Weak magnetic diffusivity corrections to kinematic dynamo

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Cvitanović and Ott [1] proposal for including effects of magnetic diffusivity on kinematic dynamo is investigated, inconclusively.

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I. INTRODUCTION

My project is an investigation of effects of magnetic diffusivity on kinematic dynamo. I work and work through obscure ref. [2], and then very intuitive but more obscure unpublished paper by Cvitanović and Ott [1].

Many astrophysical magnetic fields are thought to be arise by dynamo action due to internal fluid motions. Many flows are known to give a growing $\mathbf{B}$ [3, 4]. However, even most of the simplest models shows chaotic behaviors [5]. Here we apply the periodic orbit theory to dynamo problems. The periodic orbit theory of classical chaos expresses all long time averages over chaotic dynamics in terms of cycle expansions [6, 7, 8, 9], sums over periodic orbits (cycles) ordered hierarchically according to the orbit length, stability, or action. The study of periodic orbit can be useful when they are dense. Then we can find a periodic orbit around any point, and see the evolution on the orbit easily, instead of numerical calculation. If the symbolic dynamics is known, and the flow is hyperbolic. The longer cycles are shadowed by the shorter ones, and cycle expansions converge exponentially or even super-exponentially with the cycle length [10].

In sect. II I derive the induction equation for passive diffusion less transport of magnetic flux lines and in sect. III show that chaotic stretching and folding of a magnetic field can lead to growth of average fields and dynamo action. In sect. IV I rederive the Balmforth *et al.* [2] trace formula for the deterministic dynamo action. In sect. VI I turn to the central issue of this project, the rate of smearing of periodic trajectories by diffusion, and attempt, inconclusively, to derive Ott's [1] modification of the cycle weight in presence of magnetic diffusivity. My results and the reasons why I failed to complete this project on the level of a publishable paper are discussed in sect. VIII.

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II. INDUCTION EQUATION

In the limit where current associated with charge separation and relativistic correction for motion of fluid is neglected, a reduced form of Maxwell's equation is valid. The system can be described as follows:

$$\nabla \times \mathbf{B} = \mu \mathbf{J} \qquad (1)$$
$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \qquad (2)$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,. \qquad (3)$$

Taking curl of (1), using (2), (3)

$$\begin{aligned} \text{LHS} &= \nabla \times \nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\nabla^2 \mathbf{B} \\ \text{RHS} &= \mu\sigma \nabla \times (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \\ &= \frac{1}{\eta}\left(-\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B})\right) \qquad (4) \end{aligned}$$

yields

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) - \eta \nabla^2 \mathbf{B} = 0\,, \qquad (5)$$

where $\eta = 1/\mu\sigma$ is magnetic diffusivity whose dimension is $[L^2]/[T]$. When the fluid is incompressible, $\nabla \cdot \mathbf{u} = 0$, (5) can be simplified by using

$$\begin{aligned} \nabla \times (\mathbf{u} \times \mathbf{B}) &= \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{B} \\ &\quad + (\nabla \cdot \mathbf{B})\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{B} \qquad (6) \end{aligned}$$

as the following:

$$\mathbf{B} \cdot \nabla \mathbf{u} = \frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} - \eta \nabla^2 \mathbf{B} = \frac{d\mathbf{B}}{dt} - \eta \nabla^2 \mathbf{B}\,. \qquad (7)$$

In (6), $(\mathbf{u} \cdot \nabla)\mathbf{B}$, $(\mathbf{B} \cdot \nabla)\mathbf{u}$, and $\mathbf{B}(\nabla \cdot \mathbf{u})$ represent advection, stretching, and compression, respectively [11].

When the fluid is compressible, the equation of mass conservation can be expressed as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{u} = 0 \qquad (8)$$

$$\nabla \cdot \mathbf{u} = -\frac{1}{\rho}\frac{d\rho}{dt} \tag{9}$$

Then (7) becomes

$$\frac{d\mathbf{B}}{dt} - \eta\nabla^2\mathbf{B} - \frac{d\rho}{dt}\frac{1}{\rho}\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} \tag{10}$$

Dividing both sides by ρ yields

$$\begin{aligned}\frac{\mathbf{B}}{\rho}\cdot\nabla\mathbf{u} &= \frac{1}{\rho}\frac{d\mathbf{B}}{dt} - \frac{\eta}{\rho}\nabla^2\mathbf{B} - \frac{\mathbf{B}}{\rho^2}\frac{d\rho}{dt} \\ &= \frac{d}{dt}\frac{\mathbf{B}}{\rho} - \frac{\eta}{\rho}\nabla^2\mathbf{B} \end{aligned} \tag{11}$$

Suppose at a time t two nearby points are marked as $\mathbf{x}$ and $\mathbf{x}+\delta\mathbf{l}$. Then at time $t+dt$, the points would move to $\mathbf{x}+\mathbf{u}(\mathbf{x},t)dt$ and $\mathbf{x}+\delta\mathbf{l}+\mathbf{u}(\mathbf{x}+\delta\mathbf{l},t)dt$. $\delta\mathbf{l}$ becomes $\delta\mathbf{l}+\delta\mathbf{l}\cdot\nabla\mathbf{u}dt+O(\delta\mathbf{l}^2)$ in dt. Therefore

$$\frac{d\delta\mathbf{l}}{dt} \approx \delta\mathbf{l}\cdot\nabla\mathbf{u}(\mathbf{x},t) \tag{12}$$

Note that (12) and B/ρ of (11) has same mathematical structure when there is no diffusion.

Let's consider a Lagrangian map $\mathcal{M}:\mathbf{a}\rightarrow\mathbf{x}$ where $\mathbf{a}$ is an initial position of a particle. Then we can define the Jacobian of the map $\mathcal{J}(\mathbf{a},\mathbf{x})$ by

$$J_{ij}(\mathbf{a},\mathbf{x}) = \frac{\partial x_i(\mathbf{a},t)}{\partial a_j} \tag{13}$$

At time t, $d\mathbf{l}=\mathcal{J}(\mathbf{a},t)d\mathbf{a}$ is tangent to the material line which was tangent to $d\mathbf{a}$ at $\mathbf{a}$. Again, using the fact that magnetic field behaves just like material line, B can be expressed in the form

$$\frac{\mathbf{B}}{\rho}(\mathbf{x}(\mathbf{a},t),t) = \mathcal{J}(\mathbf{a},t)\frac{\mathbf{B}}{\rho}(\mathbf{a},0) \tag{14}$$

This is called Cauchy's solution of the induction equation. Since

$$\frac{\rho(\mathbf{a},0)}{\rho(\mathbf{x},t)} = det\mathcal{J}(\mathbf{a},t) \tag{15}$$

$\mathbf{B}$ can be expressed entirely in terms of the initial field and the Jacobian. We can rewrite (15) by defining and operator $\mathcal{T}$ on an initial magnetic field as

$$(\mathcal{T}\mathcal{B})(\mathbf{x},t) = \frac{\rho(\mathbf{x},t)}{\rho(\mathcal{M}^{-1}\mathbf{x},0)}\mathcal{J}(\mathcal{M}^{-1}\mathbf{x},t)\mathbf{B}(\mathcal{M}^{-1}\mathbf{x},0) \tag{16}$$

Operator $\mathcal{T}$ is called induction operator for a perfect conductor. For the case of incompressible flow, the ratio of initial density and density at a later time is unity.

III. GROWTH RATE

Chaotic stretching of magnetic field together with constructive folding, can lead to growth of average fields and dynamo action. Chaotic flows are associated with positive Lyapunov exponents, which measure asymptotic growth of vectors. For a given spatial point $\mathbf{a}$ and velocity vector $\mathbf{v}$, we can define the *Lyapunov exponent* as

$$\Lambda_{Lyap} = \limsup_{t\rightarrow\infty}\frac{1}{t}\log\left|\mathcal{J}^T(\mathbf{a})\mathbf{v}\right| . \tag{17}$$

Lyapunov exponent is almost same in a given region and gives the rates for almost all vectors stretch. However, there are examples that the growth rate on the fast dynamo is greater than the Lyapunov exponent. Therefore we need a different quantity to determine the upper bound of fast dynamo growth rate. Since the Lyapunov exponent is almost same, we can take the average over a single chaotic region as

$$\Lambda_{Lyap} = \lim_{t\rightarrow\infty}\frac{1}{t}\left\langle \log\left|\mathcal{J}^T(\mathbf{a})\mathbf{v}\right|\right\rangle \tag{18}$$

By Oseledets' multiplicative ergodic theorem [12], the Lyapunov exponent exists for almost all $\mathbf{v}$ and $\mathbf{a}$ in a chaotic region. Now let's consider the rate of stretching of a finite curve in the same chaotic region. After some time, the curve will be spread out all over the volume. Its length will be given by average of $\mathcal{J}^T(\mathbf{a})\mathbf{v}$ over the region as

$$h_{line} = \lim_{t\rightarrow\infty}\frac{1}{t}\log\left\langle\left|\mathcal{J}^T(\mathbf{a})\mathbf{v}\right|\right\rangle . \tag{19}$$

Convexity of log function implies $h_{line} \geq \Lambda_{Lyap}$ and equality holds only if the stretching is uniform. Newhouse [13] and Yomdin [14] showed that the topological entropy gives the maximum growth rate of finite k-dimensional volume in a C^∞ flow. For $2d$ area preserving flow $h_{line}=h_{top}$, but for $3d$ flow $h_{line}\leq h_{top}$.

Given a flow $\mathcal{M}^t$ on a compact metric space, $t>0$ and $\epsilon>0$, we say two points $\mathbf{x}$ and $\mathbf{y}$ are $(t,\epsilon)-$ *separated* if

$$d(\mathcal{M}^\tau\mathbf{x},\mathcal{M}^\tau\mathbf{y}) > \epsilon\,, for\, some\, \tau\, with\, 0<\tau<t \tag{20}$$

where $d(\mathbf{x},\mathbf{y})$ is the distance between $\mathbf{x}$ and $\mathbf{y}$. We then define $N_{sep}(t,\epsilon)$ to be the maximum number of points $\mathbf{x}_i$ which are mutually $(t,\epsilon)-$ *separated.* If we observe the system for a time t and resolve distance of ϵ, we can at most observe $N_{sep}(t,\epsilon)$ distinct orbits. We define the *topological entropy* of flow $\mathbf{u}$ by

$$h_{top} = \lim_{\epsilon\rightarrow 0+}\limsup_{t\rightarrow\infty}\frac{1}{t}logN_{sep}(t,\epsilon)\,, \tag{21}$$

which is the rate at which the system reveals information about its structure as t increases. Positive topological entropy is typical for a chaotic flow, since it indicates sensitive dependence on initial conditions. Vishik [15], Klapper and Young [16] have shown topological entropy is an upper bound on the fast dynamo growth rate.

IV. TRACE FORMULA

Dynamic properties of dynamo can be studied by investigating the behavior on a periodic orbit and the link is the trace formula. Consider a steady incompressible flow $\mathbf{v}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x})$ and its corresponding eigenfunction of induction equation as $\mathbf{B}(\mathbf{x}, t) = \mathbf{b}(\mathbf{x})e^{st}$, where $\mathbf{x}(t) = f^t(\mathbf{a})$ and $\mathbf{x}(t = 0) = \mathbf{a}$ is the flow. Then (14) becomes

$$\begin{aligned} \mathbf{B}(\mathbf{x}, t) &= e^{st}\mathbf{b}(\mathbf{x}) \\ &= \mathcal{J}(\mathbf{a}, t)\mathbf{B}(\mathbf{a}, 0) \\ &= \int d^3\mathbf{x}'\delta(\mathbf{x}' - f^{-t}(\mathbf{x}))\mathcal{J}(\mathbf{x}', t)\mathbf{b}(\mathbf{x}')\,, \end{aligned} \tag{22}$$

For simplicity, define $L^t_{ij}(\mathbf{x}, \mathbf{x}')$ as

$$L^t_{ij}(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x}' - f^{-t}(\mathbf{x}))\frac{\partial x_i}{\partial x'_j}\,. \tag{23}$$

Then each component of (22) can be rewritten as

$$e^{st}b_i(\mathbf{x}) = \int d^3\mathbf{x}' L^t_{ij}(\mathbf{x}, \mathbf{x}')b_j(\mathbf{x}')\,. \tag{24}$$

Suppose L^t_{ij} has a sequence of eigenvalues $e^{s_0 t}$, e^{s_1}, $\cdots$ with $s_0 \geq s_1 \geq s_2 \geq \cdots$, then long term evolution is dominated by dominant eigenvalue $e^{s_0 t}$.

As any pair of nearby two points on a periodic orbit has same period, the distance between two point is conserved. Therefore Jacobian matrix has eigenvalue 1 along the flow of periodic orbit. Since incompressible flow is a area preserving mapping, $det\mathcal{J}(\mathbf{a}, t) = 1$, which implies other two eigenvalues are inverse of the other. By convention we take larger eigenvalue between those two and call it Λ_p.

We can separate a flow into transverse $\mathbf{x}_{||}$ and longitudinal coordinates $\mathbf{x}_\perp$. Then we can write trace of $\mathcal{L}^t$ as

$$\begin{aligned} \operatorname{tr}\mathcal{L}^t = \int d\mathbf{x}'_\perp d\mathbf{x}'_{||}\delta\left(\mathbf{x}'_\perp - f^t_\perp(\mathbf{x})\right) \\ \delta\left(\mathbf{x}'_{||} - f^t_{||}(\mathbf{x})\right)\sum_i \frac{\partial x_i}{\partial x'_i}\,. \end{aligned} \tag{25}$$

Transverse integration is defined on the Poincaré surface. Linearization of the periodic flow transverse to the orbit yields

$$\int d\mathbf{x}'_\perp \delta\left(\mathbf{x}'_\perp - f^{rT_p}_\perp(\mathbf{x})\right) = \frac{1}{|det(1 - \mathcal{J}^r_p)|}\,, \tag{26}$$

where $\mathcal{J}_p$ is p-cycle transverse Jacobian matrix. $\sum_i \partial x_i/\partial x'_i$ term of (25) is same as $tr\mathcal{J} = 1 + tr\mathcal{J}_p$. To compute the longitudinal component, we use a parametrized coordinate $x_{||}$ by flight time

$$x_{||}(\tau) = \int_0^\tau d\sigma v(\sigma)_{\quad mod\ \ L_p}\,, \tag{27}$$

where $v(\sigma) = v(x_{||}(\sigma))$ and L_p is the length of circuit of periodic orbit. Then integral along longitudinal coordinate can be rewritten as

$$\begin{aligned} &\int_0^{L_p} d\mathbf{x}_{||}\delta\left(\mathbf{x}'_{||} - f^t_{||}(\mathbf{x})\right) \\ &= \int_0^{T_p} d\tau v(\tau)\delta\left(\int_\tau^{t+\tau} d\sigma v(\sigma)_{\quad mod\ \ L_p}\right)\,. \end{aligned} \tag{28}$$

All the zeros for the term within delta function do not depend on τ. Using

$$\int dx\delta\left(h(x)\right) = \sum_{x:h(x)=0}\frac{1}{|h'(x)|}\,, \tag{29}$$

(28) can be rewritten as

$$\begin{aligned} &\int_0^{L_p} d\mathbf{x}_{||}\delta\left(\mathbf{x}'_{||} - f^t_{||}(\mathbf{x})\right) \\ &= \sum_{r=1}^\infty \delta(t - rT_p)\int_0^{T_p} d\tau v(\tau)\frac{1}{v(\tau + t)} \\ &= T_p\sum_{r=1}^\infty \delta(t - rT_p)\,. \end{aligned} \tag{30}$$

Using (26) and (30), (25) becomes

$$\begin{aligned} \operatorname{tr}\mathcal{L}^t &= \sum_{n=0}^\infty m_n e^{s_n t} \\ &= \sum_p T_p \sum_{r=1}^\infty \frac{1 + \Lambda^r_p + \frac{1}{\Lambda^r_p}}{\Lambda^r_p + \frac{1}{\Lambda^r_p} - 2}\delta(t - rT_p)\,. \end{aligned} \tag{31}$$

Therefore dynamics evolution is determined by properties of periodic orbit. The period and eigenvalues of Jacobian set the dynamics around the periodic orbit.

V. TOPOLOGICAL ENTROPY BOUND

Let $\Gamma(s)$ be the Laplace transformation of $tr\mathcal{L}^t$. By (31), we have

$$\Gamma(s) = \begin{cases} \sum_{n=0}^\infty \frac{m_n}{s - s_n} & s > s_0 \\ \infty & s \leq s_0 \end{cases} \tag{32}$$

$$= \sum_p T_p \sum_{r=1}^\infty \frac{1 + \Lambda^r_p + \frac{1}{\Lambda^r_p}}{\Lambda^r_p + \frac{1}{\Lambda^r_p} - 2}e^{-srT_p}\,. \tag{33}$$

Let s_0 be the element of $\{s_n\}$ with largest real part. Then for large time, $tr\mathcal{L}^t \approx e^{s_0 t}$ and we can rewrite (33) as

$$\Gamma(s) \approx \sum_P T_p \sum_{r=1}^\infty \sigma^r_p e^{-srT_p}\,, \tag{34}$$

where σ_p^r is defined as

$$\sigma_p^r = \frac{1+\Lambda_p^r+\frac{1}{\Lambda_p^r}}{\Lambda_p^r+\frac{1}{\Lambda_p^r}-2} \approx \frac{\Lambda_p^r}{|\Lambda_p^r|} = \pm 1\,. \tag{35}$$

Then (34) is bounded as

$$\Gamma(s) \le \sum_p T_p \sum_{r=1}^{\infty} e^{-Re(s)rT_p}\,. \tag{36}$$

The rate of growth of the number of periodic orbits with the cycle period topological entropy, can be calculated by

$$\sum_p T_p \sum_{r=1}^{\infty} e^{-hrT_p}\,. \tag{37}$$

As h decreases from ∞, this quantity goes under a transition from a finite value to ∞ and the value of h at transition is called topological entropy h_{top}. Thus we have

$$|\Gamma(s)| < \infty \quad for \quad Re(s) > h_{top}\,. \tag{38}$$

To see how close the growth rate gets to the topological entropy, consider steady flow described by the following map [17]:

$$\mathcal{M}(x,y) = \left\{\left|\frac{x}{2} - u\left(y-\frac{1}{2}\right)\right|, 2\left|y - u\left(y-\frac{1}{2}\right)\right|\right\} \tag{39}$$

$$T(x,y) = \begin{cases} T_a & if \quad x < \frac{1}{2} \\ T_b & if \quad x > \frac{1}{2} \end{cases} \tag{40}$$

where u denotes the unit step function.

This map can be understood as a deformation of the unit square as the fluid flows from $z=0$ and $z=\tilde{L}$. When $T_a \ne T_b$, the flux does not cancel each other perfectly. Now consider the surface $y = 1/2$ and the total magnetic flux through this surface per unit length in z by $\Phi(z,t)$. We seek for exponential solutions, so set

$$\Phi(z,t) = \phi(z)e_{st} \tag{41}$$

where s can be a complex number. The flux at $z = L$ and $t = t$ can be represented as

$$\Phi(L,t) = \Phi(0, t-T_a) - \Phi(0, t-T_b)\,. \tag{42}$$

The flux at $z = L$ is sum of fluxes at $z = 0$, due to the deformation, with different signs. Also, the two parts arrive at $z = L$ with different time interval. Since the flow is periodic in z with period L, we can set $\phi(z)$ as

$$\phi(z) = p(z)e^{ikz}\,, \tag{43}$$

where $p(z) = p(z + L)$. Then (42) becomes

$$e^{ikL} = e^{-sT_a} - e^{-sT_b}\,. \tag{44}$$

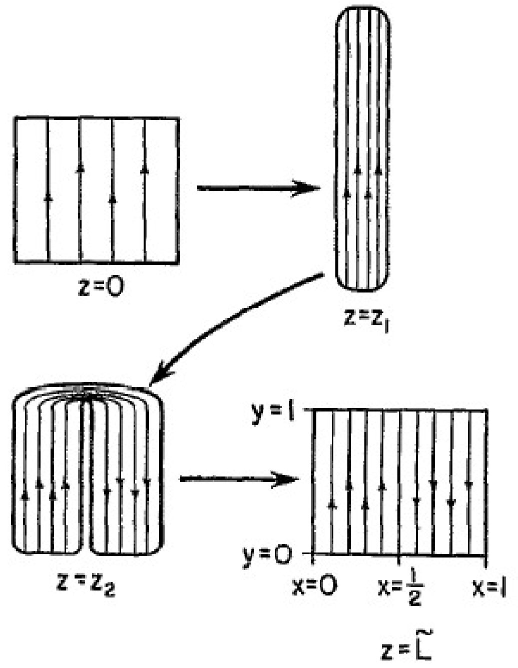

FIG. 1: Deformation of the unit square corresponding to the baker's map

Topological entropy is the exponential rate of increase of the flux without cancellation [18, 19]. Then (44) is changed into

$$1 = e^{-h_{top}T_a} + e^{-h_{top}T_b}\,. \tag{45}$$

Assuming $T_a < T_b$, then

$$1 \approx e^{-h_{top}T_a}\,. \tag{46}$$

Therefore the flux growth given by (44) can almost reach the bound set by topological entropy.

VI. SMEARING OF PERIODIC TRAJECTORIES BY DIFFUSION

Consider a factorized hyperbolic flow,

$$\frac{du}{dt} = \lambda u + d_u \tag{47}$$

$$\frac{ds}{dt} = -\lambda s + d_s \tag{48}$$

$$\frac{dz}{dt} = \mathbf{x}(u,s) + d_{||}\,. \tag{49}$$

Assume the diffusion is isotropic:

$$\langle d_i(t)\rangle = 0$$
$$\langle d_i(t)d_j(t')\rangle = 2D\delta_{ij}\delta(t-t') \tag{50}$$

Assume that z coordinate is periodic with period T. Integration of (49) yields

$$u(T) = \int_0^T dt e^{\lambda(T-t)} d_u(t)\,, \qquad (51)$$

assuming $u(t = 0) = 0$. The average of displacement from the periodic orbit would be 0, because of the isotropic diffusion. To see the effect of diffusion, we should consider the square of displacement:

$$\begin{aligned} \left\langle u(T)^2 \right\rangle &= \int_0^T dt \int_0^T dt' e^{\lambda(2T-t-t')} d_u(t) d_u(t') \\ &= \frac{D}{\lambda}\left(e^{2\lambda T} - 1\right)\,. \end{aligned} \qquad (52)$$

For contracting axis, it becomes

$$\left\langle s(T)^2 \right\rangle = \frac{D}{\lambda}\left(1 - e^{-2\lambda T}\right)\,. \qquad (53)$$

Therefore mean square thickness $\left\langle \delta\mathbf{x}_\perp^2 \right\rangle$ after one period is given as

$$\begin{aligned} \left\langle \delta\mathbf{x}_\perp^2 \right\rangle &= \left\langle u(T)^2 + s(T)^2 \right\rangle \\ &= 2\frac{D}{\lambda} sinh(\lambda T) \end{aligned} \qquad (54)$$

Along u, diffusion can be extended, but along s the effect of diffusion is suppressed due to contraction.

Let the velocity field evolution be given by

$$\frac{d\mathbf{x}}{dt} = \mathcal{A}\mathbf{x} + \mathcal{N}(\mathbf{x}) + v_{||}\hat{\mathbf{x}_{||}} + \mathbf{d}(\mathbf{x}, t)\,, \qquad (55)$$

where $\mathcal{A}$ is the stability matrix, $\mathcal{N}(\mathbf{x})$ contains all higher order nonlinear terms, and $\mathbf{d}(\mathbf{x}, t)$ is noise by diffusion. Pick $\mathbf{x}(t)$ along a noiseless prime cycle of period T and consider linearized flow for a small deviation $\mathbf{x} + \delta\mathbf{x}(t)$, with period $T + \delta T$. Then the position after one cycle can be given as

$$\begin{aligned} &\delta\mathbf{x}(T + \delta T) = e^{\int_0^{T+\delta T} dt \mathcal{A}(\mathbf{x}(t))}\delta(0) \\ &+ \int_0^{T+\delta T} dt e^{\int_t^{T+\delta T} d\tau \mathcal{A}(\mathbf{x}(\tau))} \mathbf{d}(\mathbf{x}, t) \end{aligned} \qquad (56)$$

where the integrals are time ordered. Average displacement in transverse directions can be calculated as

$$\begin{aligned} \left\langle \delta\mathbf{x}(T+\delta T)_\perp^2 \right\rangle &= \int\int_0^{T+\delta T} dt\, dt' \,\Big\langle \mathbf{d}(\mathbf{t'})^{\mathbf{T}} \\ &\quad e^{-\int_{T+\delta T}^{t'} d\tau A(\mathbf{x}(\tau))} \mathcal{P}_\perp^T \\ &\quad \mathcal{P}_\perp e^{\int_t^{T+\delta T} d\tau A(\mathbf{x}(\tau))} \mathbf{d}(\mathbf{t}) \Big\rangle \\ &= 2D \int_0^{T+\delta T} dt \ \mathrm{tr}_\perp \\ &\quad \Big\{ e^{-\int_{T+\delta T}^{t'} d\tau \mathcal{A}(\mathbf{x}(\tau))} \\ &\quad e^{\int_t^{T+\delta T} d\tau A(\mathbf{x}(\tau))} \Big\}\,, \end{aligned} \qquad (57)$$

[21]
where $tr_\perp$ is the trace in two dimensional subspace of locally transverse coordinate and $\mathcal{P}_\perp$ is the projection operator into locally transverse coordinate system. Trace in (57) is due to $\langle d_i(t) d_j(t') \rangle = 2D\delta_{ij}\delta(t-t')$. Integration within the trace in (57) represent evolution of $\mathcal{A}$ from time t to $T + \delta T$ along the periodic orbit and following the same path to t. Therefore above will depend on the initial point $\mathbf{x}(0)$ on periodic orbit.

To derive $\Gamma(s)$, we need to compute $\left\langle (\delta T_p)^2 \right\rangle$. δT_p can be expressed as

$$\delta T_p = \frac{L_p}{v_{||}}\left(\frac{\delta L_p}{L_p} - \frac{\delta v_{||}}{v_{||}}\right)\,, \qquad (58)$$

where L_p is the physical length of the prime cycle and $T_p = L_p/v_{||}$. Here we only considered the first order correction. Then $\left\langle (\delta T_p)^2 \right\rangle$ becomes

$$\begin{aligned} \left\langle (\delta T_p)^2 \right\rangle &= \left\langle \frac{\delta {L_p}^2}{{v_{||}}^2} \right\rangle \\ &\quad -2\left\langle \frac{\delta L_p \delta v_{||}}{{v_{||}}^3} L_p \right\rangle + \left\langle \frac{\delta {v_{||}}^2}{{v_{||}}^4} {L_p}^2 \right\rangle . \end{aligned} \qquad (59)$$

By assumption of periodicity and incompressibility, $\mathbf{v}_{||} = const$. From (50), $\left\langle (\delta L_p)^2 \right\rangle = 2DT_p$. Since there is no correlation of diffusion at different time, $\left\langle \delta L_p \delta v_{||} \right\rangle = 0$. For small diffusion,

$$\delta v_{||} = \frac{\partial v_{||}}{\partial s}\delta s + \frac{\partial v_{||}}{\partial u}\delta u + d_{||}\,, \qquad (60)$$

$$\delta v_{||}^2 = \left(\frac{\partial v_{||}}{\partial s}\right)^2 \delta s^2 + \left(\frac{\partial v_{||}}{\partial u}\right)^2 \delta u^2 + 2\frac{\partial v_{||}}{\partial s}\frac{\partial v_{||}}{\partial u}\delta s \delta u\,, \qquad (61)$$

where s and u are locally transverse spatial coordinates. Therefore (59) can be rewritten as

$$\left\langle (\delta T_p)^2 \right\rangle = \frac{2DT_p}{{v_{||}}^2} + \frac{T_p^2}{v_{||}^2}\left\langle \delta v_{||}^2 \right\rangle\,. \qquad (62)$$

Note that last term of (62) is proportional to T_p^2, not T_p

From (34), noisy case becomes

$$\begin{aligned} \Gamma(s) &= \sum_p \sum_{r=1}^{\infty} \sigma_p^r e^{-rsT_p} \\ &\approx \sum_p \sum_{r=1}^{\infty} \sigma_p^r e^{-rs\langle T_p \rangle - \frac{rs^2}{2}\left\langle \delta T_p^2 \right\rangle} \\ &= \sum_p \sum_{r=1}^{\infty} \sigma_p^r e^{-rsT_p - \frac{rs^2}{2}\left\langle \delta T_p^2 \right\rangle}\,. \end{aligned} \qquad (63)$$

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[21] RJ: $tr_\perp$ will pick up two component with factor $2D$.

For $T_p \gg 1$, the leading contribution of noise comes from $v_{||}^2$. Therefore noise depends mainly on the dynamics along the periodic orbit.

VII. SIMPLE SIMULATION

Let's consider a simple model:

$$\begin{aligned} \dot{r} &= \alpha(r - r_c)^2 + d_r \\ \dot{\theta} &= \omega_0 + \frac{d_\theta}{r} \\ \dot{z} &= -\alpha z \;-^2 + d_z \end{aligned} \tag{64}$$

where each noise d_i follows Gaussian distribution with standard deviation σ.

Method: $r_c = 100$, $\omega = 0.01$, period $T = 200\pi$. Each data set correponds to 1000 periodic orbits, and $< \delta T^2 >$ is averaged if the particle starting from $r = 100, \theta = 0, z = 0$ comes back within $|\delta r| < 3$ and $|\delta z| < 3$. Diffusion constant D is given by $\sqrt{\pi\sigma}$ where σ is the standard deviation of Gaussian noise. The result are (values on the right is $< \delta T^2 >$:

$$\begin{aligned} \sigma = 0.001 &\rightarrow 9.78 \times 10^{-7} \\ \sigma = 0.01 &\rightarrow 6.05 \times 10^{-5} \\ \sigma = 0.1 &\rightarrow 6.45^{-3} \\ \sigma = 1 &\rightarrow 6.05 \times 10^{-1} \\ \sigma = 5 &\rightarrow 1.24 \times 10^{2} \\ \sigma = 10 &\rightarrow 7.36 \times 10^{2} \end{aligned} \tag{65}$$

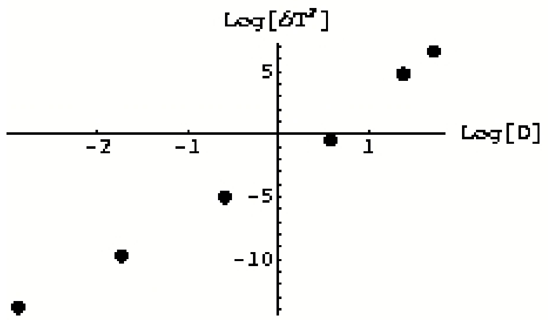

FIG. 2: Relation between $< \delta T^2 >$ and D

Fig. **??** suggest $< \delta T^2 >$ will not be quite saturated in modle of (64).

I tried another model:

$$\begin{aligned} \dot{r} &= r - r_c + d_r \\ \dot{\theta} &= \omega_0 + \frac{d_\theta}{r} \\ \dot{z} &= z + d_z\,, \end{aligned} \tag{66}$$

with sane coefficient with (65). The results are:

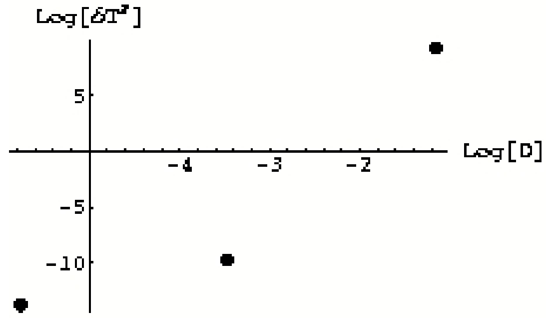

FIG. 3: Relation between $< \delta T^2 >$ and D

VIII. CONCLUSIONS AND DISCUSSION

I rederived some of basic equations on the dynamics of fast dynamo. Also, I was able to see the upper of growth rate becomes the topological entropy. From (62), we can see that the leading term is coming from solely the velocity along the periodic orbits. The effect of diffusion on the transverse direction contribute higher order terms, which is not obvious. But the condition of periodic orbit and isotropic diffusion reduces the effect of diffusion on transverse direction.

However, there are still many problems are remaining. It is not sure whether or not we can use Ott's formulation of noise in general cases. In general, the Poincaré section away from the orbit is not calculable. Here we assumed transverse plane of a point of periodic orbit is Poincaré section, but it is possible it is highly deformed to make it difficult to determine δT_p. Also, if the diffusion is large, our formula will not work. So it would be important to set a limit where my work remains valid. Also there are still lots of ambiguities left for the dynamics on transverse plane. In real dynamo, the diffusion will not be isotropic and the situation gets much more complicated.

Acknowledgments

I would like to thank Predrag Cvitanović and Domenico Lippolis for numerous helpful suggestions. This work is part of the course Special Problem PHYS 8901 Spring 2007 by P. Cvitanović.

I enjoyed working on this problem. This field was quite new and I had to swallow many books and papers, because there is not a standard book summarizing all these. I started working on ref. [1] in about midway through this semester. It seems Ott has great intuition. I tried to make it clear somehow mathematically, but not quite satisfied yet. There was a lot to consider to make it rigorous, and something new to take into account popped up continuously. But this might be why this problem is interesting and it was my great pleasure to discuss those with Predrag.

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APPENDIX A: BAKER'S MAP

Folded baker's map has the form

$$(x,y) \rightarrow \begin{cases} (2x, \frac{y}{2}) & (0 \leq x < \frac{1}{2}) \\ (2-2x, 1-\frac{y}{2}) & (\frac{1}{2} \leq x < 1) \end{cases} \tag{A1}$$

This is a representation of a complete cancellation of flux following stretching. A stacked baker's map is defined as

$$(x,y) \rightarrow \begin{cases} (2x, \frac{y}{2}) & (0 \leq x < \frac{1}{2}) \\ (2x-1, \frac{1+y}{2}) & (\frac{1}{2} \leq x < 1) \end{cases} \tag{A2}$$

and is 2 dimensional model for exponential growth of field. One application of map doubles the embedded flux.

A mixing map M has the property that sub volume become uniformly spread over whole volume D. An area preserving map M is mixing if

$$\mu(M^n U \cap V) \rightarrow \mu(U)\mu(V) \tag{A3}$$

as $n \rightarrow \infty$ where μ is measure to represent area for any two sub domain U and V in D. For any continuous function $f(x,y)$ defined on the unit square, the map M is ergodic if [20]

$$\lim_{N\rightarrow\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(M^n(x,y)) = \int_D f(x,y)dxdy \tag{A4}$$

for almost all points (x,y) in the unit square. Ergodicity is equivalent to the absence of sets invariant under M having area other than 0 or 1. If M is a mixing map, setting U and V as a invariant set under M gives

$$\mu(M^n U \cap U) = \mu(U) \rightarrow \mu(U)^2 \tag{A5}$$

The area invariant set is either 0 or 1 thus a mixing map is ergodic.

APPENDIX B: LAGRANGIAN CHAOS

When solution of the equation describing the trajectory of fluid elements,

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x}(t),t) \tag{B1}$$

are chaotic, it is called Lagrangian chaos. It is also called as chaotic advection. Chaotic velocity field implies that velocity of an individual particle $\frac{d\mathbf{x}}{dt}$ is not integrable. To produce a chaotic advection, the Eulerian velocity which is velocity field at a spatially fixed point is not necessarily turbulent. Chaotic advection can be made even when Eulerian velocity is periodic in time.

Due to chaos, nearby trajectory typically separates exponentially with Lyapunov coefficient λ

$$\lambda = \lim_{t\rightarrow\infty} \lim_{\delta_0\rightarrow 0} \frac{1}{t} ln \frac{\delta(t)}{\delta_0} \tag{B2}$$

where $\delta(t) = |\mathbf{x}_1(t) - \mathbf{x}_2(t)|$. As an element of fluid gets elongated, the thickness of filament is decreasing exponentially $exp(-\lambda t)$. The thinning process is stopped at a diffusion scale $\sqrt{\lambda t}$.

A periodic orbit is a solution (x,T), $x \in \mathbb{R}^d$, $T \in \mathbb{R}$ of the *periodic orbit condition*

$$f^T(x) = x\,, \qquad T > 0 \tag{B3}$$

for a given flow or discrete time mapping $x \mapsto f^t(x)$. Our goal is to determine periodic orbits of flows defined by first order ODEs

$$\frac{dx}{dt} = v(x)\,, \qquad x \in \mathcal{M} \subset \mathbb{R}^d\,, \qquad (x,v) \in \mathbf{T}\mathcal{M} \tag{B4}$$

in d dimensions. Here $\mathcal{M}$ is the phase space (or state space) in which evolution takes place, and the vector field $v(x)$ is smooth (sufficiently differentiable) almost everywhere.