Exercises

Exercise 18.1 **Cycle expansions.** *Write programs that implement* binary *symbolic dynamics cycle expansions for (a) dynamical zeta functions, (b) spectral determinants. Combined with the cycles computed for a 2-branch repeller or a 3-disk system they will be useful in problem that follow.*

Exercise 18.2 **Escape rate for a 1-d repeller.** *(Continuation of exercise 17.1 - easy, but long)*
Consider again the quadratic map (17.31)

$$f(x) = Ax(1-x)$$

on the unit interval, for definitiveness take either $A = 9/2$ or $A = 6$. Describing the itinerary of any trajectory by the binary alphabet $\{0, 1\}$ ('0' if the iterate is in the first half of the interval and '1' if is in the second half), we have a repeller with a complete binary symbolic dynamics.

(a) *Sketch the graph of f and determine its two fixed points $\overline{0}$ and $\overline{1}$, together with their stabilities.*

(b) *Sketch the two branches of f^{-1}. Determine all the prime cycles up to topological length 4 using your pocket calculator and backwards iteration of f (see sect. 12.2.1).*

(c) *Determine the leading zero of the zeta function (17.15) using the weigths $t_p = z^{n_p}/|\Lambda_p|$ where Λ_p is the stability of the p cycle.*

(d) *Show that for $A = 9/2$ the escape rate of the repeller is $0.361509\ldots$ using the spectral determinant, with the same cycle weight. If you have taken $A = 6$, the escape rate is in $0.83149298\ldots$, as shown in solution 18.2. Compare the coefficients of the spectral determinant and the zeta function cycle expansions. Which expansion converges faster?*

(Per Rosenqvist)

Exercise 18.3 **Escape rate for the Ulam map.** *(Medium; repeat of exercise 12.1) We will try to compute the escape rate for the Ulam map (12.22)*

$$f(x) = 4x(1-x),$$

using the method of cycle expansions. The answer should be zero, as nothing escapes.

(a) *Compute a few of the stabilities for this map. Show that $\Lambda_0 = 4$, $\Lambda_1 = -2$, $\Lambda_{01} = -4$, $\Lambda_{001} = -8$ and $\Lambda_{011} = 8$.*

(b) *Show that*

$$\Lambda_{\epsilon_1 \ldots \epsilon_n} = \pm 2^n$$

and determine a rule for the sign.

(c) *(hard) Compute the dynamical zeta function for this system*

$$\zeta^{-1} = 1 - t_0 - t_1 - (t_{01} - t_0 t_1) - \cdots$$

You might note that the convergence as function of the truncation cycle length is slow. Try to fix that by treating the $\Lambda_0 = 4$ cycle separately. (Continued as exercise 18.12.)

Exercise 18.4 **Pinball escape rate, semi-analytical.** *Estimate the 3-disk pinball escape rate for $R : a = 6$ by substituting analytical cycle stabilities and periods (exercise 9.3 and exercise 9.4) into the appropriate binary cycle expansion. Compare with the numerical estimate exercise 15.3.*

Exercise 18.5 **Pinball escape rate, from numerical cycles.** *Compute the escape rate for $R : a = 6$ 3-disk pinball by substituting list of numerically computed cycle stabilities of exercise 12.5 into the binary cycle expansion.*

Exercise 18.6 **Pinball resonances, in the complex plane.** *Plot the logarithm of the absolute value of the dynamical zeta function and/or the spectral determinant cycle expansion (18.5) as contour plots in the complex s plane. Do you find zeros other than the one corresponding to the complex one? Do you see evidence for a finite radius of convergence for either cycle expansion?*

Exercise 18.7 **Counting the 3-disk psudocycles.** *(Continuation of exercise 13.12.) Verify that the number of terms in the 3-disk pinball curvature expansion (18.35) is given by*

$$\begin{aligned}
\prod_p (1+t_p) &= \frac{1-3z^4-2z^6}{1-3z^2-2z^3} \\
&= 1+3z^2+2z^3+\frac{z^4(6+12z+2z^2)}{1-3z^2-2z^3} \\
&= 1+3z^2+2z^3+6z^4+12z^5 \\
&\quad +20z^6+48z^7+84z^8+184z^9+\ldots
\end{aligned}$$

This means that, for example, c_6 has a total of 20 terms, in agreement with the explicit 3-disk cycle expansion (18.36).

Exercise 18.8 **3–disk unfactorized zeta cycle expansions.** *Check that the curvature expansion (18.2) for the 3-disk pinball, assuming no symmetries between disks, is given by*

$$\begin{aligned}
1/\zeta &= (1-z^2t_{12})(1-z^2t_{13})(1-z^2t_{23}) \\
&\quad (1-z^3t_{123})(1-z^3t_{132})(1-z^4t_{1213}) \\
&\quad (1-z^4t_{1232})(1-z^4t_{1323})(1-z^5t_{12123})\cdots \\
&= 1-z^2t_{12}-z^2t_{23}-z^2t_{31}-z^3(t_{123}+t_{132}) \\
&\quad -z^4[(t_{1213}-t_{12}t_{13})+(t_{1232}-t_{12}t_{23}) \\
&\quad +(t_{1323}-t_{13}t_{23})] \\
&\quad -z^5[(t_{12123}-t_{12}t_{123})+\cdots]-\cdots
\end{aligned} \tag{18.35}$$

The symmetrically arranged 3-disk pinball cycle expansion of the Euler product (18.2) (see table 13.4 and figure 9.3) is given by:

$$\begin{aligned}
1/\zeta &= (1-z^2t_{12})^3(1-z^3t_{123})^2(1-z^4t_{1213})^3 \\
&\quad (1-z^5t_{12123})^6(1-z^6t_{121213})^6 \\
&\quad (1-z^6t_{121323})^3\cdots \\
&= 1-3z^2\,t_{12}-2z^3\,t_{123}-3z^4\,(t_{1213}-t_{12}^2) \\
&\quad -6z^5\,(t_{12123}-t_{12}t_{123}) \\
&\quad -z^6\,(6\,t_{121213}+3\,t_{121323}+t_{12}^3-9\,t_{12}t_{1213}-t_{123}^2) \\
&\quad -6z^7\,(t_{1212123}+t_{1212313}+t_{1213123}+t_{12}^2t_{123} \\
&\quad -3\,t_{12}t_{12123}-t_{123}t_{1213}) \\
&\quad -3z^8\,(2\,t_{12121213}+t_{12121313}+2\,t_{12121323} \\
&\quad +2\,t_{12123123}+2\,t_{12123213}+t_{12132123} \\
&\qquad +3\,t_{12}^2t_{1213}+t_{12}t_{123}^2-6\,t_{12}t_{121213} \\
&\qquad -3\,t_{12}t_{121323}-4\,t_{123}t_{12123}-t_{1213}^2)-\cdots
\end{aligned} \tag{18.36}$$

Remark 18.7 Unsymmetrized cycle expansions. The above 3-disk cycle expansions might be useful for cross-checking purposes, but, as we shall see in chapter **??**, they are not recommended for actual computations, as the factorized zeta functions yield much better convergence.

Exercise 18.9 4–disk unfactorized dynamical zeta function cycle expansions *For the symmetriclly arranged 4-disk pinball the symmetry group is C_{4v}, of order 8. The degenerate cycles can have multiplicities 2, 4 or 8 (see table 13.2):*

$$\begin{aligned}
1/\zeta &= (1-z^2t_{12})^4(1-z^2t_{13})^2(1-z^3t_{123})^8 \\
&\quad (1-z^4t_{1213})^8(1-z^4t_{1214})^4(1-z^4t_{1234})^2 \\
&\quad (1-z^4t_{1243})^4(1-z^5t_{12123})^8(1-z^5t_{12124})^8 \\
&\quad (1-z^5t_{12134})^8(1-z^5t_{12143})^8 \\
&\quad (1-z^5t_{12313})^8(1-z^5t_{12413})^8\cdots
\end{aligned} \tag{18.37}$$

and the cycle expansion is given by

$$\begin{aligned}
1/\zeta &= 1-z^2(4\,t_{12}+2\,t_{13})-8z^3\,t_{123} \\
&\quad -z^4(8\,t_{1213}+4\,t_{1214}+2\,t_{1234}+4\,t_{1243} \\
&\quad -6\,t_{12}^2-t_{13}^2-8\,t_{12}t_{13}) \\
&\quad -8z^5(t_{12123}+t_{12124}+t_{12134}+t_{12143}+t_{12313} \\
&\quad +t_{12413}-4\,t_{12}t_{123}-2\,t_{13}t_{123}) \\
&\quad -4z^6(2\,S_8+\,S_4+\,t_{12}^3+3\,t_{12}^2\,t_{13}+\,t_{12}t_{13}^2 \\
&\quad -8\,t_{12}t_{1213}-4\,t_{12}t_{1214} \\
&\quad -2\,t_{12}t_{1234}-4\,t_{12}t_{1243} \\
&\quad -4\,t_{13}t_{1213}-2\,t_{13}t_{1214}-\,t_{13}t_{1234} \\
&\quad -2\,t_{13}t_{1243}-7\,t_{123}^2)-\cdots
\end{aligned} \tag{18.38}$$

where in the coefficient to z^6 the abbreviations S_8 and S_4 stand for the sums over the weights of the 12 orbits with multiplicity 8 and the 5 orbits of multiplicity 4, respectively; the orbits are listed in table 13.4.

Exercise 18.10 Tail resummations. *A simple illustration of such tail resummation is the ζ function for the Ulam map (12.22) for which the cycle structure is exceptionally simple: the eigenvalue of the $x_0 = 0$ fixed point is 4, while the eigenvalue of any other n-cycle is $\pm 2^n$. Typical cycle weights used in thermodynamic averaging are $t_0 = 4^\tau z$, $t_1 = t = 2^\tau z$, $t_p = t^{n_p}$ for $p \neq 0$. The simplicity of the cycle eigenvalues enables us to evaluate the ζ function by a simple trick: we note that if the value of any n-cycle eigenvalue were t^n, (17.21) would yield $1/\zeta = 1 - 2t$. There is only one cycle, the x_0 fixed point, that has a different weight $(1 - t_0)$, so we factor it out, multiply the rest by $(1 - t)/(1 - t)$, and obtain a rational ζ function*

$$1/\zeta(z) \;=\; \frac{(1-2t)(1-t_0)}{(1-t)} \tag{18.39}$$

Consider how we would have detected the pole at $z = 1/t$ without the above trick. As the $\overline{0}$ fixed point is isolated in its stability, we would have kept the factor $(1-t_0)$ in (18.7) unexpanded, and noted that all curvature combinations in (18.7) which include the t_0 factor are unbalanced, so that the cycle expansion is an infinite series:

$$\prod_p (1 - t_p) \;=\; (1 - t_0)(1 - t - t^2 - t^3 - t^4 - \ldots) \tag{18.40}$$

(we shall return to such infinite series in chapter 22). The geometric series in the brackets sums up to (18.39). Had we expanded the $(1 - t_0)$ factor, we would have noted that the ratio of the successive curvatures is exactly $c_{n+1}/c_n = t$; summing we would recover the rational ζ function (18.39).

Exercise 18.11 Escape rate for the Rössler system. *(continuation of exercise 12.7) Try to compute the escape rate for the Rössler system (2.17) using the method of cycle expansions. The answer should be zero, as nothing escapes. Ideally you should already have computed the cycles and have an approximate grammar, but failing that you can cheat a bit and peak into table 12.1.*

Exercise 18.12 Ulam map is conjugate to the tent map. *(Continuation of exercise 18.3 / repeat of exercise 7.3 and exercise 12.2; requires real smarts, unless you look it up) Explain the magically simple form of cycle stabilities of exercise 18.3 by constructing an explicit smooth conjugacy (7.1)*

$$g^t(y_0) = h \circ f^t \circ h^{-1}(y_0)$$

that conjugates the Ulam map (12.22) into the tent map (10.6).