Gone fishin'

a blog

`spieker/blog.tex`, rev. 37: last edit by Dustin Spieker, 12/07/2009

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May 13, 2011

Contents

Chapter 1

Research blog on fluid flows

[1]

Only dead fish go with the flow
— J Hightower, Texas politician

1.1 LaTeX blog

My main research blog is

www.channelflow.org/dokuwiki/doku.php/gtspring2009:spieker_blog.

Here I enter LaTeX text that can be used for my thesis or our papers.

10/21/09 DWS I'm getting an error inserting my references into my blog. I moved all of the references listed from John E's and John G's svn repositories to a file called elton.bib and just started building my own library to cite.

10/22/09 PC building your own BibTeX is not good for our collaboration - all existing references should be cited by their current names, I have created bibtex/spieker.bib for you from halcrow/bibtex/halcrow.bib. Before adding a BibTeX entry to your spieker.bib, check whether it is already there, so you do not rename an existing one.

10/21/09 DWS I've found that a lot of the tools I've read about apply nicely to small systems, but there doesn't seem to be a bridge between, say, the 16-D Kuramoto-Sivashinky equation and the 60000-D Navier-Stokes equations. Today I read the Christiansen paper [2] provided on Predrag's homepage. I did find this paper quite a bit more helpful than others I've read (like "Observations of Order and Chaos in Nonlinear Systems" by Harry L. Swinney). I really like the Feigenbaum tree provided in Figure

[1] `spieker/dailyBlog.tex`, rev. 29: last edit by Dustin Spieker, 11/22/2009

1, and I think, eventually, we should be able to produce a similar tree for Navier-Stokes with Re as the independent parameter. I also got to thinking about the $I - D = 0$ Poincaré section for plane Couette flow.

10/22/09 PC I copied to the blog sect. 5.3 I wrote for you in the ChaosBook.org. When you edit this text, please do it in

dasbuch/book/chapters/appendStability.tex,

not here, so we do not have diverging edits of the same text. Compile from dasbuch/book/ using

pdflatex bibtex pdflatex pdeflatex book

10/21/09 DWS I'll look into making the change from $s1, s2, s3$ to the isotropy groups on monday, and update that on the $P96.66$ table. I need to read the Halcrow thesis before that. The parameters provided are accurate, and I can provide more if need be. If these eigenvectors are orthogonal, does that void the relevance of putting the corresponding eigenvalues into a table or not?

10/22/09 PC The eigenvalues are invariant, and need to be tabulated. If the eigenvectors are Gram-Schmidt orthogonalized, they have no particular physical significance, save for the leading one. The Floquet eigenvectors are what we need.

10/21/09 DWS I'm going to go ahead and calculate the eigenvalues of the $P96.66$ solution over the weekend on post those in my blog next week as well.

10/28/09 DWS I am starting an application for a DOE fellowship, due November 30th. I know that I should get my mentor in Los Alamos as one of my references, but I'm not exactly sure who else to ask for recommendations from. Is it kosher to ask your own advisor for one?

10/31/09 PC Nobody is Jewish, but hey, I and John G. will write letters for you anyway. Who is your mentor in Los Alamos? For us it would be best to develop a collaboration with Bob Ecke, if it is someone else skype John G and me to discuss.

10/28/09 DWS So, according to John G., the eigenvectors calculated by Arnoldi iteration are the non-orthogonal Floquet eigenvectors, so that is good.

10/28/09 DWS I'm confused as to assigning values to $\sigma_p^{(j)}$ of complex eigenvector pairs. You prescribe $\sigma_p^{(j)} = T_p \omega_p^{(j)}$, but based on the fact that $T_p = 1/\omega_p^{(j)}$, wouldn't $\sigma_p^{(j)}$ just be ± 1 for all complex eigenvalue pairs? I guess I don't understand what to calculate for complex eigenvalue pairs in the table.

10/31/09 PC Period of a periodic orbit is not given by $T_p = 1/\omega_p^{(j)}$, that is period of spiralling around an equilibrium, nothing to do with periodic orbits, except as a rough estimate of a time scale of a periodic orbit that spends most of its time close to an equilibrium. I do not write $\sigma_p^{(j)} = T_p\omega_p^{(j)}$, my sign refers to the Floquet multiplier, not to its logarithm; check `ChaosBook.org/chapters/invariant.pdf`. You would think that a -1 sign in front of a Floquet multiplier of a periodic orbit could be absorbed into the exponent as an extra π phase in $\omega_p^{(j)}$, but the problem is that what is in the exponent is the $iT_p\omega_p^{(j)}$, so there is no natural way to include this sign into our list of exponents. By the way, as I have not run into this in actual tabulations of Floquet exponents, I might be dead wrong; now that I think of it I am most likely wrong, for complex Floquet multipliers there is no overall sign, the $T_p\omega_p^{(j)}$ phase can be anything in the $[0, \pi]$ interval. Would be good to straighten me out now, so I do not write something stupid in ChaosBook.tex.

10/30/09 DWS I am currently reading about the isotropy groups in Halcrow's thesis ref. [11] so that I can write down the isotropy groups of the eigenvectors of the periodic orbits I have found. I feel like I need a better understanding of group theory to continue on in this project, so at first I looked in birdtracks.org, and I had no idea what was going on. I then asked Domenico what he thought the best path of learning would be for me and gave me his copy of *Group Theory and Quantum Mechanics* by Tinkham, and told to read chapters 2 and 3.

10/31/09 PC Lets keep your dokuwiki blog the main daily blog (John G prefers that), Enter here LaTeX text that can be used for your thesis or our papers. My response to above is on your dokuwiki blog.

10/30/09 DWS I found a new equilibrium today in the Ω_{W03} cell which I have labeled E_{16}, see figure 1.3. I found it by integration of an initial condition restricted to $R = \{e, \sigma_{xz}\} \times \{e, \sigma_z\}$. When I initiated the hookstep search, though, I did not constrain the solution search to any isotropy group, so the solution I found may or may not be in R. I will hopefully be able determine the isotropy groups of the three solutions I have found after I learn a bit more group theory.

11/16/09 PC I know that both for pipe and planes equilibria and relative equilibria people like to state $||u||$,

but why? I do not see what does this number mean physically, and I do not see how its value correlates with with any interesting property of a solution. Dissipation rate D, by contrast, is a physical property which reflects the amount of the vorticity of a solution.

11/16/09 PC A LaTeX remark: use `\emph{...}` for emphasis; it toggles the font within italics text. `\textit{...}` is not distinguished, if the font is changed to italics. Happens sometimes...

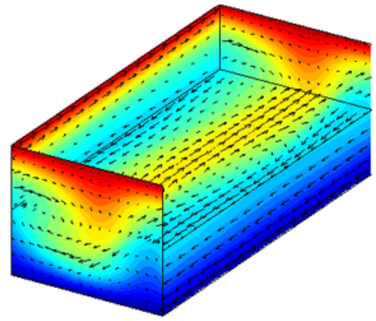

Figure 1.1: Equilibrium E_{14} in Ω_{w03} found on 09/24/2009 with dissipation $D = 1.603$ and $||u|| = .241$.

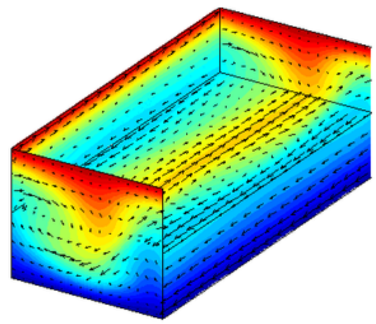

Figure 1.2: Equilibrium E_{15} in Ω_{w03} found on 09/28/2009 with dissipation $D = 1.763$ and $||u|| = .268$

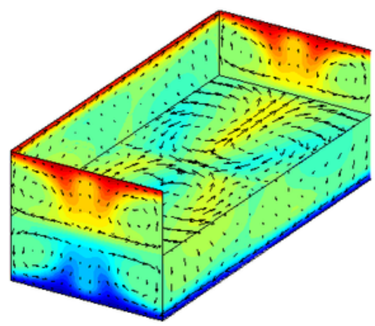

Figure 1.3: Equilibrium E_{16} in Ω_{w03} found on 10/30/2009 with dissipation $D = 3.492$ and $||u|| = .333$.

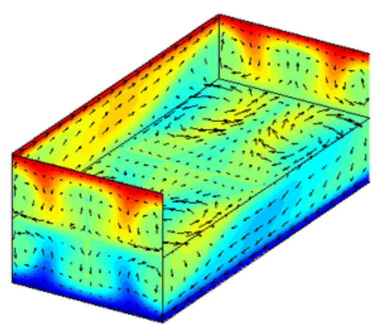

Figure 1.4: Equilibrium E_{17} in Ω_{w03} found on 11/06/2009 with dissipation $D = 2.979$ and $||u|| = .291$.

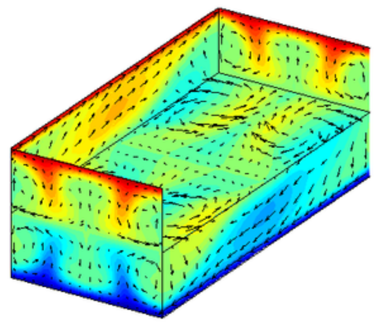

Figure 1.5: Equilibrium E_{18} in Ω_{w03} found on 11/10/2009 with dissipation $D = 3.659$ and $||u|| = .325$.

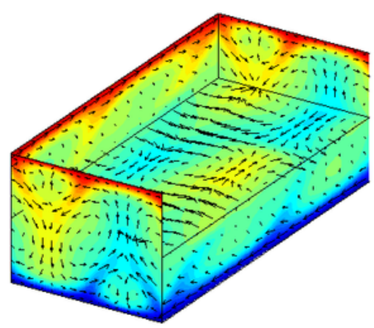

Figure 1.6: Equilibrium E_{19} in Ω_{w03} found on 11/12/2009 with dissipation $D = 5.542$ and $||u|| = 0.427$.

Chapter 2

The importance of symmetry

2.1 Basics of group theory

Suppose that a set of elements $A, B, C, ...$ has a form of group multiplication that associates a third member of the set with two other elements of the set; a group, $G = \{g_1, g_2, ..., g_n\}$, is defined as the members of that set that satisfy four conditions.

The first condition is called *closure*, namely that multiplication between any two elements in a group produces an element in the same group:

$$g_i \circ g_j \in G$$

The second condition is called *associativity*: the order in which group multiplication is performed does not matter. If parenthesis denote the first operation to be performed:

$$(g_i \cdot g_j) \circ g_k = g_i \circ (g_j \circ g_k)\,.$$

The third condition is that there must exist an *identity element* in the group, usually denoted by e, such that for all $g_i \in G$

$$e \circ g_i = g_i \circ e = g_i\,.$$

The fourth condition is the existence of an *inverse*. For every element in a group $g \in G$, there must exist an inverse $h = g^{-1} \in G$ such that

$$g \circ h = h \circ g = e\,,$$

where e once again represents the identity element. As an example, the real line with the exception of zero, $\mathbb{R} \cap 0$, forms an order ∞ group under multiplication. [1] It satisfies all of the above conditions. Including 0 in the set would fail to

[1]Predrag: This example is probably more confusing than helpful, the group is continuous but not a Lie group (no generators, I believe). I think you can also show that is a group/not a group under addition.

Table 2.1: Group multiplication table for a square. The order of group operations is multiplying the column by the row.

	E	A	B	C	D	F	G	H
E	E	A	B	C	D	F	G	H
A	A	E	G	F	H	C	B	D
B	B	G	E	H	F	D	A	C
C	C	H	F	E	G	B	D	A
D	D	F	H	G	E	A	C	B
F	F	D	C	A	B	G	H	E
G	G	B	A	D	C	H	E	F
H	H	C	D	B	A	E	F	G

satisfy the existence of an inverse condition and the set would no longer be a group.

Group theory is much more developed than the simple properties listed above, but with these definitions alone, we might begin to apply our knowledge of groups toward analysis of dynamical systems. First we must define few terms that will come up frequently in our considerations. It will be useful to consider a simple symmetric geometric object to reference throughout the definitions. Consider a square that is symmetric under reflection by π about axes A, B, C, D and rotations about the center of the square by $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$ called F, G, H respectively. We also include the identity symmetry E. With these 8 symmetries in place, we can work out the group multiplication table 2.1. [2]

Now we might begin our definitions. The first term to come up regularly is *conjugacy*. Suppose there exists a group $G = \{g_1, g_2, ..., g_n\}$. Element $\mathcal{Q}$ is said to be conjugate to P if $g_i P g_i^{-1} = \mathcal{Q}$ or $g_i^{-1} Q g_i = P$ for all elements $g_i \in G$. [3] For example, in the square example, both E and G are conjugate to themselves (but not to each other).

Suppose that we carry out the out calculations of the form $g_i^{-1} A g_i$ for the group of elements of our square symmetry. One would find

$$\begin{aligned}&\{AAA^{-1}, BAB^{-1}, CAC^{-1}, DAD^{-1}, EAE^{-1}, FAF^{-1}, GAG^{-1}, HAH^{-1}\}\\ &= \{A, A, B, A, A, B, A, B\} \qquad (2.1)\end{aligned}$$

Notice that only the elements A and B turn up. If we do the same calculation for B, we find that

$$\begin{aligned}&\{ABA^{-1}, BBB^{-1}, CBC^{-1}, DBD^{-1}, EBE^{-1}, FBF^{-1}, GBG^{-1}, HBH^{-1}\}\\ &= \{B, B, A, B, B, A, B, A\} \qquad (2.2)\end{aligned}$$

[2]Predrag: No need to fix this now in table 2.1, but the multiplication tables convention is to put identity into the first row/column, then other elements class by class, in increasing class size. This will be useful later on, when you write down character tables which tell you the weight for a given class within a given irreducible representation.

[3]Predrag: wrong as it stands - 'all elements' defins the class, conjugacy works only for specific elements, as you show in (2.1)

Again, note that only the elements A and B result. This property of A and B links them in a way that is called a *class*, which is our next definition. A class is a set of group elements $\mathcal{C} \in G$, such that for any $g_i \subset G$

$$g_i \mathcal{C} g_i^{-1} = \mathcal{C}\,.$$

For our square symmetry group, we find there are 5 such classes: $\mathcal{C}_1 = \{E\}$, $\mathcal{C}_2 = \{G\}$, $\mathcal{C}_3 = \{F, H\}$, $\mathcal{C}_4 = \{A, B\}$, $\mathcal{C}_5 = \{C, D\}$. Note that all of the classes, with the exception of $\mathcal{C}_1$, do not contain the identity element. This means that classes, with the exception of the class that contain the identity as its sole element, cannot be groups.

This brings us to our next definition: *subgroups*. A subgroup has all of the same properties as a group, but is part of a larger group. Just by looking at our multiplication table, one is able to see that there are a number of subgroups: one of order 4, 5 of order 2, and the obvious order one identity subgroup.

2.2 Discrete symmetries of PCF

The Dirichlet boundary condition on the top and bottom plates leads to rotational invariance in both the spanwise and streamwise directions, which leads to three rotational symmetries in plane Couette flow

$$\begin{aligned}
\sigma_z[u,v,w](x,y,z) &= [u,v,-w](x,y,-z) && (2.3)\\
\sigma_x[u,v,w](x,y,z) &= [-u,-v,w](-x,-y,z) && (2.4)\\
\sigma_{xz}[u,v,w](x,y,z) &= [-u,-v,-w](-x,-y,-z) && (2.5)
\end{aligned}$$

2.3 Continuous symmetries

Both the translational symmetries of the infinite-extent pipes and planes flows, and their finite cell versions with periodic boundary conditions imposed in both the streamwise and spanwise directions are examples of continuous symmetries. Factoring out continuous symmetries in spatially extended systems can be useful in reducing the state space and visualizing the dynamics of infinite dimensional systems. How this state space reduction is implemented is discussed in detail in refs. [19, 4] for the complex Lorenz flow.

2.3.1 PCF symmetries and isotropy subgroups

[4] On an infinite domain and in the absence of boundary conditions, the Navier-Stokes equations are equivariant under any $3D$ translation, $3D$ rotation, and $\mathbf{x} \to -\mathbf{x}$, $\mathbf{u} \to -\mathbf{u}$ inversion through the origin [6]. In plane Couette flow, the counter-moving walls restrict the rotation symmetry to rotation by π about

[4] Predrag: copied this section from ref. [8]

the z-axis. We denote this rotation by σ_x and the inversion through the origin by σ_{xz}. The suffixes indicate which of the homogeneous directions x, z change sign and simplify the notation for the group algebra of rotation, inversion, and translations presented in sects. 2.3.2 and 2.3.3. The σ_{xz} and σ_x symmetries generate a discrete dihedral group $D_1 \times D_1 = \{e, \sigma_x, \sigma_z, \sigma_{xz}\}$ of order 4, where

$$\begin{aligned} \sigma_x\,[u, v, w](x, y, z) &= [-u, -v, w](-x, -y, z) \\ \sigma_z\,[u, v, w](x, y, z) &= [u, v, -w](x, y, -z) \\ \sigma_{xz}\,[u, v, w](x, y, z) &= [-u, -v, -w](-x, -y, -z)\,. \end{aligned} \tag{2.6}$$

The walls also restrict the translation symmetry to $2D$ in-plane translations. With periodic boundary conditions, these translations become the $SO(2)_x \times SO(2)_z$ continuous two-parameter group of streamwise-spanwise translations

$$\tau(\ell_x, \ell_z)[u, v, w](x, y, z) = [u, v, w](x + \ell_x, y, z + \ell_z)\,. \tag{2.7}$$

The equations of plane Couette flow are thus equivariant under the group $\Gamma = O(2)_x \times O(2)_z = D_{1,x} \ltimes SO(2)_x \times D_{1,z} \ltimes SO(2)_z$, where $\ltimes$ stands for a semi-direct product, x subscripts indicate streamwise translations and sign changes in x, y, and z subscripts indicate spanwise translations and sign changes in z.

The solutions of an equivariant system can satisfy all of the system's symmetries, a proper subgroup of them, or have no symmetry at all. For a given solution $\mathbf{u}$, the subgroup that contains all symmetries that fix $\mathbf{u}$ (that satisfy $s\mathbf{u} = \mathbf{u}$) is called the isotropy (or stabilizer) subgroup of $\mathbf{u}$ [14, 16, 10, 9]. For example, a typical turbulent trajectory $\mathbf{u}(\mathbf{x}, t)$ has no symmetry beyond the identity, so its isotropy group is $\{e\}$. At the other extreme is the laminar equilibrium, whose isotropy group is the full plane Couette symmetry group Γ.

In between, the isotropy subgroup of the Nagata equilibria and most of the equilibria reported here is $S = \{e, s_1, s_2, s_3\}$, where

$$\begin{aligned} s_1\,[u, v, w](x, y, z) &= [u, v, -w](x + L_x/2, y, -z) \\ s_2\,[u, v, w](x, y, z) &= [-u, -v, w](-x + L_x/2, -y, z + L_z/2) \\ s_3\,[u, v, w](x, y, z) &= [-u, -v, -w](-x, -y, -z + L_z/2)\,. \end{aligned} \tag{2.8}$$

These particular combinations of flips and shifts match the symmetries of instabilities of streamwise-constant streaky flow [23, 25] and are well suited to the wavy streamwise streaks observable in simulations, with suitable choice of L_x and L_z. But S is one choice among a number of intermediate isotropy groups of Γ, and other subgroups might also play an important role in the turbulent dynamics. In this section we provide a partial classification of the isotropy groups of Γ, sufficient to classify all currently known invariant solutions and to guide the search for new solutions with other symmetries. We focus on isotropy groups involving at most half-cell shifts. The main result is that among these, up to conjugacy in spatial translation, there are only five isotropy groups in which we should expect to find equilibria.

2.3.2 Flips and half-shifts

[5] A few observations will be useful in what follows. First, we note the key role played by the rotation and reflection symmetries σ_x and σ_z (2.6) in the classification of solutions and their isotropy groups. The equivariance of plane Couette flow under continuous translations allows for traveling-wave solutions, i.e., solutions that are steady in a frame moving with a constant velocity in (x, z). In state space, relative equilibria either trace out circles or wind around tori, and these sets are both continuous-translation and time invariant. The sign changes under σ_x, σ_z, and σ_{xz}, however, imply particular centers of symmetry in x, z, and both x and z, respectively, and thus fix the translational phases of fields that are fixed by these symmetries. Thus the presence of σ_x or σ_z in an isotropy group prohibits relative equilibria in x or z, and the presence of σ_{xz} prohibits any form of relative equilibrium. Guided by this observation, we will seek equilibria only for isotropy subgroups that contain the σ_{xz} inversion symmetry.

Second, the periodic boundary conditions impose discrete translation symmetries of $\tau(L_x, 0)$ and $\tau(0, L_z)$ on velocity fields. In addition to this full-period translation symmetry, a solution can also be fixed under a rational translation, such as $\tau(mL_x/n, 0)$ or a continuous translation $\tau(\ell_x, 0)$. If a field is fixed under continuous translation, it is constant along the given spatial variable. If it is fixed under rational translation $\tau(mL_x/n, 0)$, it is fixed under $\tau(mL_x/n, 0)$ for $m \in [1, n-1]$ as well, provided that m and n are relatively prime. For this reason the subgroups of the continuous translation $SO(2)_x$ consist of the discrete cyclic groups $C_{n,x}$ for $n = 2, 3, 4, \ldots$ together with the trivial subgroup $\{e\}$ and the full group $SO(2)_x$ itself, and similarly for z. For rational shifts $\ell_x/L_x = m/n$ we simplify the notation a bit by rewriting (2.7) as

$$\tau_x^{m/n} = \tau(mL_x/n, 0)\,,\ \tau_z^{m/n} = \tau(0, mL_z/n)\,. \tag{2.9}$$

Since $m/n = 1/2$ will loom large in what follows, we omit exponents of $1/2$:

$$\tau_x = \tau_x^{1/2}\,,\ \tau_z = \tau_z^{1/2}\,,\ \tau_{xz} = \tau_x\tau_z\,. \tag{2.10}$$

If a field $\mathbf{u}$ is fixed under a rational shift $\tau(L_x/n)$, it is periodic on the smaller spatial domain $x \in [0, L_x/n]$. For this reason we can exclude from our searches all equilibrium whose isotropy subgroups contain rational translations in favor of equilibria computed on smaller domains. However, as we need to study bifurcations into states with wavelengths longer than the initial state, the linear stability computations need to be carried out in the full $[L_x, 2, L_z]$ cell. For example, if EQ is an equilibrium solution in the $\Omega_{1/3} = [L_x/3, 2, L_z]$ cell, we refer to the same solution repeated thrice in $\Omega = [L_x, 2, L_z]$ as "spanwise-tripled" or $3\times EQ$. Such solution is by construction fixed under the $C_{3,x} = \{e, \tau_x^{1/3}, \tau_x^{2/3}\}$ subgroup.

Third, some isotropy groups are conjugate to each other under symmetries of the full group Γ. Subgroup H' is conjugate to H if there is an $s \in \Gamma$ for which

[5] Predrag: copied this section from ref. [8]

$H' = s^{-1}Hs$. In spatial terms, two conjugate isotropy groups are equivalent to each other under a coordinate transformation. A set of conjugate isotropy groups forms a conjugacy class. It is necessary to consider only a single representative of each conjugacy class; solutions belonging to conjugate isotropy groups can be generated by applying the symmetry operation of the conjugacy.

In the present case conjugacies under spatial translation symmetries are particularly important. Note that $O(2)$ is not an abelian group, since reflections σ and translations τ along the same axis do not commute [13]. Instead we have $\sigma\tau = \tau^{-1}\sigma$. Rewriting this relation as $\sigma\tau^2 = \tau^{-1}\sigma\tau$, we note that

$$\sigma_x\tau_x(\ell_x, 0) \quad = \quad \tau^{-1}(\ell_x/2, 0)\,\sigma_x\,\tau(\ell_x/2, 0)\,. \tag{2.11}$$

The right-hand side of (2.11) is a similarity transformation that translates the origin of coordinate system. For $\ell_x = L_x/2$ we have

$$\tau_x^{-1/4}\,\sigma_x\,\tau_x^{1/4} = \sigma_x\tau_x\,, \tag{2.12}$$

and similarly for the spanwise shifts / reflections. Thus for each isotropy group containing the shift-reflect $\sigma_x\tau_x$ symmetry, there is a simpler conjugate isotropy group in which $\sigma_x\tau_x$ is replaced by σ_x (and similarly for $\sigma_z\tau_z$ and σ_z). We choose as the representative of each conjugacy class the simplest isotropy group, in which all such reductions have been made. However, if an isotropy group contains both σ_x and $\sigma_x\tau_x$, it cannot be simplified this way, since the conjugacy simply interchanges the elements.

Fourth, for $\ell_x = L_x$, we have $\tau_x^{-1}\,\sigma_x\,\tau_x = \sigma_x$, so that, in the special case of half-cell shifts, σ_x and τ_x commute. For the same reason, σ_z and τ_z commute, so the order-16 isotropy subgroup

$$G = D_{1,x} \times C_{2,x} \times D_{1,z} \times C_{2,z} \subset \Gamma \tag{2.13}$$

is abelian.

2.3.3 PCF: The 67-fold path

[6] We now undertake a partial classification of the lattice of isotropy subgroups of plane Couette flow. We focus on isotropy groups involving at most half-cell shifts, with $SO(2)_x \times SO(2)_z$ translations restricted to order 4 subgroup of spanwise-streamwise translations (2.10) of half the cell length,

$$T = C_{2,x} \times C_{2,z} = \{e, \tau_x, \tau_z, \tau_{xz}\}\,. \tag{2.14}$$

All such isotropy subgroups of Γ are contained in the subgroup G (2.13). Within G, we look for the simplest representative of each conjugacy class, as described above.

Let us first enumerate all subgroups $H \subset G$. The subgroups can be of order $|H| = \{1, 2, 4, 8, 16\}$. A subgroup is generated by multiplication of a

[6] Predrag: copied this section from ref. [8]

set of generator elements, with the choice of generator elements unique up to a permutation of subgroup elements. A subgroup of order $|H| = 2$ has only one generator, since every group element is its own inverse. There are 15 non-identity elements in G to choose from, so there are 15 subgroups of order 2. Subgroups of order 4 are generated by multiplication of two group elements. There are 15 choices for the first and 14 choices for the second. However, each order-4 subgroup can be generated by $3 \cdot 2$ different choices of generators. For example, any two of $\tau_x, \tau_z, \tau_{xz}$ in any order generate the same group T. Thus there are $(15 \cdot 14)/(3 \cdot 2) = 35$ subgroups of order 4.

Subgroups of order 8 have three generators. There are 15 choices for the first generator, 14 for the second, and 12 for the third. There are 12 choices for the third generator and not 13, since if it were chosen to be the product of the first two generators, we would get a subgroup of order 4. Each order-8 subgroup can be generated by $7 \cdot 6 \cdot 4$ different choices of three generators, so there are $(15 \cdot 14 \cdot 12)/(7 \cdot 6 \cdot 4) = 15$ subgroups of order 8. In summary: there is the group G itself, of order 16, 15 subgroups of order 8, 35 of order 4, 15 of order 2, and 1 (the identity) of order 1, or 67 subgroups in all [11]. This is whole lot of isotropy subgroups to juggle; fortunately, the observations of sect. 2.3.2 show that there are only 5 *distinct conjugacy classes* in which we can expect to find equilibria.

The 15 order-2 groups fall into 8 distinct conjugacy classes, under conjugacies between $\sigma_x\tau_x$ and σ_x and $\sigma_z\tau_z$ and σ_z. These conjugacy classes are represented by the 8 isotropy groups generated individually by the 8 generators σ_x, σ_z, σ_{xz}, $\sigma_x\tau_z$, $\sigma_z\tau_x$, τ_x, τ_z, and τ_{xz}. Of these, the latter three imply periodicity on smaller domains. Of the remaining five, σ_x and $\sigma_x\tau_z$ allow relative equilibria in z, σ_z and $\sigma_z\tau_x$ allow relative equilibria in x. Only a single conjugacy class, represented by the isotropy group

$$\{e, \sigma_{xz}\}\,, \tag{2.15}$$

breaks both continuous translation symmetries. Thus, of all order-2 isotropy groups, we expect only this group to have equilibria. EQ_9, EQ_{10}, and EQ_{11} described below are examples of equilibria with isotropy group $\{e, \sigma_{xz}\}$.

Of the 35 subgroups of order 4, we need to identify those that contain σ_{xz} and thus support equilibria. We choose as the simplest representative of each conjugacy class the isotropy group in which σ_{xz} appears in isolation. Four isotropy subgroups of order 4 are generated by picking σ_{xz} as the first generator, and σ_z, $\sigma_z\tau_x$, $\sigma_z\tau_z$, or $\sigma_z\tau_{xz}$ as the second generator (R for reflect-rotate):

$$\begin{aligned}
R \;\; &= \{e, \sigma_x, \sigma_z, \sigma_{xz}\} &&= \{e, \sigma_{xz}\} \times \{e, \sigma_z\} \\
R_x &= \{e, \sigma_x\tau_x, \sigma_z\tau_x, \sigma_{xz}\} &&= \{e, \sigma_{xz}\} \times \{e, \sigma_x\tau_x\} \\
R_z &= \{e, \sigma_x\tau_z, \sigma_z\tau_z, \sigma_{xz}\} &&= \{e, \sigma_{xz}\} \times \{e, \sigma_z\tau_z\} \\
R_{xz} &= \{e, \sigma_x\tau_{xz}, \sigma_z\tau_{xz}, \sigma_{xz}\} &&= \{e, \sigma_{xz}\} \times \{e, \sigma_z\tau_{xz}\} \simeq S\,.
\end{aligned} \tag{2.16}$$

These are the only isotropy groups of order 4 containing σ_{xz} and no isolated translation elements. Together with $\{e, \sigma_{xz}\}$, these 5 isotropy subgroups represent the 5 conjugacy classes in which expect to find equilibria.

The R_{xz} isotropy subgroup is particularly important, as the [17] equilibria belong to this conjugacy class [23, 3, 25], as do most of the solutions reported here. The *NBC* isotropy subgroup of ref. [18] and S of ref. [7] are conjugate to R_{xz} under quarter-cell coordinate transformations. In keeping with previous literature, we often represent this conjugacy class with $S = \{e, s_1, s_2, s_3\} = \{e, \sigma_z\tau_x, \sigma_x\tau_{xz}, \sigma_{xz}\tau_z\}$ rather than the simpler conjugate group R_{xz}. Schmiegel's I isotropy group is conjugate to our R_z; ref. [18] contains many examples of R_z-isotropic equilibria. R-isotropic equilibria were found by [21] for plane Couette flow in which the translation symmetries were broken by a streamwise ribbon. We have not searched for R_x-isotropic solutions, and are not aware of any published in the literature.

The remaining subgroups of orders 4 and 8 all involve $\{e, \tau_i\}$ factors and thus involve states that are periodic on half-domains. For example, the isotropy subgroup of EQ_7 and EQ_8 studied below is $S \times \{e, \tau_{xz}\} \simeq R \times \{e, \tau_{xz}\}$, and thus these are doubled states of solutions on half-domains. For the detailed count of all 67 subgroups, see ref. [11].

Exercise 2.1 ***?-path for duct flows*** **PC** *to* **DWS***: this is optional, but it might have customers. Repeat in this section Halcrow* et al. *classification of isotropy groups for the duct flows (square profile, as opposed to the circular profile of the pipe). Reason: number of groups (Nagata, Botero, Kawahara, etc - see Marburg turbulence conference ETC12 proceedings [5], in* `ChaosBook.org/library`*, and my channelflow.org blog) are repeating plane Couette kind of investigations for the duct problem. None of them have worked detailed group theory, or the state space visualizations. For us it is simpler than the pipe, as there is only one, streamwise continuous symmetry, and I believe it is pure* $SO(2)$*, not* $O(2)$.

Chapter 3

Pipe flows

11/16/09 PC transferred halcrow/blog/TEX/pipe.tex to here. [1]

3.1 Pipe flow blog

The main pipe flow research blog is

www.channelflow.org/dokuwiki/doku.php/chaosbook:pipes .

Here we mostly enter LaTeX text that can be used for Spieker thesis or our papers.

09/07/07 Kerswell Yohann Duguet is currently looking for periodic orbits in pipe flow.

10/29/07 PC Ooops - I have to publish this soon... I have a draft of a paper ChaosBook.org/~predrag/papers/preprints.html#trace
that young lions intensely dislike (they say group theory is incomprehensible) so I'll have to rewrite it before submitting it. But the point is:

look for RELATIVE periodic orbits in the pipe flow, relative both in the shift down the pipe and rotation around the pipe axis. [2]

Likelihood of finding a periodic orbit is *ZERO*, methinks. One expects some only if in addition to a continuous symmetry one has a discrete symmetry which is not a subgroup of the continuous symmetry. I believe you do not have any such for pipe flow. [3] Duct flow would have them.

However, Kuramoto-Sivashinsky and plane Couette flow do have discrete symmetries in addition - that is why there are some equilibria (as opposed

[1] `spieker/blog/TEX/pipe.tex`, rev. 47: last edit by Predrag Cvitanović, 01/05/2010

[2] Predrag: 11/15/09 I was wrong. Because of the azimuthal reflection symmetry, they are always traveling downpipe, be either fixed (antisymmetric subspace under reflection) or traveling azimuthally, in which case they come in clockwise / anti-clockwise pairs.

[3] Predrag: 11/15/09 I was wrong, there is an obvious azimuthal reflection symmetry. Still, because of downstream SOn2, no equilibria, no periodic orbits.

to relative equilibria) and some periodic orbits in these cases. They belong to discrete symmetry subspaces. Atypical as they are (no turbulent solution will be confined to these subspaces) they are important for periodic orbit theory, as there the shortest orbits dominate.

Divakar has plane Couette flow (relative) periodic orbits under control, though that is only a start - we do not understand the topology of the flow at all yet.

I have not looked at the isotropic turbulence orbits paper (van Veen et al 2006) in detail as I do not think there is any interesting physics in such models; if they find any equilibrium, periodic orbit solutions, they should be in the discrete symmetry reduced subspaces.

The deepest thinker on relative periodic orbits is in your neck of the woods, Stephen Creagh in Nottingham. Thomas Bartsch at Laughborough is also very good.

Please bug me about confusing things in the draft, the idea is very simple and I have to write it clearly so people do not get discouraged by group theory jargon.

09/07/07 Kerswell I can see (very roughly) how the cycle expansion becomes one over relative periodic orbits but I don't see how this precludes (true) periodic orbits. I can see that relative periodic orbits should be more generic but this doesn't prevent (true) periodic orbits existing, does it? In fact we already know of lots of periodic orbits in pipe flow (the relative equilibriums) albeit of special type. I must admit that it isn't the continuous rotational or (axial) translational symmetry that made me worry about the existence of periodic orbits but rather the lack of mirror symmetry (there is net advection along the pipe).

I can sympathize with your young lions - the group theory analysis occurs at breakneck speed! How about developing a simple example in tandem with the general analysis in your preprint? I have struggled with all the notation and missed steps (as, indeed, has Carl D who kept smiling as he read this manuscript....)

11/16/09 PC Relative equilibria are *not* periodic orbits, they are group orbits, morally the same as equilibria.

10/07/07 DV (Divakar's relative periodic orbits symmetries) Connecting the spectrum to correlation decay is a project that would be doable and also quite interesting. I will be glad to follow up for Couette/pipe flow. This could be a deeper way to look at transition thresholds than to just look at the lower branch states or edge states.

3.2 Quotienting pipe flows

06/14/07 PC Yohann understood immediately why and how to quotient (pipe flow)/$\mathrm{O}(2) \times \mathrm{SO}(2)$ so there is a hope that someone will do it Predragian way. For reasons that puzzle me the Atlanta Local has not been able to quotient ZM this way (though reformulation in terms of invariant polynomial does work, albeit on a hypersurface defined by a conserved syzygy - maybe that is the right way to do it).

09/12/09 PC Yohann implemented my suggestion. When he ran it, he got a growth in in numerical errors and gave up. I guess that's the difference between a physicist and an applied mathematician; when things do not work a physicist is in heaven, and starts working really hard. A sensible person gives up, goes does something else that is known to work. But I digress. Unfortunately I made a wrong mistake, and the totally sensible method I suggested to him is what Rowlwy and Marsden call the 'method of connections.' We know that it does not reduce periodic orbits to relative periodic orbits - it accrues a 'geometrical phase' that we all understand for the Foucault pendulum, but I do not understand what that is good for in pipes and planes.

Still, I do not understand why he got a growth of numerical errors - unlike the 'method of slices,' the 'method of connections' should not run into any artificial singularities. And it certainly should have worked for the relative equilibria, there is no geometrical phase there.

3.3 Symmetries of pipe flow solutions

11/16/09 PC to Dustin: repeat in this section the Halcrow *et al.* classification of isotropy groups [8], but this time for the pipe flow. It is easier than the plane Couette flow, and is needed for the forthcoming Gibsonization of pipe state space visualizations. This you need to do as part of your thesis work - to be sure that the symmetries are under control. [4]

11/21/09 PC With all due respect to the established tradition, the registry of symmetries of Gibson, Halcrow and Cvitanović [8] seems more rational to me then the various shift-reflects, so I think we may be so bold as to use a similar (much simpler) list for pipes, as long as we clearly describe the conjugacies that map solutions in the literature to our classification. Our justification would be that we view pipes and planes as related problems.

Example 3.1 ***Symmetries of pipe flow solutions.*** *The Navier-Stokes equations for pipe flow are formulated in the cylindrical-polar coordinates, where* (s, ϕ, z) *are the radius, azimuthal angle and the stream-wise (axial) positions, respectively. The fluid velocity field* $\mathbf{u}$ *is represented by* $[u, v, w, p](s, \phi, z)$, *with* u, v *and* w *respectively*

[4] Predrag: The formulation of the exercise is a bit confusing, edit it as you see fit.

the radial, azimuthal and stream-wise velocity components, and p the pressure. A pressure gradient drives the flow in the stream-wise, increasing z direction. In numerical simulations the infinite pipe is represented by the L-cell with periodic boundary condition in the streamwise direction.

Denote group actions by $\tau(\theta, 0)$ for an azimuthal rotation by angle θ about the symmetry axis of the pipe, σ for a reflection about the $\phi = 0$ azimuthal angle, and $\tau(0, \ell)$ for a streamwise translation by ℓ:

$$\begin{aligned} \tau(\theta,\ell)\,[u,v,w,p](s,\phi,z) &= [u,v,w,p](s,\phi+\theta,z+\ell) \\ \sigma\,[u,v,w,p](s,\phi,z) &= [u,-v,w,p](s,-\phi,z) \end{aligned} \tag{3.1}$$

The equations for pipe flow are equivariant under azimuthal rotations, azimuthal reflection (reversal of the direction of azimuthal rotation), and streamwise translations. The symmetry group of pipe flow is thus $\Gamma = O(2)_\phi \times SO(2)_z = D_1 \ltimes SO(2)_\phi \times SO(2)_z$, where $\ltimes$ stands for a semi-direct product, z subscript indicates streamwise translation, ϕ subscript indicates azimuthal rotation, and D_1 azimuthal reflection. While the flow equations are invariant under Γ, the solutions are typically not. At the two extremes, the symmetry of a generic turbulent state is the trivial symmetry group $\{e\}$, whereas the Hagen parabolic profile is invariant under all of Γ.

Because of the continuous translational and rotational symmetries, we expect to find relative equilibrium and relative periodic orbits that propagate in both the streamwise direction (z, or axial traveling waves), and about the symmetry axis of the pipe (ϕ, or rotational waves). The discrete reflection symmetry about $\phi = 0$ can be used to prohibit relative equilibria and relative periodic orbits from circling the symmetry axis of the pipe, hence we also expect streamwise relative periodic orbits and relative equilibria which with zero azimuthal velocity.

In the special case of half-cell shifts, σ and τ_ϕ commute, so the order-8 isotropy subgroup

$$G = D_1 \times C_{2,\phi} \times C_{2,z} \subset \Gamma \tag{3.2}$$

is abelian.

Three types of solutions are invariant under stream-wise, azimuthal 1/2-shifts

$$\{e, \tau_\phi\}\,, \{e, \tau_z\}\,, \{e, \tau_{z\phi}\}\,, \tag{3.3}$$

where the solutions are invariant under half-shifts τ_z, τ_ϕ, defined in (2.10).

The periodicity in the azimuthal direction imposes discrete rotational symmetries $\tau(2\pi/m, 0)$ on velocity fields. Not only can solutions have the trivial discrete symmetry of rotation by 2π, but solutions can also be invariant under rotations of integer divisors of 2π. For example, a solution is said to be invariant under $C_{\phi,5}$ if the solution is invariant under rotations of $2\pi/5$.

While solutions that are invariant under $C_{2,\phi}$ are numerous, solutions have been found to be invariant all the way up to $C_{6,\phi}$ [26], where

$$C_{n,\phi} = \{e, \tau_\phi^{1/n}, \tau_\phi^{2/n}, \cdots, \tau_\phi^{1-1/n}\} \subset SO(2)_\phi\,.$$

Due to existence of such solutions, symmetry considerations, though more general, must be made to classify the isotropy subgroups of more highly symmetric solutions. For solutions with invariant under the action of m-fold azimuthal rotational *symmetry, $C_{m,\phi}$, we can form two n-order invariant subgroups:*

$$
\begin{aligned}
R_\phi^m &= \{e, \sigma\tau(2\pi/m, 0)), \sigma\tau(4\pi/m, 0)), \cdots, \sigma\tau((m-1)2\pi/m, 0)\} \\
R_{z\phi}^m &= \{e, \sigma\tau(2\pi/m, 1/2), \sigma\tau_(4\pi/m, 1/2), \cdots, \sigma\tau((m-1)2\pi/m, 1/2)\}
\end{aligned}
\qquad (3.4)
$$

One important feature of property of solutions invariant under $1/m$ shifts $\in C_m$ is that we do not expect relative periodic orbits to be invariant under $1/m$ shifts. Relative periodic orbits are characterized by $\mathbf{u}^T(\mathbf{x}+\mathbf{c}t) = \mathbf{u}$. For a relative periodic orbit therefore, for all t, we don't expect the typical puffs and slugs of pipe flow to change length scales and therefore not obey

$$
\tau^{1/m}\mathbf{u}^T(\mathbf{x}+\mathbf{c}t) = \mathbf{u} \; \forall \; t \qquad (3.5)
$$

This leaves the four conjugacy-inequivalent types of possible symmetries of pipe flow solutions:

$$
\begin{aligned}
R &= \{e, \sigma\} \\
R_z &= \{e, \sigma\tau_z\} \\
R_\phi &= \{e, \sigma\tau_\phi\} \\
R_{z\phi} &= \{e, \sigma\tau_{z\phi}\}
\end{aligned}
\qquad (3.6)
$$

Conjugation can eliminate two of these isotropy subgroups. Firstly, it can be easily shown that

$$
\sigma\tau_\phi(\theta, 0) \quad = \quad \tau_\phi^{-1}(\theta/2, 0)\, \sigma\, \tau_\phi(\theta/2, 0)\,. \qquad (3.7)
$$

The right hand side of the equation is a similarity transformation that simply moves the origin of our coordinate system. Therefore, any isotropy subgroup that contains $\sigma\tau_\phi$ can be replaced by one that simply contains σ. If an isotropy subgroup contains both $\sigma\tau_\phi$ and σ, conjugation would not simplify the subgroup as it would only switch the two elements. In our half-cell rotation case, $\tau(\pi, 0) = \tau_\phi$, we have

$$
\sigma\tau_\phi = \tau_{\frac{\phi}{2}}^{-1}\sigma\tau_{\frac{\phi}{2}} \qquad (3.8)
$$

herefore, the isotropy subgroup R_ϕ $\{e, \sigma\tau_\phi\}$ is conjugate to the simpler isotropy subgroup R $= \{e, \sigma\}$. Similarly,

$$
\sigma\tau(\theta, \ell_z) \quad = \quad \tau^{-1}(\theta/2, 0)\, \sigma\, \tau(0, \ell_z)\, \tau(\theta/2, 0)\,. \qquad (3.9)
$$

In the specific case of half-cell rotations and translations, we have

$$
\sigma\tau_{z\phi} \quad = \quad \tau_{\frac{\phi}{2}}^{-1}\, \sigma\, \tau_z\, \tau_{\frac{\phi}{2}} \qquad (3.10)
$$

and it follows that isotropy subgroup $R_{z\phi} = \{e, \sigma\tau_{z\phi}\}$ is conjugate to the simpler R_z $= \{e, \sigma\tau_z\}$. We therefore have only two second order isotropy subgroups to consider: R_z and R.

(D.W. Spieker and P. Cvitanović)

Remark 3.1 Pipe flow. Example 3.1 follows Wedin and Kerswell [26], who have found solutions with radial symmetries all the way up to $C_{6,\phi}$.

Pringle *et al.* [1] define following symmetries:

$$\begin{aligned}
\mathbf{R}_m &: [u,v,w,p](s,\phi,z) \to [u,v,w,p](s,\phi+2\pi/m,z) \\
\mathbf{S} &: [u,v,w,p](s,\phi,z) \to [u,-v,w,p](s,-\phi,z+\pi/\alpha) \\
\mathbf{\Omega_m} &: [u,v,w,p](s,\phi,z) \to [u,v,w,p](s,\phi+\pi/m,z+\pi/\alpha) \\
\mathbf{Z}_{j\pi/2m} &: [u,v,w,p](s,\phi,z) \to [u,-v,w,p](s,-\phi+,z)
\end{aligned} \tag{3.11}$$

$\mathbf{R}_m$ is the same as our $\tau_\phi^{1/m}$. $\mathbf{S}$ is the same as $\sigma\,\tau_z$. $\mathbf{\Omega_m}$ is the same as our $\tau_\phi^{1/m}\,\tau_z$. [5]

Kerswell & Tutty JFM 2006 In this paper [15], I think they very succinctly enumerate only solutions invariant under the action of $\mathcal{S}$: [6]

Pipe flows that are invariant under the 'rotate-and-reflect' symmetry of Wedin and Kerswell [26], defined in ref. [22] by:

$$[u,v,w,p](s,\phi,z) \to [u,-v,w,p](s,-\phi,z+\pi\Lambda)\,. \tag{3.12}$$

This symmetry (here the R_z symmetry) reflects the velocity field about the plane $\phi=0$ (or $\phi=\pi$) and shifts it by half the pipe length.

the action of the shift and rotate symmetry

$$\mathcal{S}:[u,v,w,p](s,\phi,z) \to [u,-v,w,p](s,-\phi,z+\pi/\alpha) \tag{3.13}$$

$\mathcal{S}$ is the same as our $\sigma\,\tau_z$.

Viswanath and Cvitanović [22] denote the length of the periodic domain in the z direction by $2\pi\Lambda$.

(D.W. Spieker and P. Cvitanović)

Exercise 3.1 ***Symmetries of pipe flow solutions.***

(a) Describe pipe flow, define the appropriate coordinates and velocity fields.

(b) Describe the symmetry group Γ of pipe flow. What discrete subgroups are physically important at low Re numbers? [7]

(c) What kinds of relative equilibrium (traveling wave) and relative periodic orbit (modulated traveling wave) solutions do you expect?

[5] Dustin: I'm a little confused by the relation 2.5 in [1], and I don't think it's consistent with 2.4 in the same paper. Based on my calculation, $\mathbf{Z}_{j\pi/2m} \neq \mathbf{Z}_\phi$ in the case of $j=2$, unless the two symmetries are not meant to be the same thing.

[6] Dustin: I think we should adopt using wavenumbers in our symmetry notation, based on these two papers

[7] Predrag: to extent possible, look how things are defined in sect. 2.3.1, and use the same notation and macros. That will save time later

(d) *Do you expect to find any equilibrium and periodic orbit solutions? What symmetry is prerequisite to existence of equilibrium and periodic orbit solutions? What kind of streamwise solutions do you expect?*

(e) *Explain why 1/2, 1/3 azimuthal and streamwise (discrete cyclic C_m subgroups of $SO(2)$) shifts play a special role for equilibrium and periodic orbit solutions.* [8] *Describe the m-fold 'rotationally symmetric solutions $\mathcal{R}_m$' of Wedin and Kerswell [26].*

(f) *Interpret solutions invariant under $1/m$ shifts $\in C_m$.* [9]

(g) *Enumerate possible symmetries of solutions, as is done for the plane Couette flow in ref. [8], see sect. 2.3.1. Describe their relation to the 'rotate-and-reflect' symmetries of Wedin and Kerswell [26], defined in ref. [22] by:*

$$[u, v, w, p](s, \phi, z) \to [u, -v, w, p](s, -\phi, z + \pi\Lambda)\,. \tag{3.14}$$

This symmetry reflects the velocity field about the plane $\phi = 0$ (or $\phi = \pi$) and shifts it by half the pipe length.
Does (2.11) eliminate any of (3.19) groups by conjugation?

(h) *So far we have considered only azimuthal $m = 2$ case (half-rotations). Complete the classification for any m. $m > 2$ are already observed in pipes at low Re.*

Solution 3.1 - Symmetries of pipe flow solutions.

(a) *We formulate Navier-Stokes equations for pipe flow in the usual cylindrical polar coordinate system (s, ϕ, z), following Wedin and Kerswell [26]. The axial (stream-wise) direction is z, and u, v are radial and azimuthal velocity components respectively. Any velocity field, $\mathbf{u}$ can therefore be represented by $[u, v, w, p](s, \phi, z)$. p is a quantity specified in the system, because, unlike in plane Couette flow, where streamwise shearing of the flow at the upper and lower walls drives the flow, a pressure gradient in pipe flow drives the flow. Viswanath and Cvitanović [22] denote the length of the periodic domain in the z direction by $2\pi\Lambda$.*

(b) *The pipe flow is invariant under azimuthal rotations, azimuthal reflection (reversal of the direction of azimuthal rotation), and streamwise translations. We denote group actions by $\tau(\theta, 0)$ for an azimuthal rotation by angle θ about the symmetry axis of the pipe, σ for a reflection about the $\phi = 0$ azimuthal angle, and $\tau(0, \ell)$ for a streamwise translation by ℓ, (in numerical simulations replaced by the L-cell periodic boundary condition in the streamwise direction):* [10]

$$\begin{aligned} \tau(\theta, \ell)\,[u, v, w, p](s, \phi, z) &= [u, v, w, p](s, \phi + \theta, z + \ell) \\ \sigma\,[u, v, w, p](s, \phi, z) &= [u, -v, w, p](s, -\phi, z) \end{aligned} \tag{3.15}$$

[8] Predrag: These arguments are perhaps already in sect. 2.3.1. Make them your own, also edit the plane Couette flow text in a way that makes sense to you.

[9] Dustin: " I firstly want to consider the order two isotropy subgroups. In order to form order two subgroups, one needs to find symmetries that are their own inverses. σ already satisfies this property, as do τ_ϕ and $\tau_{\frac{\ell_z}{2}}$." **PC:** Need a physical reason to study only 1/2-shift for such low Re, discuss it.

[10] Predrag: Replacing the azimuthal rotation $\mathcal{R}_\theta$ by $\tau(\theta, 0)$, $\mathcal{R}_\pi$ by τ_ϕ, is probably not an improvement, recheck other people's choices.

[11] *The equations of pipe flow are thus equivariant under the group $\Gamma = O(2)_\phi \times SO(2)_z = D_2 \ltimes SO(2)_\phi \times SO(2)_z$, where $\ltimes$ stands for a semi-direct product, z subscript indicates streamwise translation, and ϕ subscript indicates azimuthal rotation. It is worth noting that while the equations are invariant under Γ, velocity fields are not. A typical turbulent flow will only belong to the trivial symmetry group $\{e\}$ whereas the trivial solution of the laminar state in pipe flow would be invariant under all of Γ.*

As explained in sect. 2.3.2, in the special case of half-cell shifts, σ and τ_ϕ commute, so the order-8 isotropy subgroup

$$G = D_{1,\phi} \times C_{2,\phi} \times C_{2,z} \subset \Gamma \tag{3.16}$$

is abelian.

(c) *Because of the continuous translational and rotational symmetries of pipe flow, we expect to find relative equilibrium and relative periodic orbits that propagate in both the streamwise (z) direction and rotationally about the symmetry axis of the pipe (ϕ).*

(d) *We expect equilibria and periodic orbits in isotropy subgroups where translational phases have been fixed by invariance under a change of sign. In pipe flow, the discrete reflection symmetry about $\phi = 0$ which would prohibit relative equilibria and relative periodic orbits from circling the symmetry axis of the pipe. No such phase-fixing reflective symmetry is present in the streamwise direction, meaning that periodic orbits and equilibria, in all likelihood, will not be present in pipe flow. Therefore, any discussion, hence forth, about solutions will be referring to relative equilibrium and relative periodic orbits.*

(e)

(f) *Three types of solutions are invariant under stream-wise, azimuthal 1/2-shifts*

$$\{e, \tau_\phi\}, \{e, \tau_z\}, \{e, \tau_{z\phi}\}, \tag{3.17}$$

where the solutions invariant under half-shifts τ_z, τ_ϕ, defined in (2.10), represent ??? of solutions???. Periodic boundary conditions prescribed in the azimuthal direction impose discrete rotational symmetries $\tau(2\pi/m, 0)$ on velocity fields. Not only can solutions have the trivial discrete symmetry of rotation by 2π, but solutions can also be invariant under rotations of integer divisions of 2π. For example, a solution is said to be invariant under $C_{\phi,5}$ if the solution is invariant under rotations of $2\pi/5$.

One important feature of property of solutions invariant under $1/m$ shifts $\in C_m$ is that we do not expect relative periodic orbits to be invariant under $1/m$ shifts. Relative periodic orbits are characterized by $\mathbf{u}^T(\mathbf{x}+\mathbf{c}t) = \mathbf{u}$. For a relative periodic orbit therefore, for all t, we don't expect the typical puffs and slugs of pipe flow to change length scales and therefore not obey

$$\tau^{1/m}\mathbf{u}^T(\mathbf{x}+\mathbf{c}t) = \mathbf{u} \;\forall\; t \tag{3.18}$$

[11] Dustin: " Akin to the analysis of ref. [8], we seek isotropy subgroups to narrow our search for invariant objects in the pipe flow system." **PC:** For numerical reasons yes, but it is deeper than that; in the reduced state space only symmetry-quotiented solutions contribute.

(g) *This leaves the four conjugacy-inequivalent types of possible symmetries of pipe flow solutions:* [12]

$$
\begin{aligned}
R &= \{e, \sigma\} \\
R_z &= \{e, \sigma\tau_z\} \\
R_\phi &= \{e, \sigma\tau_\phi\} \\
R_{z\phi} &= \{e, \sigma\tau_{z\phi}\}
\end{aligned}
\tag{3.19}
$$

Conjugation can eliminate two of these isotropy subgroups. Firstly, it can be easily shown that

$$
\sigma\tau_\phi(\theta, 0) \quad = \quad \tau_\phi^{-1}(\theta/2, 0)\, \sigma\, \tau_\phi(\theta/2, 0)\,. \tag{3.20}
$$

The right hand side of the equation is a similarity transformation that simply moves the origin of our coordinate system. Therefore, any isotropy subgroup that contains $\sigma\tau_\phi$ can be replaced by one that simply contains σ. If an isotropy subgroup contains both $\sigma\tau_\phi$ and σ, conjugation would not simplify the subgroup as it would only switch the two elements. In our half-cell rotation case, $\tau(\pi, 0) = \tau_\phi$, we have

$$
\sigma\tau_\phi = \tau_{\frac{\phi}{2}}^{-1} \sigma \tau_{\frac{\phi}{2}} \tag{3.21}
$$

herefore, the isotropy subgroup R_ϕ $\{e, \sigma\tau_\phi\}$ is conjugate to the simpler isotropy subgroup $R\ = \{e, \sigma\}$. Similarly,

$$
\sigma\tau(\theta, \ell_z) \quad = \quad \tau^{-1}(\theta/2, 0)\, \sigma\, \tau(0, \ell_z)\, \tau(\theta/2, 0)\,. \tag{3.22}
$$

In the specific case of half-cell rotations and translations, we have

$$
\sigma\tau_{z\phi} \quad = \quad \tau_{\frac{\phi}{2}}^{-1}\, \sigma\, \tau_z\, \tau_{\frac{\phi}{2}} \tag{3.23}
$$

and it follows that isotropy subgroup $R_{z\phi} = \{e, \sigma\tau_{z\phi}\}$ is conjugate to the simpler $R_z\ = \{e, \sigma\tau_z\}$. We therefore have only two second order isotropy subgroups to consider: R_z and R. Pipe flows that are invariant under the action of the shift and rotate symmetry (??) given by Wedin and Kerswell [26] are equivalently invariant under the R_z isotropy subgroup.

(h) *While solutions that are invariant under $C_{2,\phi}$ are numerous, solutions have been found to be invariant all the way up to $C_{6,\phi}$ [26], where*

$$
C_{n,\phi} = \{e, \tau_\phi^{1/n}, \tau_\phi^{2/n}, \cdots, \tau_\phi^{1-1/n}\} \subset SO(2)_\phi\,.
$$

Due to existence of such solutions, symmetry considerations, though more general, must be made to classify the isotropy subgroups of more highly symmetric solutions. For solutions with invariant under the action of n-fold azimuthal rotational symmetry, $C_{n,\phi}$, we can form two n-order invariant subgroups:

$$
\begin{aligned}
R_\phi^n &= \{e, \sigma\tau(2\pi/n, 0)), \sigma\tau(4\pi/n, 0)), \cdots, \sigma\tau((n-1)\pi/n, 0)\} \quad (3.24) \\
R_{z\phi}^n &= \{e, \sigma\tau(2\pi/n, 1/2), \sigma\tau_(4\pi/n, 1/2), \cdots, \sigma\tau((n-1)\pi, 1/2)\} \quad (3.25)
\end{aligned}
$$

(D.W. Spieker and P. Cvitanović)

[12] Dustin: 'Seven distinct second order subgroups can be formed from these three symmetries." **PC:** I find only four. Try to find sensible names for them.

3.4 Tracking symmetry notation

Relating pipe symmetries in literature to our minimalist description

Pringle et al. 2009 The main symmetries presented in this paper [1] are

$$\begin{aligned}
\mathbf{R}_m &: [u,v,w,p](s,\phi,z) \to [u,v,w,p](s,\phi+2\pi/m,z) && (3.26)\\
\mathbf{S} &: [u,v,w,p](s,\phi,z) \to [u,-v,w,p](s,-\phi,z+\pi/\alpha) && (3.27)\\
\mathbf{\Omega_m} &: [u,v,w,p](s,\phi,z) \to [u,v,w,p](s,\phi+\pi/m,z+\pi/\alpha) && (3.28)\\
\mathbf{Z}_{j\pi/2m} &: [u,v,w,p](s,\phi,z) \to [u,-v,w,p](s,-\phi+,z) && (3.29)
\end{aligned}$$

$\mathbf{R}_m$ is the same as our $\tau_\phi^{1/m}$. $\mathbf{S}$ is the same as $\sigma\,\tau_z$. $\mathbf{\Omega_m}$ is the same as our $\tau_\phi^{1/m}\,\tau_z$. [13]

Kerswell & Tutty JFM 2006 In this paper [15], I think they very succinctly enumerate only solutions invariant under the action of $\mathcal{S}$: [14]

$$\mathcal{S} : [u,v,w,p](s,\phi,z) \to [u,-v,w,p](s,-\phi,z+\pi/\alpha) \qquad (3.30)$$

$\mathcal{S}$ would be the same as our $\sigma\,\tau_z$.

[13] Dustin: I'm a little confused by the relation 2.5 in [1], and I don't think it's consistent with 2.4 in the same paper. Based on my calculation, $\mathbf{Z}_{j\pi/2m} \neq \mathbf{Z}_\phi$ in the case of $j=2$, unless the two symmetries are not meant to be the same thing.

[14] Dustin: I think we should adopt using wavenumbers in our symmetry notation, based on these two papers

Chapter 4

Duct flows

4.1 Duct flow symmetries

We formulate the incompressible Navier-Stokes equations in a rectangular duct using cartesian position and velocity coordinates because of the intrensic geometry of a rectangular duct. Any velocity field, $\mathbf{u}$, can therefore be represented by $[u, v, w, p](x, y, z)$, where u, v, w represent velocity components in the streamwise (x), cross-sectional vertical (y), and spanwise (y) directions respectively. The p term represents the pressure exerted in the duct, as the gradient of the pressure is what drives the flow.

For numerical simulations, we impose periodic boundary conditions in the streamwise direction, so for a duct of length L, $[u, v, w, p](x, y, z) = [u, v, w, p](x, y, z + L)$. This periodic boundary condition on the velocity field leads to invariance of flows under continuous translations in the streamwise direction of the velocity field. Along with this continous, streamwise translational symmetry, we expect invariant structures in the Duct Flow phase space to have reflection symmetry about vertical and horizontal symmetry axes of the duct. The full symmetry group of duct flow is therefore given by $\Gamma = \mathrm{D}_{2,y} \times \mathrm{D}_{2,z} \times SO(2)_x$.[1] We represent the action of these symmetries with the following notation

$$
\begin{aligned}
\sigma_y : [u, v, w, p](x, y, z) &\rightarrow [u, -v, w, p](x, -y, z) && (4.1)\\
\sigma_z : [u, v, w, p](x, y, z) &\rightarrow [u, v, -w, p](x, y, -z) && (4.2)\\
\sigma_{yz} : [u, v, w, p](x, y, z) &\rightarrow [u, -v, -w, p](x, -y, -z) && (4.3)\\
\tau_x^{l_x} : [u, v, w, p](x, y, z) &\rightarrow [u, v, w, p](x + l_x, y, z) && (4.4)
\end{aligned}
$$

[2]

[1] Dustin: If it isn't obvious, I don't know the exact notation for these kind of arguments and I tried to extract it from previous examples. References/Help would be appreciated.

[2] Dustin: It seems like there should be invariance under $\frac{\pi}{2}$ rotations, not just rotation by π with σ_{yz}. Body forces like gravity would break the rotational symmetry, but can anyone refute/verify this? It seems like we should two seperate group theory decompositions for square and just rectangular ducts.

Take note that all symmetry operations commute with each other. Because of the continuous translational symmetry in the streamwise direction of the duct, we expect and have already seen traveling wave (relative equilibria) and modulated traveling wave (relative periodic solutions) to propogate in the streamwise direction. We do not, however, expect equilibria or periodic orbits for lack of a translational phase fixing reflection symmetry in the streamwise direction.

4.1.1 Isotropy Subgroups of Duct Flow

We expect there to be traveling wave solutions that are invariant under the group action of a number of isotropy subgroups. Enumerated below are the conjugacy inequivalent, order two isotropy subgroups:

$$\begin{aligned} R_y &= \{e, \sigma_y\} && (4.5)\\ R_z &= \{e, \sigma_z\} && (4.6)\\ R_{yz} &= \{e, \sigma_{yz}\} && (4.7)\\ S_y &= \{e, \sigma_y \tau_x\} && (4.8)\\ S_z &= \{e, \sigma_z \tau_x\} && (4.9)\\ S_{yz} &= \{e, \sigma_{yz} \tau_x\} && (4.10) \end{aligned}$$

Here, τ_x, is shorthand for $\tau_x^{l_x/2}$, where l_x is periodic length of the duct. I will predominately consider discrete streamwise shifts of half box lengths because symmetry under the action of discrete streamwise translational symmetry means that the velocity field simply tiles the streamwise domain. For example, if a solution is invariant under the group action of $C_x^{l_x/3} = \{e, \tau_x^{l_x/3}, \tau_x^{2l_x/3}\}$, that means the velocity field is is simply repeat of itself every third of a box length. We do consider $\tau^{l_x/2}$ because we are interested in bifurcations of solutions as functions of streamwise wave number $\alpha = 2\pi/l_x$.

If one is searching for traveling wave solutions and not relative periodic orbits, one should focus on looking for solutions to Navier-Stokes that are invariant under S-type isotropy groups. If one is searching for relative periodic orbits and not traveling wave solutions, one should focus on looking for solutions invariant under R-type isotropy groups. Having classified the kind of isotropy subgroups where relative equilibria and relative periodic orbits would be present, we can now enumerate the order-4 isotropy subgroups. For relative periodic orbits, I expect only one order-4 isotropy subgroup.

$$I = \{e, \sigma_y\} \times \{e, \sigma_{yz}\} = \{e, \sigma_y, \sigma_z, \sigma_{yz}\} \qquad (4.11)$$

Chapter 5

Tables

[1]

5.1 Solution properties

Fluid states are characterized by their energy $E = \frac{1}{2}\|\mathbf{u}\|^2$ and energy dissipation rate $D = \|\nabla \times \mathbf{u}\|^2$, defined in terms of the inner product and norm [2]

$$(\mathbf{u}, \mathbf{v}) = \frac{1}{V}\int_\Omega d\mathbf{x}\,\mathbf{u}\cdot\mathbf{v}\,, \quad \|\mathbf{u}\|^2 = (\mathbf{u}, \mathbf{u})\,. \tag{5.1}$$

The rate of energy input is $I = 1/(L_x L_z)\int\int dxdz\,\partial u/\partial y$, where the integral is taken over the upper and lower walls at $y = \pm 1$. Normalization of these quantities is set so that $I = D = 1$ for laminar flow and $\dot{E} = I - D$. It is often convenient to consider fields as differences from the laminar flow, since these differences constitute a vector space, and thus can be added together, multiplied by scalars, etc. We indicate such differences with tildes: $\tilde{\mathbf{u}} = \mathbf{u} - y\hat{\mathbf{x}}$. Note that the total velocity field $\mathbf{u}$ does *not* form a vector space: the sum of any two total plane Couette velocity fields violates the $u = \pm 1$ boundary conditions at the moving walls.

Most of this study is conducted at $Re = 400$ in one of the two small aspect-ratio cells [3]

$$\begin{aligned}
\Omega_{\text{W03}} &= [2\pi/1.14, 2, 2\pi/2.5] &&\approx [5.51, 2, 2.51] \approx [190, 68, 86] \text{ wall units}\\
\Omega_{\text{HKW}} &= [2\pi/1.14, 2, 2\pi/1.67] &&\approx [5.51, 2, 3.76] \approx [190, 68, 128] \text{ wall units,}
\end{aligned} \tag{5.2}$$

[1] `spieker/tables.tex`, rev. 24: last edit by Predrag Cvitanović, 11/17/2009

[2] Predrag: copied this paragraph from ref. [8]

[3] Predrag: copied this paragraph from ref. [8]. Please update with the more logical choice of canonical cell defined by Gibson (Chanellflow.org wiki 2009), which I understand is the cell size you will work in.

Table 5.1: Properties of known equilibria of plane Couette flow. $Re = 400$ except for E_6 which has $Re = 330$. Data on equilibria E_0 to E_{13} taken from ref. [8] (last updated by DWS on 10/30/09).

	$\|\|\cdot\|\|$	D	H	E	$\sum\lambda_+$	K-Y
E_0	0	1	Γ	0.1667	0	
E_1	0.209125	1.42926	S	0.1363	0.050123	
E_2	0.385806	3.04367	S	0.0780	0.13007	
E_3	0.125884	1.31768	S	0.1382	0.051762	
E_4	0.168116	1.45368	S	0.1243	0.10849	
E_5	0.218648	2.02013	S	0.1073	0.24575	
E_6	0.275125	2.81845	S	0.0972	-	
E_7	0.093586	1.25225	$S \times \{e, \tau_{xz}\}$	0.1469	0.080364	
E_8	0.346590	1.76977	$S \times \{e, \tau_{xz}\}$	0.1204	-	
E_9	0.156523	1.40475	$\{e, \sigma_{xz}\}$	0.1290	-	
E_{10}	0.328542	2.37207	$\{e, \sigma_{xz}\}$	0.1080	-	
E_{11}	0.404869	3.43223	$\{e, \sigma_{xz}\}$	0.0803	-	
E_{12}	0.303660	2.07134	$\{e, \sigma_{xz}\}$	0.1159	-	
E_{13}	0.404894	3.36118	$\{e, \sigma_{xz}\}$	0.0813	-	
E_{14}	0.240510	1.60344	$\{e, \sigma_{xz}\}$	0.1289	0.033852	
E_{15}	0.268269	1.76297	$\{e, \sigma_{xz}\}$	0.1242	0.084577	
E_{16}	0.333003	3.49153	R	0.0808	0.49640	
E_{17}	0.291661	2.97947	R	0.0884	0.70251	
E_{18}	0.325111	3.65963	R	0.0882	1.0039	
E_{19}	0.427491	5.54202	Rz	0.0658	1.1759	

Table 5.2: The first 29 least stable Floquet exponents, $\lambda = \mu \pm i\,\omega$ of equilibrium E_{16} for Ω_{W03} cell plane Couette flow, $Re = 400$, together with the symmetries of corresponding eigenvectors. The exponents are ordered by the decreasing real part.

j	$\mu^{(j)}_{EQ5}$	$\omega^{(j)}_{EQ5}$	$s_1 s_2 s_3$
1,2	0.08914764	0.1856416	S S S
3,4	0.07640964	0.2884856	S A A
5,6	0.06740066	0.2710526	A S A
7	0.05862082		S S S
8,9	0.03653549	0.06517621	A A S
10,11	0.03443270	0.1016705	S A A
12,13	0.03344769	0.1896621	- - A
14,15	0.02490208	0.1269611	A A S
16,17	0.02313113	0.1920084	A S A
18,19	0.02046181	0.005704649	A A S
20,21	0.01809113	0.1956752	S A A
22,23	0.01018023	0.1554619	S A A
24,25	0.003639039	0.3038429	A S A
26	5.362729e-05		S A A
27	-6.193149e-07		S A A
28,29	-0.006766147	0.08891588	

where the wall units are in relation to a mean shear rate of $\langle \partial u/\partial y \rangle = 2.9$ in non-dimensionalized units computed for a large aspect-ratio simulation at $Re = 400$. Empirically, at this Reynolds number the Ω_{HKW} cell sustains turbulence for very long times [12], whereas the Ω_{W03} cell exhibits only short-lived transient turbulence [7]. The z length scale $L_z = 4\pi/5$ of Ω_{W03} was chosen as a compromise between the $L_z = 6\pi/5$ of Ω_{HKW} and its first harmonic $L_z/2 = 3\pi/5$ [24]. Unless stated otherwise, all calculations are carried out for Re $= 400$ and the Ω_{W03} cell. In the notation of this paper, the solutions presented in [17] have wavenumbers $(\alpha, \gamma) = (0.8, 1.5)$ and fit in the cell $[2\pi/0.8, 2, 2\pi/1.5] \approx [7.85, 2, 4.18]$.[4] Ref. [25] showed that these solutions first appear at critical Reynolds number of 127.7 and $(\alpha, \gamma) = (0.577, 1.15)$. Ref. [18]'s study of plane Couette solutions and their bifurcations was conducted in the cell of size $\Omega = [4\pi, 2, 2\pi] \approx [12.57, 2, 6.28]$.

An overview of the properties of known equilibria of plane Couette flow at $Re = 400$ is given in table 5.1.

5.2 New solutions

I list here properties of the new solutions reported in this thesis.

[4]Note also that Reynolds number in [17] is based on the full wall separation and the relative wall velocity, making it a factor of four larger than the Reynolds number used in this thesis.

Table 5.3: The first 27 least stable Floquet exponents, $\lambda = \mu \pm i\,\omega$ of equilibrium E_5 for Ω_{W03} cell plane Couette flow, $Re = 400$. The exponents are ordered by the decreasing real part. The two zero exponents, to the numerical precision of our computation, arise from the two translational symmetries. For details, see ref. [11].

j	$\mu^{(j)}_{EQ5}$	$\omega^{(j)}_{EQ5}$	$s_1 s_2 s_3$
1,2	0.07212161	0.04074989	S S S
3	0.06209526		S A A
4	0.06162059		A S A
5,6	0.02073075	0.07355143	S S S
7	0.009925378		S A A
8,9	0.009654012	0.04551274	A A S
10,11	0.009600794	0.2302166	S A A
12,13	1.460798e-06	1.542103e-06	- - A
14,15	-0.0001343539	0.231129	A A S
16	-0.006178861		A S A
17,18	-0.007785718	0.1372092	A A S
19	-0.01064716		S A A
20,21	-0.01220116	0.2774336	S S S
22,23	-0.01539667	0.2775381	S A A
24,25	-0.03451081	0.08674062	A S A
26,27	-0.03719139	0.215319	S A A

The Floquet exponents of equilibrium E_{16} and the symmetries of corresponding eigenvectors are reported in table 5.2. [5] [6] [7]

5.3 Eigenspectra: what to make out of them?

Well Mack the Finger said to Louie the King
I got forty red white and blue shoe strings
And a thousand telephones that don't ring
Do you know where I can get rid of these things?
— Bob Dylan, *Highway 61 Revisited*

Table 5.3, taken from ref. [11], is an example of how to tabulate the leading Floquet eigenvalues of the stability matrix of an equilibrium or relative equilibrium. The isotropy subgroup $G_{E,j}$ of the corresponding eigenfunction should also be noted. If the isotropy is trivial, $G_{E,j} = \{e\}$, it is omitted from the table. The isotropy subgroup G_E of the solution itself needs to be noted, and for relative equilibrium the velocity c along the group orbit. In addition, if the least stable (i.e., the most unstable) eigenvalue is complex, it is helpful to state

[5] Predrag: Is table 5.2 equilibrium E_{16} or E_{18}?
[6] Dustin: Equilibrium E_{17} in Ω_{W03} found on 11/06/2009
[7] Dustin: Equilibrium E_{18} in Ω_{W03} found on 11/10/2009

the period of the spiral-out motion (or spiral-in, if stable), $T_{\mathrm{E}} = 2\pi/\omega_{\mathrm{E}}^{(1)}$. [8]

Table 5.4, taken from ref. [20], is an example of how to tabulate the leading Floquet exponents of the stability matrix of an periodic orbit or relative periodic orbit. For a periodic orbit one states the period and the isotropy group of the orbit; for a relative periodic orbit one states in addition all shift parameters $\theta = (\theta_1, \theta_2, \cdots \theta_N)$. It is also useful to state $\Lambda_p = \prod \Lambda_{p,e}$, the product of expanding Floquet multipliers, as $1/|\Lambda_p|$ is the geometric weight of cycle p in a cycle expansion (remember that each complex eigenvalue contributes twice). We often do care about $\sigma_p^{(j)} = \Lambda_{p,j}/|\Lambda_{p,j}| \in \{+1, -1\}$, the sign of the jth Floquet multiplier, or, if $\Lambda_{p,j}$ is complex, its phase $T_p \omega_p^{(j)}$. [9]

Surveying this multitude of Floquet exponents is aided by a plot of the complex exponent plane (μ, ω). An example are the stability eigenvalues of equilibrium E_8 from ref. [8], plotted in figure 5.1. To decide how many of the these are "physical" in the PDE case (where number of exponents is always infinite, in principle), it is useful to look at the $(j, \mu^{(j)})$ plot. However, intelligent choice of the j-axis units can be tricky for high-dimensional problems. For Kuramoto-Sivashinsky system the correct choice are the wave-numbers which, due to the O(2) symmetry, come in pairs. For plane Couette flow the good choice is not known as yet; one needs to group $\mathrm{O}(2) \times \mathrm{O}(2)$ wave-numbers, as well as take care of the wall-normal node counting. [10]

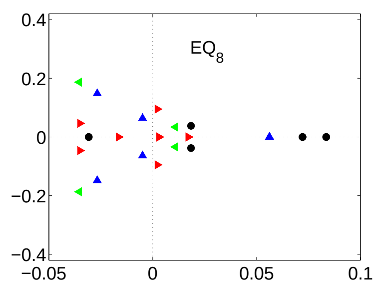

Figure 5.1: Eigenvalues of equilibrium E_8, plotted according to their isotropy groups: ● $+++$, the S-invariant subspace, ▶ $+--$, ◀ $-+-$, and ▲ $--+$, where $\pm$ symbols stand for symmetric/antisymmetric in s_1, s_2, and s_3 respectively. From ref. [8]. For numerical values of all stability eigenvalues see `channelflow.org`.

[8] Predrag: get rid of s_1, s_2, s_3 - too primitive

[9] Predrag: made up numbers in (5.4) - Dustin, please create the real table

[10] Predrag: add here the example from the Siminos blog: KS Lyapunov spectrum, with wavenumbers correctly doubled

Table 5.4: The first 25 least stable Floquet exponents, $\lambda = \mu \pm i\,\omega$ of periodic orbit $T = 59.77$ for Ω_{W03} cell(??) plane Couette flow, $Re = 400$, together with the symmetries of corresponding eigenvectors. The exponents are ordered by the decreasing real part. The one zero eigenvalue, to the numerical precision of our computation, arises from the spanwise translational SO(2) symmetry. For details, see ref. [20].

j	$\sigma_p^{(j)}$	$\mu_p^{(j)}$	$\omega_p^{(j)}$	$G_{p,j}$
1,2		0.02109336	0.0009197817	D_1
3,4	1	0.01700347	0.03051915	?
5	-1	0.003316719	0.05256136	
6		0.002889900		
7,8	-1	0.0003734452	0.03273554	
9		7.8838787e-05		
10		3.1675168e-06		
11		-3.480887e-05		
12		-0.0003734452	0.05256136	
13,14		-0.006341600	0.01839710	
15		-0.007849677		
16,17		-0.01017134	0.03474715	
18		-0.01029024		
19,20		-0.01212553	0.04625345	
21,22		-0.01342037	0.04664652	
23		-0.01602901		
24,25		-0.01908709	0.0114218	

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