Introduction to Gribov Ambiguity

Kamal Sharma

written as final termpaper for QFT course.

1.**GribovAmbiguity** All modern theories trying to explain fundamental physics are field theories which describe the configuration of a field and its dynamics as the interaction between fields and their evolution in space and time. These fields may be scalar, vector or tensor fields and they transform under a gauge transformation accordingly to give different configurations of the field. However, for some of these configurations the physical observables, which are in a direct way the reality that we perceive/observe, do not change under gauge transformations. For example, in electromagnetism, the following transformations,

$$\mathbf{A} \to \mathbf{A} + \nabla\psi \tag{3.1a}$$

$$\phi \to \phi - \frac{\partial\psi}{\partial t} \tag{3.1b}$$

keep the observables $\mathbf{E}$ and $\mathbf{B}$ (electric and magnetic fields respectively) unchanged. This is an example of a gauge transformation.

What is gauge freedom and gauge fixing? The invariance of physically observables quantities with respect to a gauge transformation implies that the system has redundant degrees of freedom in field variables. All field configurations that transform into one another through gauge transformations are physically equivalent and, therefore, for correct predictions, should be counted as one. Gauge fixing is the mathematical procedure of selecting an equivalence class for each set of physically identical field configurations. A coherent and consistent prescription of selecting the representative configurations (also known as gauge fixing) out of all possible detailed configurations is required to make

What is a gauge theory? According to (symmgauge freedom) there are two types of theories that can be called 'gauge theories'. The Yang-Mills theories and constrained Hamiltonian theories. The Hamiltonian theories subsume the Yang-Mills theories. Such theories have a common striking feature known as gauge freedom. Gauge freedom is a fancy way of saying that the theories has two kinds of variables – physical and unphysical– due to which the initial value problem in such theories are ill-defined. This means that a set of initial conditions does not uniquely determine the evolution of all the dynamical variables of the theory. The set of 'physical' variables will evolve the same way but the 'unphysical' variables can evolve in infinite number of arbitrary ways thus allowing infinite number of solutions for the same initial conditions. It is possible to interpret classical gauge theories as deterministic only if we consider the physical dynamical variables as the complete description of physical reality. If two states differ only in their unphysical variables, they represent the same physical configuration of the field.

Speaking in mathematical dialect, a gauge theory is a field theory in which the lagrangian is equivariant under a lie group of continuous transformations. Such a group of transformations is called a symmetry group of the theory. A gauge field is a vector field associated with the generators of the symmetry group and the particles that arise due to quantization of the gauge field are called gauge bosons.

Consequences of Gauge Symmetry The presence of gauge symmetry has significant consequences on the results of the theory. Let us consider a free non-relativistic particle in one dimensional space. The hamiltonian of such a system is given just by the kinetic energy of the particle $H = -\frac{1}{2}\frac{d^2}{dx^2}$. The energy eigenvalue spectrum of this problem is continuous in the absence of symmetry. In the case of a periodic boundary condition when the points x and $x + pL, p\epsilon I$ are physically identified with each other, the wavefunction has a gauge symmetry $\Psi(x) = \Psi(x + pL)$. The spectrum now becomes discrete and this is how gauge symmetry affects the physial configuration space. This is a simple example which illustrates that a gauge theory must obey some constraints that identify the physically identical configurations and that it makes the spectrum more restrictive.

Abelian and non-Abelian gauge theories The continuous symmetry operations on the lagrangian which keep the action invariant constitute a Lie group. The generators of infinitesimal transformations of such a Lie group define the algebra of such a symmetry group. If the generators of the gauge symmetry group commute then the theory is said to be Abelian, otherwise non-Abelian.

Gribov copies In 1978, Gribov showed that for a non-Abelian, for example SU(2) and SU(3), the local gauge group imposes more stringent constraints than it does on an Abelian gauge theory. The **transversality condition** $\partial.A = 0$ fixes the gauge uniquely for Abelian gauge theories but in the case of non-Abelian gauge theories, there exist distinct phase space configurations A and A' related by a finite gauge transformations $A' = UA$ such that $\partial \cdot A = 0$ and $\partial \cdot A' = 0$ where $A \neq A'$.These distinct configurations are called Gribov copies and in a non-Abelian gauge theory they have an additional constraint of being physically identical. The subspace of the full state space that contains only physically distinct configurations of the field is called a fundamental modular region and is free of any Gribov copies. In 1978, Singer**insert reference** showed that in non-Abelian theories Gribov copies are unavoidable and that the physical configuration space is topologically non-trivial. On the other hand, for an Abelian theory the physical configuration space is a linear vector space.

Quantum Chromodynamics and Confinement Quantum Chromodynamics is a theory which describes the strong interactions at sub-atomic level. Strong interaction is the force between quarks and gluons. At very high energies these sub-atomic particles are asymptotically free which means that they behave like free particles. But these particles have not been observed. This is because in low energy conditions they interact with each other and form bound states called hadrons like proton and neutron. This phenomena is called *confinement*.

Inspite of QCD being qualitatively similar to QED, the problem of confinement is not very well understood. In QED, the method of perturbative expansion using feynman diagrams has proved very successful because the small value of the QED coupling constant makes the contribution of higher terms more and more negligible. In QCD the value of the coupling constant is a function of energy and becomes larger as the energy is lowered. This phenomena called "infrared slavery" is responsible for the failure of perturbation theory for low energy phenomena. For understanding the low energy behaviour of the theory, various non-perturbative methods have been developed to describe confinement. Different methods work well in different conditions and so QCD is a patchwork of different methods that work in different conditions.

Dynamical implications of non-Abelian nature of gauge symmetry group The existence of Gribov copies and suppression of infrared modes due to their closeness to Gribov horizon leads to an interesting feature.The calculation of the gluon propagator under a non-Abelian theory leads to expulsion of the gluon from the physical spectrum of the solution. This is seen as non-existence of a physical pole in the gluon propagator.

Gauge orbit and Physical configuration Each physical configuration of the field A_phys is associated with a corresponding gauge orbit which is collection of all physically identical configurations. The physicalconfiguration space is the space of all gauge orbits modulo the group of local gauge transformations $\mathcal{G} = U$,

$$P = \mathcal{A}/\mathcal{G}.$$

Introduction to Yang-Mills theory As QCD is a specific case of a general Yang-Mills theory, it is a good idea at the general theory. Consider the compact group SU(N) of NxN unitary matrices U of determinant one. These matrices can be expressed as

$$U = exp(-ig\theta_a X_a),$$

where X^a are the generators of SU(N) group. If these generators X^a follow commutation relations

$$[X^a, X^b] = if_{abc}X^c,$$

and the $SU(N)$ corresponds to a single Lie group. These generators are defined to be hermitian and normalizable as follows:

$$X^\dagger = X,$$

$$Tr[X_a X_b] = \frac{\delta_{ab}}{2}.$$

Now, the generators X_a belong to the adjoint representation of the group $SU(N)$, i.e.

$$UX_aU^\dagger = X_b(D^A)_{ba},$$

with $(D^A(X_a))_{bc} = -if_{abc}$. Here f^{abc} are the structure constants of $SU(N)$ and have the following property,

$$f^{abc}f^{dbc} = N\delta^{ad}.$$

We can construct a lagrangian which by design would be invariant under the above defined $SU(N)$ group. The Yang-Mills action for this lagrangaian would be

$$S_{YM} = \int d^4x \frac{1}{2} Tr F_{\mu\nu} F_{\mu\nu},$$

whereby $F_{\mu\nu}$ is the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu],$$

and A_μ are the gluon fields that belong to adjoint representation of $SU(N)$ symmetry, i.e.

$$A_\mu = A_\mu^a X^a.$$

The field strength is given by

$$F_{\mu\nu} = \partial_\mu A_\mu^a + g f_{akl} A_\mu^k A_\nu^l.$$

A_μ under the $SU(N)$ symmetry transforms as

$$A'_\mu = U A_\mu U^\dagger - \frac{i}{g} (\partial_\mu U) U^\dagger.$$

We find that

$$F'_{\mu\nu} = U F_{\mu\nu} U^\dagger,$$

and can now see that Yang-Mills action is invariant under $SU(N)$ symmetry. The infinitesimal transformations can thus be written as

$$\delta A_\mu^a = -D_\mu^{ab} \theta^b,$$

with D_μ^{ab} the covariant derivative in the adjoint representation

$$D_\mu^{ab} = \partial_\mu \delta^{ab} - g f^{abc} A_\mu^c.$$

There is also a matter part of the action but we will work with pure Yang-Mills action.

Faddeev-Popov Ghosts Faddeev-Popov ghosts or ghost fields are additional fields which are introduced into gauge field theories to maintiain the consistency of path integral formulation. In order for the quantum field theories to deliver unambiguous and sensible results we need to avoid overcounting of feynman diagrams that correspond to physically equivalent processes. In a gauge field theory, each physical configuration has infinite number of full state space configurations all of which lie on a gauge orbit. Selecting a representative configuration from this equivalence class is required in order for path integral method to work. But usually there is no such prescription of selecting such a representative configurations. However, it is possible to modify the action by adding extra terms called *ghost-fields* that break gauge symmetry. In general, ghost fields can add or break gauge symmetry in a field theory. This method is called Faddeev-Popov procedure. Ghost fields are mathematical tools and represent virtual particles in feynman diagrams. They are also essential for unitarity.

Faddeev-Popov Quantization To understand the Gribov problem we first need to have a look at Faddeev-Popov quantization **reference 53**. The Yang-Mills action as discussed earlier is given by

$$S_{YM} = \int d^d x \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}.$$

Our naive assumption that the generating functional $Z(J)$ would be given by

$$Z(J) = \int [dA] exp[-S_{YM} + \int dx J^a_\mu A^a_\mu].$$

But this functional is not well defined. We can look at the quadratic part of the action,

$$Z(J)_{quadr} = \int [dA] exp[-\frac{1}{4} \int dx (\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))^2 + \int dx J^a_\mu(x) A^a_\mu(x)$$

$$= \int [dA] exp[-\frac{1}{2} \int dxdy A^a_\nu(x) [\delta^{ab} \delta(x-y)(\partial^2 \delta_{\mu\nu} - \partial_\mu \partial_\nu) A^b_\mu(y)) + \int dx J^a_\mu(x) A^a_\mu(x)$$

which after a gaussian integration gives

$$Z(J)_{quadr} = (detA)^{-\frac{1}{2}} \int [dA] e^{-\frac{1}{2} \int dxdy J^a_\nu(x) A_{\mu\nu}(x,y)^{-1} J^a_\mu(y)},$$

with $A_{\mu\nu}(x, y) = \delta(x - y)(\partial^2 \delta\mu\nu - \partial mu \partial_\nu)$, is ill defined because $A_{\mu\nu}(x, y)$ is not invertible. There is something wrong with the generating functional. Following the derivation of gauge fixed action given in **ref 16**:

$$S = S_{YM} + \int dx (\bar{c} \partial_\mu D^{ab}_\mu c^b - \frac{1}{2\alpha} (\partial_\mu A^a_\mu)^2).$$

If we take the limit $\alpha \to 0$, we have the Landau Gauge which stays a fixed point under normalization. If $\alpha = 1$, it's called the Feynman gauge in which the gluon propagator is the simplest.

What is BRST symmetry? Fixing the gauge cause the local gauge symmetry to break. However, after fixing the gauge, a new symmetry called the BRST symmetry appears which is basically the symmetry of the gabhost fields.For example, inserting a *b-field*

$$S = S_{YM} + \int d^d x (b^a \partial_\mu A_\mu + \alpha \frac{b^a)^2}{2} + \bar{c}^a \partial_\mu D^{ab}_\mu c^b),$$

where Z(J) is now given as

$$Z(J) = \int [dA][dc][d\bar{c}][db] e^{[-S + \int dx J^a_{\mu a=0} A^a_\mu]}.$$

Here b is the bosonic field. The action for this theory has a new symmetry called the BRST symmetry,

$$sS = 0,$$

with

$$sA^a_\mu = -(D_\mu c)^a, sc^a = \frac{1}{2} g f^{abc} c^b c^c,$$

$$s\bar{c}^a = b^a, sb^a = 0$$

$$s\bar{\psi}\alpha = -igc^a (X^a)^{ij} \psi^j_\alpha, s\bar{\psi}^i_\alpha = -ig\psi^j_\alpha c^a (X^a)^{ji}.$$

This BRST symmetry property is the proof that Yang-Mills theory is unitary in perturbation theory. Introduction of BRST symmetry introduces extra particles called ghost particles c and $\bar{c}$. Like other ghost particles, these particles too violate spin-statistics thorem.

Gribov Problem For any kind of gauge orbit the gauge fixing condition might have one, more or no solutions, i.e. the slice might intersect the gauge orbit once, more than once or never. Consider two Gribov copies A_μ and A'_μ related by a gauge transformation

$$A'_\mu = UA_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger,$$

which obviously satisfy the transversality condition

$$\partial_\mu A_\mu = 0 \; \partial_\mu A'_\mu = 0.$$

These equations when combined and expanded to first order gives

$$-\partial_\mu(\partial_\mu \alpha + ig[\alpha, \partial_\mu]) = 0$$

which is equivalent to

$$-\partial_\mu D_\mu \alpha = 0.$$

This means that the relevant Gribov copies are in a space orthogonal to the trivial null space. The transversality condition, $\partial_\mu A_\mu$, also implies that Faddeev-Popov operator is Hermitian,

$$-\partial_\mu D_\mu = \partial_\mu D_\mu.$$

Thus existence of Gribov copies are connected to zero eigenvalues of the Faddeev-Popov operator.

Important observation For small A_μ, the equation reduces to the eigenvalue equation

$$-\partial^2_\mu \psi = \epsilon\psi,$$

has positive eigenvalues but this cannot be gauranteed for large A_μ. This means that for large A_μ the eigenvalues of Fadeev-Popov operator are zero.

Possible Solutions?
Gribov Region and Gribov Horizon
We need to improve the gauge fixing for non-Abelian theories. This can be done by finding the Gribov region Ω which is defined as subspaces with positive eigenvalues of Faddeev-Popov operator,

$$\Omega = A^a_\mu, \partial_\mu A^a_\mu = 0, \mathcal{M}^{ab} > 0,$$

where M is the Faddeev-Popov operator,

$$\mathcal{M}^{ab}(x, y) = -\partial_\mu D^{ab}_\mu \delta(x - y).$$

This is the region which obeys Landau gauge and where the FP operator is positive definite. The border of the Gribov region is the manifold where the first eigenvalue of the FP operator becomes zero. This is known as Gribov horizon. The eigenvalues become negative on the other side of the Gribov horizon. Another way of chosing a Gribocv region is to select those points on the gauge orbit which have minimum A^2. This definition also agrees with our previous definiton on Gribov region.

Properties of the Gribov region
1. It is important that each group orbit passes through the Gribov region because we want to take into account all possible physical configurations. Gribov showed that for every configuration infinitesimally close to the Gribov horizon, there exists a Gribov copy on the other side of the horizon infinitesimally close the horizon. It has been rigorously proved that every gauge orbit passes through the Gribov region.
2. Gribov region is a convex manifold. **reference 12**
3. Gribov region is bounded in every direction. Unfortunately, despite of all the nice properties, it has been discussed **reference 86** that Gribov region still contains Gribov copies.
Another possible solution, Fundamental Modular Region
Now when even the Gribov region has Gribov copies, let us define a fundamental modular region as a more restrictive subspace of all the configurations which have all absolute minima of the functional. We shall select only the configurations closest to the region. This is called the fundamental modular region or the minimal Landau gauge.
Properties of FMR
1.All gauge orbits intersect with FMR.
2.$A_\mu = 0$ belongs to FMR as 0 is the smallest norm. 3.FMR is also convex and bounded in every direction. 4.The boundary of Λ, $\delta\Lambda$ has some points common with the Gribov horizon. 5. Gribov copies exist on the boundary.

Other attempts at gauge fixing
Singer showed that suitable regularity conditions at infinity does not leave any

contiuous gauge choices. This means that there is no unique representative of the gauge orbit that is continuous in the space of gauge orbits. A gauge free of Gribov copies is singular gauge and very difficult to handle in computations and also violates Lorentz invariance.

In 2005, Ghiotti, Kalloniatis, and Williams tried to improve the Fadeev-Popov gauge fixing by including the determinant into the action but in such a method the number of Gribov copies is not accounted for and no further calculations have been done along these lines.

Slavnov, in 2008 and 2010, pointed out that if we don't take the absoulte value of the determinant of the Faddeev-Popov operator and integrate over all Gribov copies, their effects will cancel out. The disadvantage is that if we make approximations then the errors can get very large.

Stochastic quantization with stochastic gauge fixing introduces a gauge-fixing force which is tangent to the gauge orbit. More works need to be done in this method.

Summary In order to get correct predictions from non-Abelian field theories, which are susceptible to large number of gauge copies, we need to choose a representative of each gauge orbit. Some new methods and mathematical tricks have been explored but none have given a consistent recipe for selection of the representatives of these equivalence classes. However, if approximations are made in a clever manner, some of the methods can give us practically usable results. There are other methods like semi-classical approach by Gribov which I haven't mentioned.