# Diffusion - Sawtooth Map

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## 1 Deterministic Diffusion

The analysis of transport properties of chains of piecewise linear one-dimensional maps may yield interesting and unexpected results. Especially the deterministic diffusion coefficient exhibits unusual behavior.

The sawtooth-map is a one-dimensional piecewise linear map defined on the unit interval, and which has the following simple form

$$f(\hat{x}) = \begin{cases} \Lambda\hat{x} & \text{for } \hat{x} \in [0; \frac{1}{2}] \\ 1-(1-\hat{x})\Lambda & \text{for } \hat{x} \in [\frac{1}{2}; 1] \end{cases}, \qquad \Lambda > 2,$$

where $\Lambda$ is the slope of the individual parts of the map. The dynamics may now be extended to the entire real line by requiring the translation property

$$\hat{f}(\hat{x}+\hat{n}) = \hat{f}(\hat{x}) + \hat{n}, \qquad \hat{n} \in Z,$$

as is evident in Fig. 1.

The dynamics may be restricted to the elementary cell through the following operation whereby it is characterized by the *jumping number* $\hat{n}_p \in Z$ of the elementary cell cycle $p \in \{x_1, x_2, \ldots, x_{n_p}\}$.

$$f(x) = \hat{f}(\hat{x}) - [\hat{f}(\hat{x})], \qquad x = \hat{x} - [\hat{x}] \in [0,1],$$

where $[\cdots]$ denotes the integer part of the argument (Fig. 2). The dynamics is now characterized by the *jumping number* $\hat{n}_p \in Z$ of the elementary cell cycle $p \in \{x_1, x_2, \ldots, x_{n_p}\}$. In order to

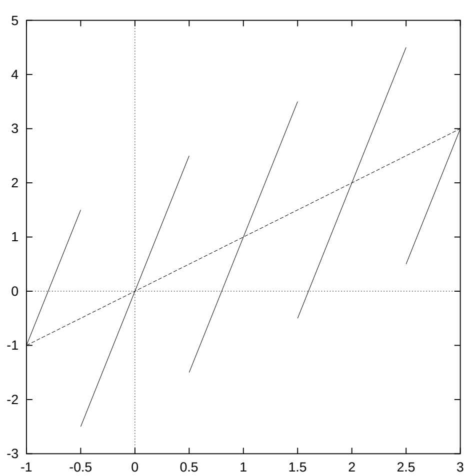


Figure 1: The Sawtooth map ($\Lambda = 5$).

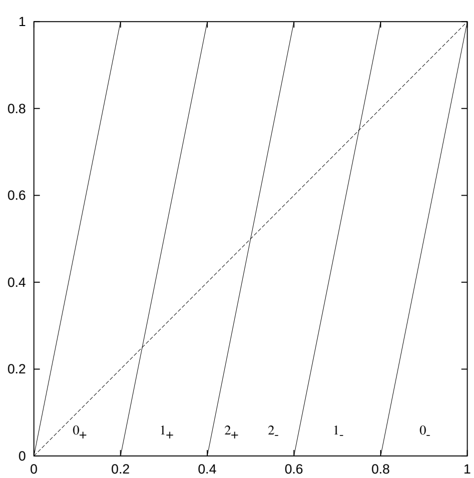


Figure 2: The restricted Sawtooth map

calculate the dynamical $\zeta$-function the cycle weights are given by

$$t_p = z^{n_p} \frac{e^{\beta \hat{n}_p}}{|\Lambda_p|}. \tag{1}$$

The diffusion constants is then given by

$$D = \frac{1}{2} \frac{\langle n \rangle_\zeta}{\langle \hat{x}^2 \rangle_\zeta}, \tag{2}$$

where the "mean cycle time"

$$\langle n \rangle_\zeta = z \frac{\partial}{\partial z} \frac{1}{\zeta(0,z)} \bigg|_{z=1}, \tag{3}$$

and the mean cycle displacement

$$\langle \hat{x}^2 \rangle_\zeta = \frac{\partial^2}{\partial \beta^2} \frac{1}{\zeta(\beta,1)} \bigg|_{\beta=0}. \tag{4}$$

# 2 Diffusion Coefficients through Periodic Orbit Theory

We now calculate the diffusion coefficients for the simplest values of $\Lambda \in [4;6]$ where the Markov partitions are finite. In order for the Markov partition to be finite the critical point ($x$=1/2) must be mapped on to the border of one of the jumping number partitions.

## 2.1 $\Lambda = 4$ ($\Lambda$ even)

In this case the critical point is mapped into a full shift. In particular we will solve the general problem when the slope $\Lambda$ is an even integer thereby yielding a full shift

$$\Lambda = 2a, \qquad a \in Z. \tag{5}$$

For all $\Lambda$ an even integer the symbolic dynamics is full unrestricted shift in $2a$ letters and the dynamical $\zeta$-function may thus be written

$$\frac{1}{\zeta(\beta,z)} = \left(1 - \sum_{\mathcal{A}} t_p\right),$$

where

$$p \in \mathcal{A} = \{0_+, 1_+, \ldots, (a-1)_+, (a-1)_-, \ldots, 1_-, 0_-\}.$$

With weights given in eq. 1 we obtain the following

$$\begin{aligned} \frac{1}{\zeta(\beta,z)} &= 1 - t_{0_+} - t_{0_-} - \cdots - t_{(a-1)_+} - t_{(a-1)_-} \\ &= 1 - \frac{2z}{\Lambda}\left(1 + \sum_{j=1}^{a-1} \cosh(\beta j)\right). \end{aligned}$$

First we observe that there is material flow conservation

$$\frac{1}{\zeta(0,1)} = 1 - \frac{2}{2a} a = 0.$$

Now we obtain for the mean cycle time

$$\begin{aligned} \langle n \rangle_\zeta &= z \frac{\partial}{\partial z} \frac{1}{\zeta(0,z)} \bigg|_{z=1} \\ &= -\frac{2}{2a} a \\ &= -1, \end{aligned}$$

and

$$\begin{aligned}
\langle \hat{x}^2 \rangle_\zeta &= \frac{\partial^2}{\partial \beta^2} \frac{1}{\zeta(\beta,1)} \bigg|_{\beta=0} \\
&= -\frac{2}{\Lambda} \sum_{j=1}^{a-1} j^2 \\
&= -\frac{2}{6\Lambda} a(a-1)(2a-1) \\
&= -\frac{2}{6\Lambda} \frac{\Lambda}{2} \frac{\Lambda-2}{2} \Lambda - 1 \\
&= -\frac{1}{12}(\Lambda-1)(\Lambda-2).
\end{aligned}$$

Thus the diffusion coefficient $D$ is easily found from eq. (2)

$$D = \frac{1}{24}(\Lambda-1)(\Lambda-2). \tag{6}$$

In particular for the case $\Lambda = 4$

$$D = \frac{1}{4}. \tag{7}$$

## 2.2 $\Lambda = 2 + \sqrt{6}$

We now wish to examine the case when the critical point maps onto the right border of $0-+$. In order for this to happen the following must be fulfilled

$$\frac{1}{2}\Lambda = 2 + x \quad \text{and} \quad \Lambda x = 1 \qquad \Rightarrow \qquad \Lambda^2 - 4\Lambda - 2 = 0 \qquad \Rightarrow \qquad \Lambda = 2 + \sqrt{6}. \tag{8}$$

In this case there are some forbidden symbolic subsequences,

$$\begin{array}{lllll}
-2_+1_+-, & -2_+2_+-, & -2_+2_--, & -2_+1_--, & -2_+0_--, \\
-2_-1_--, & -2_-2_--, & -2_-2_+-, & -2_-1_+-, & -2_-0_+-,
\end{array}$$

and the dynamical $\zeta$-function may thus be written as

$$\frac{1}{\zeta(\beta,z)} = \left(1 - \sum_{\mathcal{A}} t_p\right),$$

with orbits in the unrestricted alphabet

$$p \in \mathcal{A} = \{0_+, 1_+, 2_+0_+, 2_-0_-, 1_-, 0_-\}.$$

This yields

$$\begin{aligned}
\frac{1}{\zeta(\beta,z)} &= 1 - t_{0_+} - t_{0_-} - t_{1_+} - t_{1_-} - t_{2_+0_+} - t_{2_-0_-} \\
&= 1 - \frac{2z}{\Lambda}(1 + \cosh(\beta) - \frac{2z^2}{\Lambda^2}\cosh(2\beta).
\end{aligned}$$

Material flow is conserved according to (8)

$$\frac{1}{\zeta(0,1)} = 1 - \frac{2}{\Lambda}2 - \frac{2}{\Lambda^2} = 0.$$

We now obtain for the mean cycle time

$$\begin{aligned}
\langle n \rangle_\zeta &= -\frac{2}{\Lambda}2 - \frac{4}{\Lambda^2} \\
&= -\frac{4\Lambda + 4}{\Lambda^2},
\end{aligned}$$

and

$$\begin{aligned}\langle \hat{x}^2 \rangle_\zeta &= -\frac{2}{\Lambda} - \frac{2}{\Lambda^2}4 \\ &= -\frac{2\Lambda + 8}{\Lambda^2}.\end{aligned}$$

The diffusion coefficient is thus

$$\begin{aligned}D &= \frac{1}{2}\frac{2\Lambda + 8}{4\Lambda + 4} \\ &= \frac{1+\sqrt{6}}{4+2\sqrt{6}}.\end{aligned} \tag{9}$$

## 2.3 $\Lambda = 2 + 2\sqrt{2}$

We now wish to examine the case when the critical point maps onto the right border of $1 - +$. In order for this to happen the following must be fulfilled

$$\frac{1}{2}\Lambda = 2 + x \quad \text{and} \quad \Lambda x = 2 \qquad \Rightarrow \qquad \Lambda^2 - 4\Lambda - 4 = 0 \qquad \Rightarrow \qquad \Lambda = 2 + 2\sqrt{2}. \tag{10}$$

The forbidden symbolic subsequences are given by,

$$\begin{array}{llll} -2_+2_+-, & -2_+2_--, & -2_+1_--, & -2_+0_--, \\ -2_-2_--, & -2_-2_+-, & -2_-1_+-, & -2_-0_+-, \end{array}$$

and the dynamical $\zeta$-function may thus be written as

$$\frac{1}{\zeta(\beta, z)} = \left(1 - \sum_{\mathcal{A}} t_p\right),$$

with orbits in the unrestricted alphabet

$$p \in \mathcal{A} = \{0_+, 1_+, 2_+0_+, 2_+1_+, 2_-1_-, 2_-0_-, 1_-, 0_-\}.$$

This yields

$$\begin{aligned}\frac{1}{\zeta(\beta, z)} &= 1 - t_{0_+} - t_{0_-} - t_{1_+} - t_{1_-} - t_{2_+0_+} - t_{2_-0_-} - t_{2_+1_+} - t_{2_-1_-} \\ &= 1 - \frac{2z}{\Lambda}(1 + \cosh(\beta) - \frac{2z^2}{\Lambda^2}(\cosh(2\beta) + \cosh(3\beta)).\end{aligned}$$

Material flow is conserved according to (10)

$$\frac{1}{\zeta(0,1)} = 1 - \frac{2}{\Lambda}2 - \frac{2}{\Lambda^2}2 = 0.$$

We now obtain for the mean cycle time

$$\begin{aligned}\langle n \rangle_\zeta &= -\frac{2}{\Lambda}2 - \frac{4}{\Lambda^2}2 \\ &= -\frac{4\Lambda + 8}{\Lambda^2},\end{aligned}$$

and

$$\begin{aligned}\langle \hat{x}^2 \rangle_\zeta &= -\frac{2}{\Lambda} - \frac{2}{\Lambda^2}(4+9) \\ &= -\frac{2\Lambda + 26}{\Lambda^2}.\end{aligned}$$

The diffusion coefficient is thus

$$\begin{aligned}D &= \frac{1}{2}\frac{2\Lambda + 26}{4\Lambda + 8} \\ &= \frac{15+2\sqrt{2}}{16+8\sqrt{2}}.\end{aligned} \tag{11}$$

### 2.4 $\Lambda = 5$ ($\Lambda$ odd)

In this case the critical points is mapped onto the right border of the $2_+$ partition. In particular we will solve the general problem when the slope $\Lambda$ is an odd integer

$$\Lambda = 2a - 1, \qquad a \in Z, \tag{12}$$

for which the critical points is mapped onto the right border of $(a-1)_+$.

The forbidden subsequences for this type of shift are

$$\begin{array}{llll} -(a-1)_+(a-1)_- -, & -(a-1)_+(a-2)_- -, & \cdots \; -(a-1)_+1_- -, & -(a-1)_+0_- -, \\ -(a-1)_-(a-1)_+ -, & -(a-1)_-(a-2)_+ -, & \cdots \; -(a-1)_-1_+ -, & -(a-1)_-0_+ - . \end{array}$$

The above pruning rule does not explicitly prune the two fixed points $\overline{2}_+$ and $\overline{2}_-$ and they must therefore be explicitly included in the unrestricted grammar. This yields the following dynamical $\zeta$-function as the curvature corrections vanish

$$\frac{1}{\zeta(\beta,z)} = \left(1 - \sum_{\mathcal{A}} t_p\right)(1 - t_{2_+})(1 - t_{2_-}),$$

with the alphabet

$$p \in \mathcal{A} = \{2_+^k 0_+, 2_+^k 1_+, 2_-^k 1_-, 2_-^k 1_-\} \quad k = 0, 1, 2, \ldots.$$

Explicitly the dynamical $\zeta$-function then yields

$$\begin{aligned} \frac{1}{\zeta(\beta,z)} &= 1 - 0_+ - 0_- - 1_+ - 1_- - 2_+ - 2_- + 2_+0_- + 2_-0_+ + 2_+1_- + 2_-1_+ + 2_+2_- \\ &= 1 - \frac{2z}{\Lambda}\left(1 + \sum_{j=1}^{a-1}\cosh(\beta j)\right) + \frac{2z^2}{\Lambda}\left(\frac{1}{2} + \sum_{j=1}^{a-1}\cosh(\beta j)\right). \end{aligned}$$

We notice that the material flow is conserved

$$\frac{1}{\zeta(0,1)} = 1 - \frac{2}{\Lambda}a + \frac{2}{\Lambda^2}\left(\frac{1}{2} + a - 1\right) = 1 - \frac{2}{\Lambda}\frac{\Lambda+1}{2} + \frac{1}{\Lambda} = 0.$$

The mean cycle time may now be calculated

$$\begin{aligned} \langle n \rangle_\zeta &= z\frac{\partial}{\partial z}\frac{1}{\zeta(0,z)}\bigg|_{z=1} \\ &= -\frac{2}{\Lambda}(1 + a - 1) + \frac{4}{\Lambda^2}\left(\frac{1}{2} + \frac{\Lambda - 1}{2}\right) \\ &= \frac{1-\Lambda}{\Lambda}. \end{aligned}$$

and the cycle displacement

$$\begin{aligned} \langle \hat{x}^2 \rangle_\zeta &= \frac{\partial^2}{\partial\beta^2}\frac{1}{\zeta(\beta,1)}\bigg|_{\beta=0} \\ &= -\frac{2}{\Lambda}\sum_{j=1}^{a-1} j^2 + \frac{2}{\Lambda^2}\sum_{j=1}^{a-1} j^2 \\ &= \frac{a(a-1)(2a-1)}{3\Lambda}\frac{1-\Lambda}{\Lambda} \\ &= \frac{1}{12}(\Lambda^2 - 1)\frac{1-\Lambda}{\Lambda}. \end{aligned}$$

Thus the diffusion coefficient $D$ is

$$\begin{aligned} D &= \frac{1}{2}\frac{\langle \hat{x}^2\rangle_\zeta}{\langle n\rangle_\zeta} \\ &= \frac{1}{24}(\Lambda^2-1). \end{aligned} \tag{13}$$

In particular for the case $\Lambda = 5$

$$D = 1. \tag{14}$$

## 2.5 $\Lambda = 3+\sqrt{5}$

We now wish to examine the case when the critical point maps onto the right border of $2_-$. In order for this to happen the following must be fulfilled

$$\frac{1}{2}\Lambda = 2+x \quad \text{and} \quad \Lambda x + 1 - \Lambda = -1 \quad \Rightarrow \quad \Lambda^2 - 6\Lambda + 4 = 0 \quad \Rightarrow \qquad \Lambda = 3+\sqrt{5}. \tag{15}$$

The forbidden symbolic subsequences are given by,

$$\begin{array}{ll} -2_+1_--, & -2_+0_--, \\ -2_-1_+-, & -2_-0_+-, \end{array}$$

and again we need to specifically include the fix points $\overline{2}_+$ and $\overline{2}_-$ in the unrestricted alphabet. This yields the following dynamical $\zeta$-function

$$\frac{1}{\zeta(\beta,z)} = \left(1-\sum_{\mathcal{A}} t_p\right)(1-t_{2_+})(1-t_{2_-}),$$

with the alphabet

$$p \in \mathcal{A} = \{2_+^k 0_+, 2_+^k 1_+, 2_+^{k+1}2_-^{k+1}, 2_-^k 1_-, 2_-^k 1_-\} \quad k = 0,1,2,\ldots.$$

Explicitly the dynamical $\zeta$-function then yields

$$\begin{aligned} \frac{1}{\zeta(\beta,z)} &= 1 - 0_+ - 0_- - 1_+ - 1_- - 2_+ - 2_- + 2_+0_- + 2_-0_+ + 2_+1_- + 2_-1_+ \\ &= 1 - \frac{2z}{\Lambda}(1+\cosh(\beta)+\cosh(2\beta)) + \frac{2z^2}{\Lambda}(\cosh(\beta)+\cosh(2\beta)). \end{aligned}$$

Again we notice that material flow is conserved from (15)

$$\frac{1}{\zeta(0,1)} = 1 - \frac{2}{\Lambda}3 + \frac{2}{\Lambda^2}2 = 0.$$

The mean cycle time is now

$$\begin{aligned} \langle n\rangle_\zeta &= -\frac{2}{\Lambda}3 + \frac{4}{\Lambda^2}2 \\ &= \frac{8-6\Lambda}{\Lambda^2}, \end{aligned}$$

and the cycle displacement is given by

$$\begin{aligned} \langle \hat{x}^2\rangle_\zeta &= -\frac{2}{\Lambda}(1+4) + \frac{2}{\Lambda^2}(1+4) \\ &= \frac{10-10\Lambda}{\Lambda^2}. \end{aligned}$$

Finally the diffusion coefficient is given by

$$\begin{aligned} D &= \frac{1}{2}\frac{10-10\Lambda}{8-6\Lambda} \\ &= \frac{5+2\sqrt{5}}{6+2\sqrt{5}}. \end{aligned} \tag{16}$$

## 2.6 $\Lambda = 3 + \sqrt{7}$

We now wish to examine the case when the critical point maps onto the right border of $1_-$. In order for this to happen the following must be fulfilled

$$\frac{1}{2}\Lambda = 2 + x \quad \text{and} \quad \Lambda x + 1 - \Lambda = 0 \quad \Rightarrow \quad \Lambda^2 - 6\Lambda + 2 = 0 \quad \Rightarrow \quad \Lambda = 3 + \sqrt{5}. \tag{17}$$

The forbidden symbolic subsequences are given by,

$$-2_+0_- -, \quad -2_-0_+ -,$$

and again we need to specifically include the fix points $\overline{2}_+$ and $\overline{2}_-$ in the unrestricted alphabet. This yields the following dynamical $\zeta$-function

$$\frac{1}{\zeta(\beta,z)} = \left(1 - \sum_{\mathcal{A}} t_p\right)(1 - t_{2_+})(1 - t_{2_-}).$$

with the alphabet

$$p \in \mathcal{A} = \{2_+^k 0_+, 2_+^k 1_+, 2_+^k 1_-, 2_+^{k+1} 2_-^{k+1}, 2_-^k 1_+, 2_-^k 1_-, 2_-^k 1_-\} \quad k = 0, 1, 2, \ldots.$$

Explicitly the dynamical $\zeta$-function then yields

$$\begin{aligned} \frac{1}{\zeta(\beta,z)} &= 1 - 0_+ - 0_- - 1_+ - 1_- - 2_+ - 2_- + 2_+0_- + 2_-0_+ \\ &= 1 - \frac{2z}{\Lambda}(1 + \cosh(\beta) + \cosh(2\beta)) + \frac{2z^2}{\Lambda}\cosh(2\beta). \end{aligned}$$

Again we notice that material flow is conserved from (17)

$$\frac{1}{\zeta(0,1)} = 1 - \frac{2}{\Lambda}3 + \frac{2}{\Lambda^2} = 0.$$

The mean cycle time is now

$$\begin{aligned} \langle n \rangle_\zeta &= -\frac{2}{\Lambda}3 + \frac{4}{\Lambda^2} \\ &= \frac{4 - 6\Lambda}{\Lambda^2}, \end{aligned}$$

and the cycle displacement is given by

$$\begin{aligned} \langle \hat{x}^2 \rangle_\zeta &= -\frac{2}{\Lambda}(1 + 4) + \frac{2}{\Lambda^2} \\ &= \frac{8 - 10\Lambda}{\Lambda^2}. \end{aligned}$$

Finally the diffusion coefficient is given by

$$\begin{aligned} D &= \frac{1}{2}\frac{8 - 10\Lambda}{4 - 6\Lambda} \\ &= \frac{1}{2}\frac{11 + 5\sqrt{7}}{7 + 3\sqrt{7}}. \end{aligned} \tag{18}$$

## 2.7 $\Lambda = 6$

From eq. 6 for the case of a full shift with an even integer slope $\Lambda$ we easily obtain

$$D = \frac{5}{6}. \tag{19}$$

## 3 Numerically determined Diffusion Coefficients

A numerical investigation of the diffusion coefficient

$$D = \frac{1}{2} \lim_{n\to\infty} \frac{1}{n} \langle \hat{x}_n^2 \rangle, \tag{20}$$

has been undertaken with the use of the C program `diffusion.c` (appendix A). A strategy of averaging many points ($10^6$) over short time (100 iterations) was adopted as the loss of bits in the finite representation of real numbers on a computer for some values of $\Lambda$ is quite severe (in particular $\Lambda = 4 = 2^2$). Furthermore an ensemble average was performed in order to average over the random number generator. The results for $\Lambda \in [2;8]$ is shown below

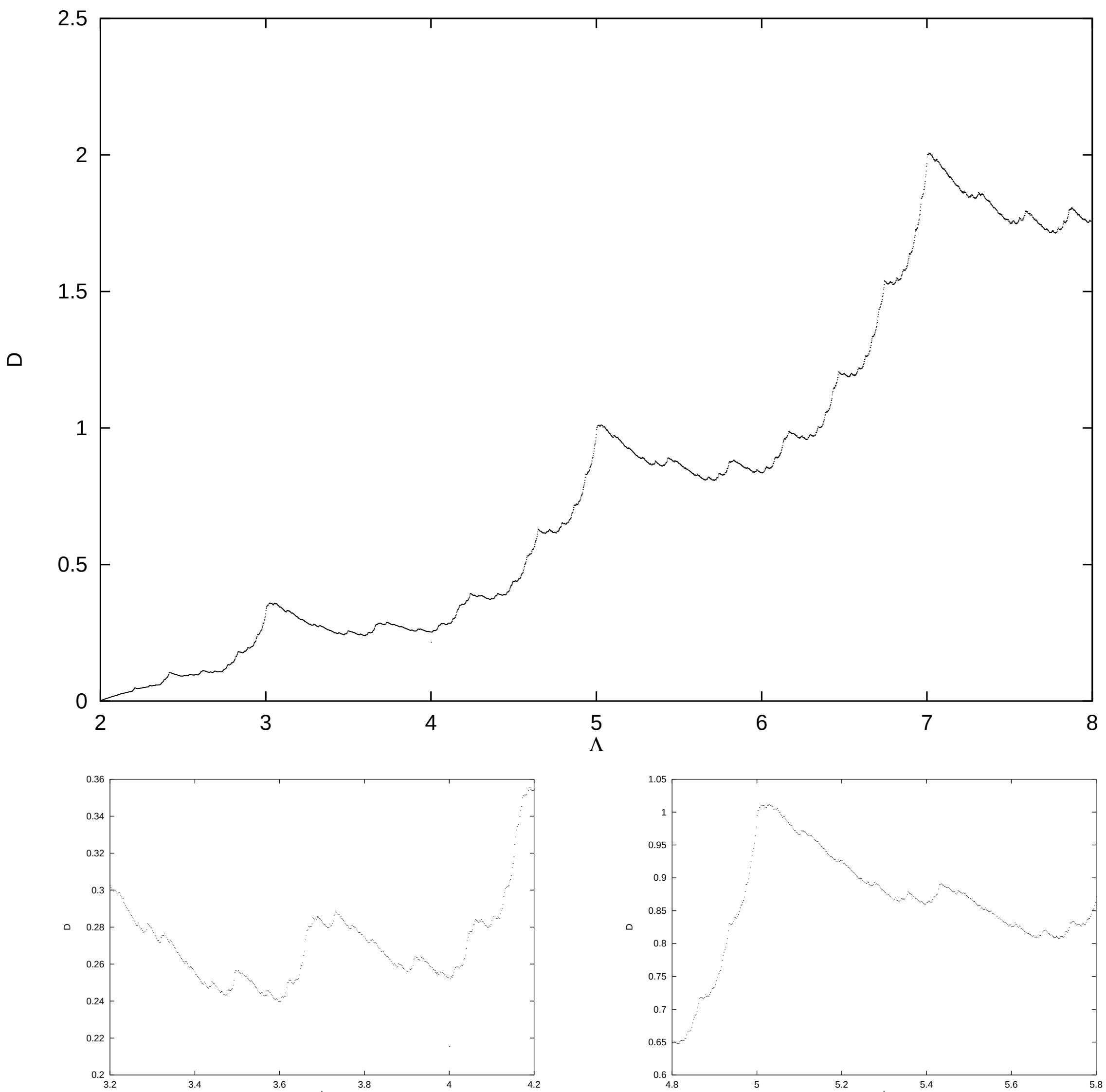


Figure 3: Top: Numerically determined diffusion coefficients for the sawtooth map for different values of $\Lambda$ ($\Lambda$ is incremented by 0.002). Below: Blowup of the above plot which shows the fractal behavior of the diffusion coefficient

We may now finally compare our exact results with the results obtained from the numerical calculation ((p) - exact from periodic orbit theory, (n) - numerically)

| Λ | | D | |
|---|---|---|---|
| | 4.000 | | 0.2500 (p) |
| | | | 0.2154 (n) |
| | 4.448 | | 0.3894 (n) |
| $2+\sqrt{6}$ | $\sim$ 4.4495 | $\frac{1+\sqrt{6}}{4+2\sqrt{6}}$ | $\sim$ 0.3876 (p) |
| | 4.450 | | 0.3891 (n) |
| | 4.828 | | 0.6529 (n) |
| $2+2\sqrt{2}$ | $\sim$ 4.8284 | $\frac{15+2\sqrt{2}}{16+8\sqrt{2}}$ | $\sim$ 0.6527 (p) |
| | 4.830 | | 0.6560 (n) |
| | 0.500 | | 1.0000 (p) |
| | | | 0.9947 (n) |
| | 5.236 | | 0.9021 (n) |
| $3+\sqrt{5}$ | $\sim$ 5.2361 | $\frac{5+2\sqrt{5}}{6+2\sqrt{5}}$ | $\sim$ 0.9045 (p) |
| | 5.238 | | 0.9008 (n) |
| | 5.644 | | 0.8137 (n) |
| $3+\sqrt{7}$ | $\sim$ 5.6458 | $\frac{11+5\sqrt{7}}{14+6\sqrt{7}}$ | $\sim$ 0.8110 (p) |
| | 6.646 | | 0.8119 (n) |
| | 6.000 | | 0.8333 (p) |
| | | | 0.8356 (n) |

Note the relatively large divergence of the numerically obtained value for $\Lambda = 4$. This is due to the aforementioned loss of information and a much better value for $\Lambda = 4$ is obtained if the averaging is done over very short time ($\sim$5).

# A Program diffusion.c

```
#include <stdlib.h>
#include <stdio.h>
#include <math.h>
#include "ran1.c"

#define     ENSEMBLES     4
#define     POINTS        1000000
#define     TIME          100

double **aloc_mem(int nx, int ny){

   int i;
   double **name;

   name = (double **)calloc(nx,sizeof(double *));
   for(i=0;i<nx;i++)
     name[i] = (double *)calloc(ny,sizeof(double *));

   return name;
}


double map(double x,double lambda){

  if(x<0.5)
    return lambda*x;
  else
    return lambda*x+1-lambda;
}

int main(){

  int i,j,k;
  long seed=-92;

  double **x,xn,d,td,ad,tad,lambda;
  FILE *res;

  x = aloc_mem(POINTS,2);

  res=fopen("difcof.dat","w");

  lambda=4.0;

  while(lambda<5.99){
    tad=0.0;
    lambda+=0.0025;

    for(k=0;k<ENSEMBLES;k++){

      d=td=ad=0.0;
```

```
      seed=-(long)floor(100*ran1(&seed));

      for(j=0;j<POINTS;j++){
x[j][0]=x[j][1]=0.0;
x[j][0]=ran1(&seed);
      }

      for(i=0;i<TIME;i++)
for(j=0;j<POINTS;j++){
  xn=map(x[j][0],lambda);
  x[j][1]+=floor(xn);
  x[j][0]=xn-floor(xn);
}

      for(j=0;j<POINTS;j++){
d=0.5*(x[j][0]+x[j][1])*(x[j][0]+x[j][1])/(double)TIME;
td+=d;
      }

      ad=td/(double)POINTS;
      tad+=ad;
    }

    fprintf(res,"%lf %lf\n",lambda,tad/(double)ENSEMBLES);
  }

  fclose(res);
  free(x);
  return 0;
}
```