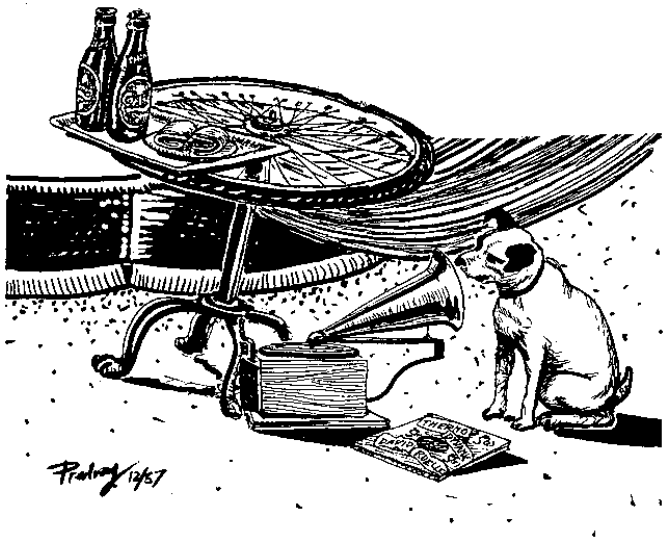
Chaos: Classical and Quantum

I: Deterministic Chaos



Predrag Cvitanović – Roberto Artuso – Ronnie Mainieri – Gregor Tanner – Gábor Vattay

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ChaosBook.org comments to: predrag@nbi.dk

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Contributors

No man but a blockhead ever wrote except for money
—Samuel Johnson

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>
> —D.H. Lawrence

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Who is the 3-legged dog reappearing throughout the book? Long ago, when we were innocent and knew not Borel measurable α to Ω sets, P. Cvitanović asked V. Baladi a question about dynamical zeta functions, who then asked J.-P. Eckmann, who then asked D. Ruelle. The answer was transmitted back: "The master says: 'It is holomorphic in a strip'." Hence His Master's Voice logo, and the 3-legged dog is us, still eager to fetch the bone. The answer has made it to the book, though not precisely in His Master's voice. As a matter of fact, the answer *is* the book. We are still chewing on it.

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