Convergence of Cycle Expansion in 1-D Mappings

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Abstract

Cycle expansion is a very useful technique to calcualte dynamic averages in chaotic system, its validity has been proved in hyperbolic systems. But in application, cycle expansion sometimes shows bad convergence and therefore is difficult to use. I applied cycle expansion to calculate Lyapunov exponent in three types of 1-dimensional mappings, parabolic, cubic and quartic, and in each type, with different pruning rules by changing the height of mapping. To evalute the convergence, a ratio between total variation of each cycle expansion line and the corresponding Lyapunov exponent has been calculated. By comparing these results, the trend that convergence goes worse while adding pruning rules is expected and verified by results, and a possible correspondence between Lyapunov exponent and convergence is presented.

I. INTRODUCTION

In statistical mechanics, the essential goal is to calculate expectation values. Expectation values can not only describe the averages of physical quantities, but also their fluctuations, and through that, can connect theoretical calculations with experimental results. In quasi-static statistical mechanics, different statistics and different ensembles are considered and the corresponding partition functions are the keys to get expectation values. But in chaotic system, this is not a good sampling any more. And we need to calculate the averages with respect to time, which are called dynamical averages. In this situation, the way we usually follow in quasi-static statistical mechanics can not be used, the substitute is a method called cycle expansion.

Cycle expansion is based on an assumption that periodic orbits are dense in phase space. Thus, for each point in phase space, there is a sequence of points converge to it, with the sequence is constructed by points on periodic orbits. Therefore, we can represent the point approximately by the sequence and the whole phase space can be approached. This theory needs all periodic orbits, and generally there are infinite of them. But obviously we cannot do that in realy application. The only thing we can do is calculating finite number of cycles (usually the short ones), and hope they can approach the long ones. The validity of this has been proved in uniformly hyperbolic systems.

Although by theory cycle expansion can give dynamical average with infinite accuracy, in practice it depends on its convergence. Because in chaotic systems, searching for periodic orbits, especially multi-step periodic orbits is not trivial. Even in 1-dimensional case, if we are looking for all the 15-step cycles in a quadratic mapping, there could be thousands of them. And missing even one single cycle will make our further calculation absolutely useless. Therefore, we want the cycle expansion converge fast, i.e. reach a constant with few cycles considered and higher-order cycles correction is comparable small. Unfortunately, convergence of cycle expansion varies a lot. In some cases, considering only short cycles can give many significant digits, but in many other situations, expansion converges badly.

In this paper, I will consider cycle expansion results of some 1-dimensional mappings. The basic chaotic system is a parabolic mapping, i.e. $f(x) = 4x(1-x)$, and comparison are got by changing both the magnitude (height) and the order of functions (cubic $f(x) = Ax(1-x)(2-x)$ and quartic $f(x) = B[16(x-0.5)^4+1]$). I will calculate the expectation value of Lyapunov exponent as an example and compare results.

II. BASIC PROPERTIES OF 1-D MAPPINGS

Numerical simulation shows that 1-D mappings are mainly chaotic when H is greater than a constant, but with some non-chaotic windows, and when H smaller the system is non-chaotic. This constant depends on the shape of mapping, and approxmatly is 0.89, 0.88 and 0.97 for parabolic, cubic and quartic mapping, respectively. (Fig. 1 and Fig. 2)

When the system is chaotic, invariant measure (Fig. 3) is worth to be calculated because it shows not only the range of mapping, but also the distribution or the density of the range. When H equals one, that is the mapping is surjective, invariant measure is smooth except at two fixed points 0 and 1. But when $H < 1$, singular points appear in $(0, 1)$, arn when H decreases, these points become more and more and the peaks become higher and higher. I expect fractal in the invariant measure, but it need more sutble work around the peak.

III. CYCLE EXPANSION

A. Periodic Orbits

To use the cycle expansion method to get dynamical averages, the first step, of course, is searching for all periodic orbits or the system. Generally, a dynamical system may have infinite number of periodic orbits, but it is impossible in pratice to find all of them. Usually, we define a truncation number and find all cycles with topological length shorter than the number. This step can be done both theoretically by symbolic dynamics and numerically by multi-shooting Newton method. In fact, we first define an "over-sufficient" symbolic dynamics, i.e. ignore pruning rules, then try to find numerical correspondence for each symbolic sequence by multi-shooting with some man-made initial values. In our 1-dimensional parabolic, cubic, and quartic cases, a natural symbolic dynamics is obviously the "0-1" sequences, with the highest value point as seperating point. Then, getting symbolic sequence is pretty straight. And we assign initial value 0.25 for all the 0s and 0.75 for all the 1s. The following table lists all periodic orbits up to 6 steps of parabolic mapping $f(x) = 4x(1 - x)$ with their stabilities.

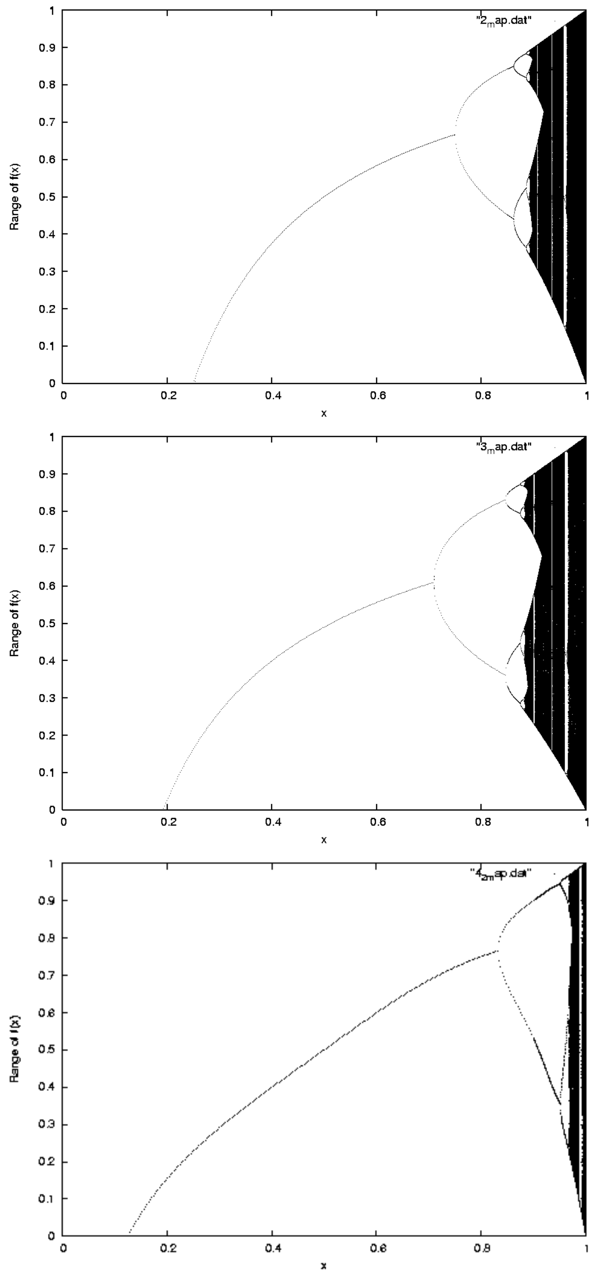

FIG. 1: Bifurcation diagrams of parabolic, cubic and quartic function, respectively. When $H = 0$, system is not chaotic and has only one attractor $x = 0$, as H increases, another attractor emerges and increases after some value. After H reaches a certain value, the non-zero attractor splits into 2 branches, then the 2 branches split into 4, and so on. When H is bigger than the critical point, chaotic phase is introduced into the system.

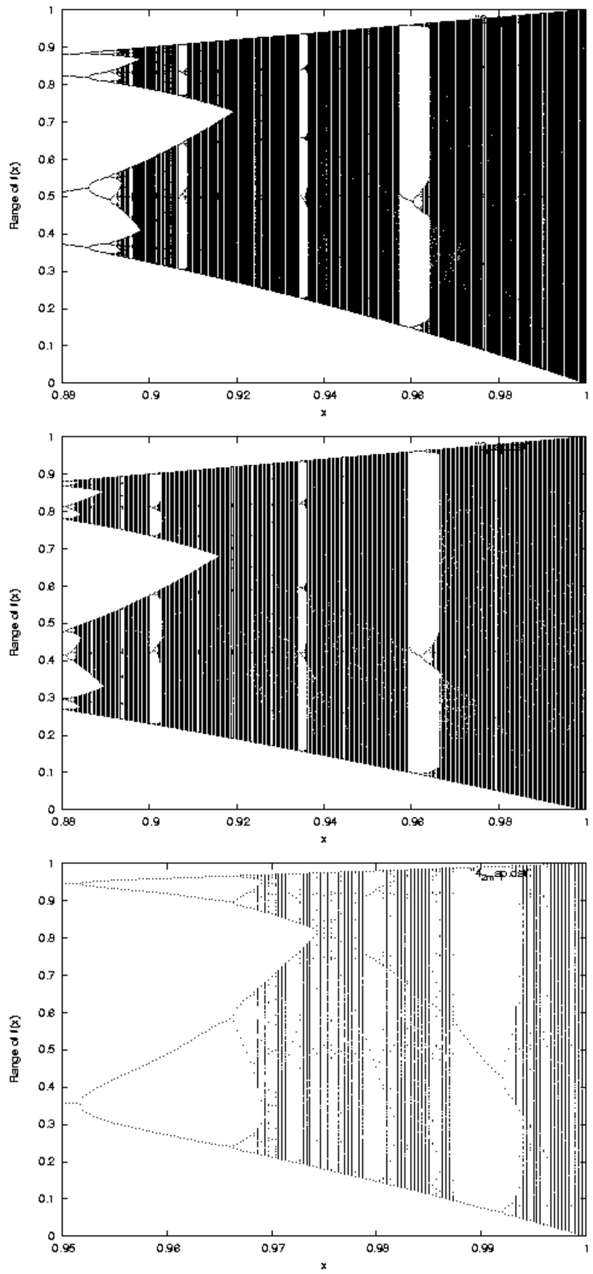

FIG. 2: These figures enlarge the chaotic regions of each kind of mapping. 3-based-branch and 5-based-branch windows appear inside the chaotic region.

This one to one correspondence from symbolic dynamics to real cycles is not always exist, because sometimes the theoretical cycle doesn't really exists because of the pruning rules. For example, all the 3-step cycles will disappear in the parabolic mappings $f(x) = 4 \times H \times x(1-x)$ when $H \leq 0.94$. (This is not the exact upper bound). To know exactly how many symbolic sequences are forbidden by pruning rules, one can compare the sequences

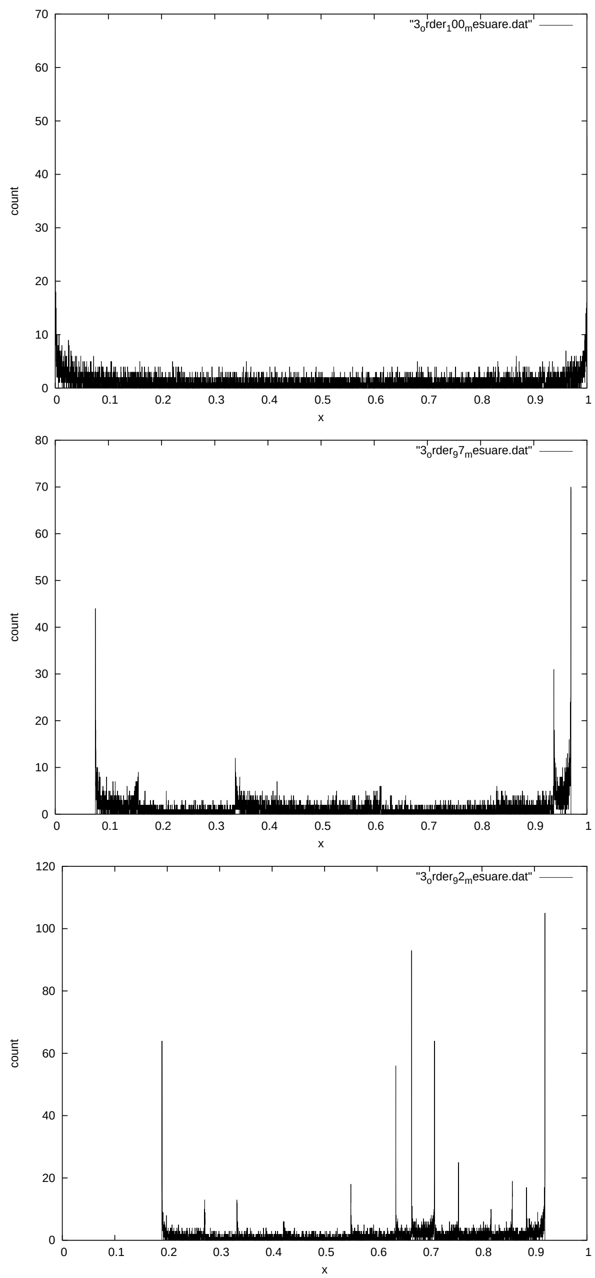

FIG. 3: Invariant measure of cubic mappings, with $H = 1.0, 0.97, 0.92$, respectively.

with leading sequence. But this method also has difficulty. To compare longer sequences, we need more and more digits in the leading sequence, but this is hard because the system is chaotic. Thus, we only check short cycles (up to 5) theoretically, and trust the program when it says there is no corresponding solution for a given symbolic sequence with longer length.

TABLE I: Periodic orbits up to 6-step of $f(x) = 4x(1-x)$.

n_p	p	x_p	Λ_p
1	0	0	4
	1	0.75	-2
2	01	0.345492	-4
3	001	0.116978	-8
	011	0.188255	8
4	0001	0.0337639	-16
	0011	0.0432273	16
	0111	0.277131	-16
5	00001	0.00903565	-32
	00011	0.010235	32
	00101	0.0896183	32
	00111	0.0793732	-32
	01011	0.382121	-32
	01111	0.235518	32
6	000001	0.00233418	-64
	000011	0.00248461	64
	000101	0.0222136	64
	000111	0.0208771	-64
	001011	0.110163	-64
	001101	0.057272	-64
	001111	0.0608892	64
	010111	0.36458	64
	011111	0.257008	-64

B. Spectral Determinant

With all these periodic orbits, we are able to apply the formulae in the Chaos book to calculate spectral determinant $\det(1 - z\mathcal{L})$ by giving some β values. In these formulae, we are supposed to make an infinite summation, and again we set a truncation number and keep all terms of z with power less than that and drop all those with higher. This will give us a polynomial of z. Finding the zero point around 1 of this polynomial, and the derivative of it with respect to *beta* will give us the dynamic average of Lyapunov exponent. Formulae in details are listed as below.

$$\begin{aligned}
\ln\det(1 - z\mathcal{L}) &= \exp\big(- \sum_{n}^{\infty} \frac{z^n}{n} Tr\mathcal{L}^n \big) \\
&= exp\big(- \sum_{p} \sum_{r=1}^{\infty} \frac{1}{r} \frac{z^{n_p r} e^{r\beta \cdot A_p}}{|\det(1 - J_p^r)|} \big)
\end{aligned} \tag{1}$$

$$< A >= - \frac{\partial z}{\partial \beta} |_{\beta=0} \tag{2}$$

And as an example, the result of a parabolic mapping $f(x) = 4 \times H \times x(1 - x)$ when $H = 0.94$ is shown in Fig. 4.

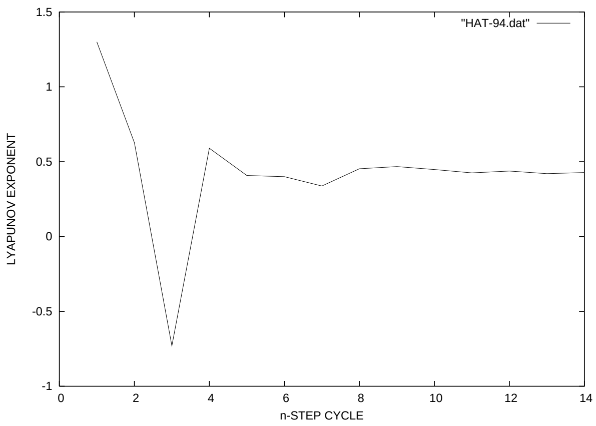

FIG. 4: Lyapunov exponent calculated by cycle expansion with consideration of up to n-step cycles for second order mapping with height equals 0.94.

IV. CONVERGENCE OF CYCLE EXPANSIONS

For 1-dimensional mappings, we have two parameters to change, the height and the shape. By changing the height, we will change the pruning rules, and therefore change the number of periodic orbits of certain length. By changing the shape, i.e., change the power of mappings from parabolic to cubic and quartic, the invariant measure will become more and

TABLE II: Lyapunov exponents of second order mappings and the quality of convergence.

H	Lyapunov Exp	Ratio
1.0	0.693500	0.0465032
0.98	0.537250	0.188460
0.97	0.510000	0.504902
0.94	0.428333	0.636187
0.92	0.341417	0.671467
0.90	0.212583	3.552720

more singular, and then, maybe have some effect of shaddowing and consequently change the convergence of cycle expansion.

First we consider the change of height of mappings. Fig. 5 shows the results of Lyapunov exponents of second order mappings with height changing from 1.0 to 0.90. From the figure we can see that the cycle expansion calculation does converge as we consider more and more cycles, and the Lyapunov exponent gets smaller and smaller when we decrease the height. Besides, the figure also shows that the convergence of each line becomes worse and worse as the height decreasing, that is, the Lyapunov exponent decreasing. It is more explicitly in Fig. 6, which enlarges the tail part of each line. We also calculate the total variation of tail of line, and get the ratio between total variation and average Lyapunov exponent. The ratios for each height are listed in Table II and they gives some numerical sense of quality of convergence. From these results, we can see a rough relationship between Lyapunov exponent and convergence of cycle expansion, although the correspondence is not accurate.

This trend of change of convergence with respect to the change of height can be understood in this way. When the height decrease, we have more and more pruning rules, and thus, less and less periodic orbits for certain length. Just as mentioned in section I, the validicity of cycle expansion is because in any chaotic system, periodic orbits are densd in phase space, and then, can approximatly reach each point in phase space if we consider all of them. But we cannot consider infinite (or even worse, uncountable) number of cycles, the only way to do is set a truncation of length, and hope the long length periodic orbits can be approximated by short cycles, which is called shaddowing. But if we have many pruning rules, then the shaddowing become worse and worse, thus the contribution of long

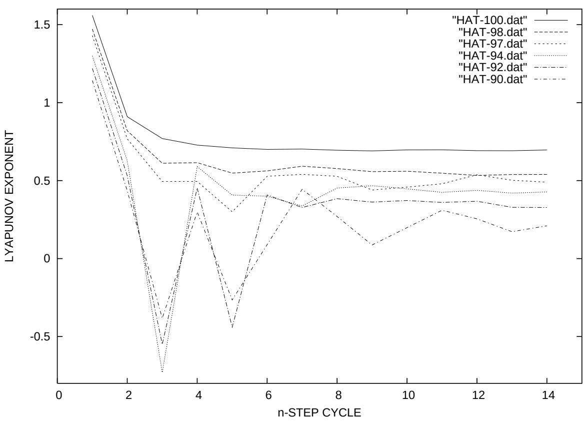

FIG. 5: Lyapunov exponent calculated by cycle expansion with consideration of up to n-step cycles for second order mappings with different heights.

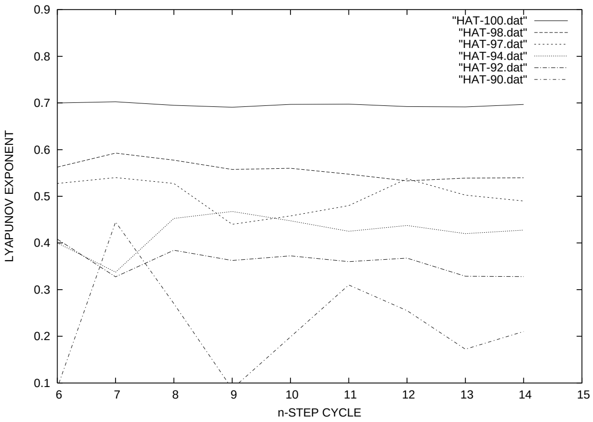

FIG. 6: Tail part of Fig. 5cycle expansion from n=6 to n=14.

length cycles cannot be represented by those of short cycles, cycle expansion will fail. And in practice, it will shows very bad convergence.

After changing the height, we also change the shape of mappings and compare the three different groups of mappings, parabolic, cubic, and quartic. Fig. 7 and Fig. 9 show the results of cubic and quartic, respectively, as Fig. 8 and Fig. 10 show the tail of them.

Compare the three kinds of mappings, the trend that convergence of cycle expansion becoming worse while decreasing the height also hold in cubic and quartic mappings. This is expected because the argument before doesn't depend on the shape. Besides, we can see that the convergence doesn't vary much in parabolic and cubic mappings, but for quartic

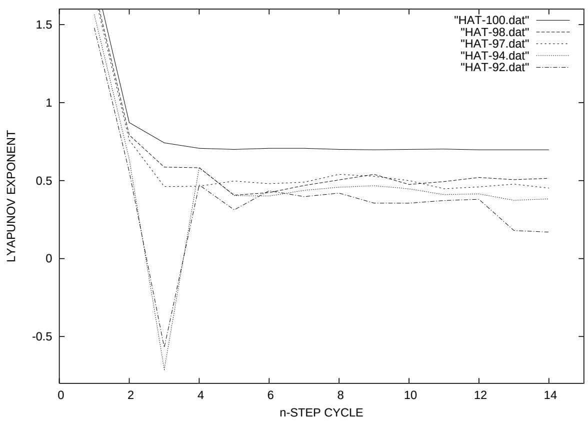

FIG. 7: Lyapunov exponent calculated by cycle expansion with consideration of up to n-step cycles for third order mappings with different heights.

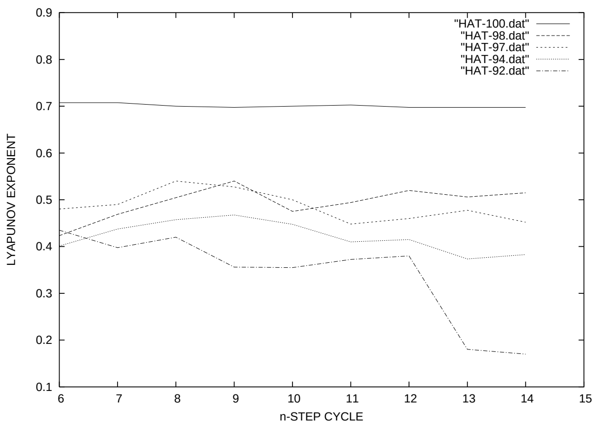

FIG. 8: Tail part of Fig. 7cycle expansion from n=6 to n=14.

mapping, cycle expansion fails much more faster than it does in parabolic and cubic cases. To see that more clearly, we also calculate the ratio between total variation and average of Lyapunov exponent, and results are given in Table III. From Fig. 11, a rough monotone relationship between the ratios and Lyapunov exponents is suggested, and Fig. 12 shows that the relationship could be exponential.

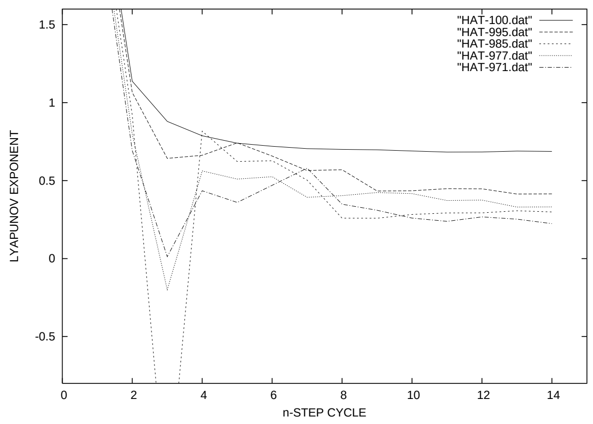

FIG. 9: Lyapunov exponent calculated by cycle expansion with consideration of up to n-step cycles for forth order mappings with different heights.

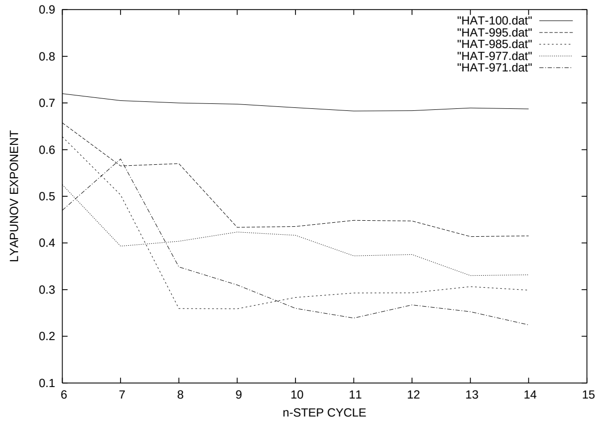

FIG. 10: Tail part of Fig. 9cycle expansion from n=6 to n=14.

V. CONCLUSION

In this paper, I report some numerical calculation of cycle expansions of some 1-dimensional mappings, and the observation of their convergence. The cycle expansion works well in full mapping, and becomes worse when decreasing the height because we insert more pruning rules and therefore dramatically decrease the number of short cycles, which are the key part because they are supposed to give good shaddowing. This result is not trivial because the "unfull" mapping is not uniform hyperbolic any more, and sometimes long cycles will also have small stability. For example, in the quartic mapping with $A = 0.971$, even for length 14-cycle, there are some cycles with $|\Lambda| \sim 23$ (such as $x_0 = 0.970651$ with

TABLE III: Lyapunov exponents of third and forth order mappings and the quality of convergence.

	H	Lyapunov Exp	Ratio
3-order	1.0	0.697500	0.0286738
	0.98	0.513667	0.485237
	0.97	0.463167	0.446384
	0.94	0.390417	0.459765
	0.92	0.243417	1.478950
4-order	1.0	0.686667	0.0666262
	0.995	0.425417	0.670519
	0.985	0.299583	1.41363
	0.977	0.34575	0.758496
	0.971	0.248167	1.66017

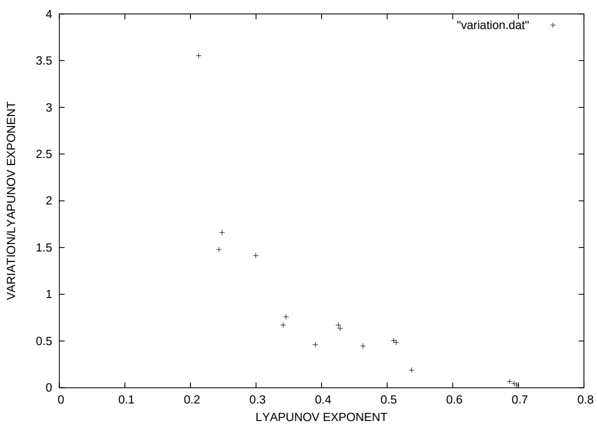

FIG. 11: The ratio of total variation over Lyapunov exponent shows a monotone dependence on Lyapunov exponent.

$\Lambda = -23.1825$), but in full mapping it should be around 2^{14}. Thus, cycle expansion may fails because we have to calculate infinite number of cycles to get all periodic orbits with small stability, similar as intermitancy. But our results show that cycle expansion still works in this case, but give bad convergence.

Another result, and maybe more interesting, is that the convergence of cycle expansion has no obvious dependence on the shape of mapping, but shows some relationship with the

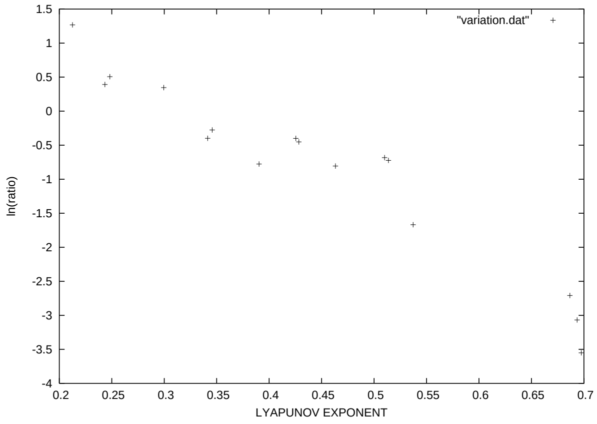

FIG. 12: The relationship is suggested to be exponential.

Lyapunove exponents, which represents the quality of chaos. An exponential correspondence is suggested and explanation still need more work.

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