A Brief Review on Complex Angular Momentum Techniques

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A brief discussion on scattering is presented. First the classical scattering is discussed and the main critical effects are pointed out. An approach to semi-classical scattering is discussed, based on the work by Berry.

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Introduction

Most of what I will write about is discussed in detail in Nussenzveig [7].

I will begin by introducing the motivation of this review on sect. I, explaining the problem of interest and some of the different areas of application of the techniques discussed here.

I will then review some of the different techniques that have been used to study the problem of scattering by a central potential. A brief review on the classical approach will be given in sect. I A where some of the main critical effects will make themselves evident.

Then, in sect. II I will review the Complex Angular Momentum (CAM) techniques to approach the problem of scattering by a sphere, and finally in sect. III I will give a summary of what was explored.

I. SCATTERING

The scattering of classical particles has been studied for many years, and the classical results are sufficient to explain a wide range of scattering fenomena observed. But the classical picture has its limitations and it becomes necesary to take into account quantum considerations to explain effects as the *Rainbow Effect* or the *Glory Effect*. The places where mathematical difficulties arise in studing classical scattering, hint us of places where interesting fenomena may occur if we approach the scattering problem considering some quantum mechanical effects.

Furthermore, while in classical scattering the interaction is only between the scattered particles and the scatterer, in quantum mechanics different scattered contributions may interfere giving raise to a different kind of effect not found in classical scattering.

Tunneling effects and classically unattainable trajectories are possible when we consider the scattering problem from a quantum mechanical point of view and even if we just approach the problem from a semi-classical perspective. Different techniques has been developed to be able to compute the differential scattering-cross sections for various problems, but one of the biggest limitations one encounters is the poor convergence of most of the methods, or the short range of validity of them.

We will discuss some of the methods used to study scattering, and will try to summarize their advantages and range of application.

A. Classical Scattering

The classical problem of scattering of a particle of mass m by a potential $V(r)$ and be solved in terms of the conservation laws of angular momentum and energy,

$$L = mr^2\dot{\phi}^2\,, \qquad E = \frac{1}{2}m\dot{r}^2 + U_L(r)\,, \tag{1}$$

with $U_L(r)$ being the effective potential defined by

$$U_L(r) = V(r) + \frac{L^2}{2mr^2}. \tag{2}$$

From the equations (1), we may find evaluate $\dot{\phi}/\dot{r}$ to find

$$\dot{\phi}/\dot{r} = \mathrm{d}\phi/\mathrm{d}r = \frac{\frac{L}{mr^2}}{\left\{\frac{2}{m}\left[E - U_L(r)\right]\right\}^{\frac{1}{2}}}\,. \tag{3}$$

Integrating r from ∞ to r_0, the largest root of $\dot{r} = 0$, which is the radius of maximum approach, and then back to ∞ and with the initial condition $\phi = \pi$ when $r \to \infty$, we obtain the deflection angle

$$\Theta(L) = \pi - 2L\int_{r_0}^{\infty} \frac{\mathrm{d}r}{r^2\left\{2m[E - U_L(r)]\right\}^{\frac{1}{2}}}\,. \tag{4}$$

It is clear that the actual solution will depend on the form of the potential, but the range of values for the deflection angle will depend on whether the potential is attractive or repulsive. For purely attractive potentials the deflection angle can have values in $[0, \pi]$. In this case the deflection angle coincides with the scattering angle θ. For repulsive potentials, it is possible that the

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particle would go around the scatterer n times before finally escaping with certain deflection, then for purelly attractive potentials the relationship that must follow is

$$\Theta + 2n\pi = \pm\theta\,, \qquad (n = 0, 1, 2, \ldots)$$

and θ is required to remain the the interval $0 \leq \theta \leq \pi$. We can also evaluate the differential scattering cross-section, for this we consider the particles with an impact parameter b (related to the angular momentum through $L = bp$, where p is the particle's linear momentum) between b and $b + \mathrm{d}b$, this corresponds to an incoming cross-sectional area $2\pi b \mathrm{d}b$. If these particles are scattered in a solid angle $\mathrm{d}\Omega = 2\pi \sin\theta \mathrm{d}\theta$ then the differential scattering cross-section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|\,. \tag{5}$$

In the case of an attractive potential, as well as in more complicated cases as mixed potentials (attractive exterior with a repulsive core, for example) many different impact parameters may lead to the same scattering angle, and it is necesary to write the scattering cross-section as a sum

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta) = \sum_j \frac{b_j(\theta)}{\sin\theta}\left|\frac{\mathrm{d}\theta}{\mathrm{d}b_j}\right|^{-1}\,, \tag{6}$$

where b_j corresponds to the jth impact parameter with corresponding scattering angle θ.

From the form of (5–6) there are at least three evident places where singularities could occur.

Rainbow Scattering

When $\theta(b)$ has a local minimum or maximum, this means the quantity $|\mathrm{d}\theta/\mathrm{d}b|_{\theta=\theta_R} = 0$. The names comes from analogy with the optical rainbow effect.

Glory Scattering

When the scattering angle $\theta = 0$ or $-n\pi$ for an non-zero impact parameter $b \neq 0, b = b_g$. The differential cross section diverges as $(\sin\theta)^{-1}$ in the forward(backward) direction for n even(odd). Despite the name, the meteorogical glory effect seems to have a different origin.

Forward Peaking

When the effectvie potential $U_L(r)$ has a long tail (or decays exponentially) there are many contributions to the $\theta = 0$ scattering angle as $b \to \infty$, so the forward differential cross-section would diverge in this case as well, since $|\mathrm{d}\theta/\mathrm{d}b|$ remains bounded. This effect is not seen in cutoff potentials though.

There is yet another singularity that may raise not from the form of (5–6) but from the form of the effective potential and the calculation of the deflection angle, when the energy for the particle has a value equal to a local maximum of the effective potential. In this case the particle would remain in an unstable circular orbit.

Orbiting

For a given L_0 such that $U_{L_0}(r)$ has a maximum at $r = r_0$, a path with energy E such that

$$U_{L_0}(r_0) = E\,, \qquad (\mathrm{d}U_{L_0}/\mathrm{d}r)_{r=r_0} = 0 \tag{7}$$

will mean that a circular orbit with radius r_0 will exist. This causes the integral (4) to diverge in the lower limit, which means that a particle comming with the right energy and angular momentum will spiral around the scatterer indefinitely.

B. Semi-Classical Scattering

As soon as one finishes studying the classical effects, one of the first questions to ask is when are these effects important when considering quantum scattering. In general, we would expect a quantum mechanical approach to reproduce the classical effects in the limit $\hbar \to 0$. To gain more insight on how to recover the classical effects from quantum mechanics, we may remember that Hamiltonian description of mechanics places a non-relativistic particle with energy E within a potential $V(r)$ following a path corresponding to that an optic light ray would follow in a medium with refractive index

$$N = \left(1 - \frac{V(r)}{E}\right)^{\frac{1}{2}}\,, \tag{8}$$

and the principle of least action is equivalent to Fermat's principle. Shrödinger's description uses the de Broglie's wavelength

$$\lambda\!\!\!^{-} \equiv 1/\kappa = \hbar/p = \hbar/\{2m[E - V(r)]\}^{\frac{1}{2}} \tag{9}$$

II. COMPLEX ANGULAR MOMENTUM

III. DISCUSSION

[1] Berry, M. V. (1969) *Uniform approximations for glory scattering and diffraction peaks*, J. Phys. B **2**, 381.

[2] Berry, M. V. & Mount, K. E. (1972) *Semiclassical approximations in wave mechanics*, Rep. Prog. Phys. **35**, 315.

[3] Cvitanović P., Artuso R., Mainieri R., Tanner G. and Vattay G. (2003) *Chaos: Classical and Quantum*, `ChaosBook.org` Niels Bohr Institute, Copenhagen.

[4] Nussenszveig, H. M. (1965) *High-frequency scattering by an impenetrable sphere*. Ann. Phys. (NY) **34**, 23.

[5] Nussenzveig, H. M. (1969) *High-Frequency Scattering by a Transparent Sphere. I. Direct Reflection and Transmission* J. Math. Phys. **10**, 82.

[6] Nussenzveig, H. M. (1988) *Uniform approximation in scattering by spheres*. J. Phys. A: Math. Gen. **21**, 81.

[7] Nussenzveig, H. M. (1992) *Diffreaction Effects in Semiclassical Scattering*. Cambridge University Press.