47). That is also the reason why the dissipative and the Hamiltonian chaotic flows are much more alike than one would have naively expected for 'compressible' *vs.* 'incompressible' flows. In hyperbolic systems what matters are the expanding directions. Whether the contracting eigenvalues are inverses of the expanding ones or not is of secondary importance. As long as the number of unstable directions is finite, the same theory applies both to the finite-dimensional ODEs and infinite-dimensional PDEs.

Résumé

Periodic orbits play a central role in any invariant characterization of the dynamics, because (a) their existence and inter-relations are a *topological*, coordinate-independent property of the dynamics, and (b) their Floquet multipliers form an infinite set of

metric invariants: The Floquet multipliers of a periodic orbit remain invariant under any smooth nonlinear change of coordinates $f \to h \circ f \circ h^{-1}$.

We shall show in chapter 10 that extending their local stability eigendirections into stable and unstable manifolds yields important global information about the topological organization of state space.

In hyperbolic systems what matters are the expanding directions. The physically important information is carried by the unstable manifold, and the expanding subvolume characterized by the product of expanding eigenvalues of J_p. As long as the number of unstable directions is finite, the theory can be applied to flows of arbitrarily high dimension.

Commentary

Remark 5.1 Floquet theory. Floquet theory is a classical subject in the theory of differential equations [2]. In physics literature Floquet exponents often assume different names according to the context where the theory is applied: they are called Bloch phases in the discussion of Schrödinger equation with a periodic potential [3], or quasimomenta in the quantum theory of time-periodic Hamiltonians.

Exercises

5.1. **A limit cycle with analytic Floquet exponent.** There are only two examples of nonlinear flows for which the stability eigenvalues can be evaluated analytically. Both are cheats. One example is the 2-*d* flow

$$\begin{aligned} \dot{q} &= p + q(1 - q^2 - p^2) \\ \dot{p} &= -q + p(1 - q^2 - p^2) \,. \end{aligned}$$

Determine all periodic solutions of this flow, and determine analytically their Floquet exponents. Hint: go to polar coordinates $(q, p) = (r\cos\theta, r\sin\theta)$. G. Bard Ermentrout

5.2. **The other example of a limit cycle with analytic Floquet exponent.** What is the other example of a nonlinear flow for which the stability eigenvalues can be evaluated analytically? Hint: email G.B. Ermentrout.

5.3. **Yet another example of a limit cycle with analytic Floquet exponent.** Prove G.B. Ermentrout wrong by solving a third example (or more) of a nonlinear flow for which the stability eigenvalues can be evaluated analytically.

References

[5.1] J. Moehlis and K. Josić, “Periodic Orbit,” www.scholarpedia.org/article/Periodic_Orbit.