Following along a technique for handling Plane Couette Flow[1]

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This document tries to fill in the gaps in the mathematical formulation of Homotopy of exact coherent structures in plane shear flows by Fabian Waleffe. It goes from Navier-Stokes to an expansion appropriate for this system.

Starting from the basic Navier-Stokes equation:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{v} + \mathbf{F}, \nabla \cdot \mathbf{v} = 0 \tag{1}$$

x will be the streamwise direction, y the wall-normal direction and z the spanwise. The velocity is expanded as a perturbation about a mean velocity (the laminar solution) $\mathbf{v} = y\hat{\mathbf{x}} \equiv \mathbf{U}_L^{(C)}$. The corresponding velocity perturbation components are u,v,w. To eliminate the pressure term, we define two operators, called the "roll-streak" projections:

$$\mathbf{P}_v = -\hat{\mathbf{y}} \cdot \nabla \times (\nabla \times (\cdot)) \tag{2}$$

$$\mathbf{P}_\eta = \hat{\mathbf{y}} \cdot \nabla \times (\cdot) \tag{3}$$

We assert that the velocity may be decomposed (called the poloidal-toroidal expansion) as

$$\mathbf{v} = \nabla \times (\nabla \times \phi\hat{\mathbf{y}}) + \nabla \times \psi\hat{\mathbf{y}} + \overline{U}\hat{\mathbf{x}} + \overline{W}\hat{\mathbf{z}} \tag{4}$$

$$v = \mathbf{v} \cdot \hat{\mathbf{y}} = -(\partial_x^2 + \partial_z^2)\phi \tag{5}$$

$$\eta \equiv \partial_z u - \partial_x w = -(\partial_x^2 + \partial_z^2)\psi \tag{6}$$

with $\overline{U}$ and $\overline{W}$ defined as the respective means of u and w over both x and z. Note the decomposition is something like $\mathbf{P_v}$, $\mathbf{P}_\eta$, and the mean flows. To show the decomposition works I'll show once v and η are known, u and w may be found from the definition of y-vorticity and incompressibility:

$$\eta = \partial_z u - \partial_x w \tag{7}$$

$$\nabla \cdot \mathbf{v} = \partial_x u + \partial_y v + \partial_z w = 0 \tag{8}$$

Taking the x and z partial derivatives and summing leaves two independant, parabolic PDEs:

$$\partial_z \eta - \partial_{xy} v = \partial_{zz} u + \partial_{xx} u \tag{9}$$

$$\partial_x \eta + \partial_{zy} v = -\partial_{xx} w - \partial_{zz} w \tag{10}$$

Now, we turn the crank on the p-t expansion:

$$u? = \partial_{xy}\phi - \partial_z \psi + \overline{U} \tag{11}$$

$$v(?) = v \tag{12}$$

$$w(?) = \partial_{zy}\phi + \partial_z \psi \tag{13}$$

So the y component is shown right away, but to see that the x and y components work apply $-(\partial_{xx} + \partial_{zz})$. With the definitions of ϕ and ψ, we get back the same set of PDEs as above. Applying $\mathbf{P}_v$ to Navier Stokes gives:

$$-\hat{\mathbf{y}} \cdot \nabla \times (\nabla \times \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right) = -\hat{\mathbf{y}} \cdot \nabla \times \left(\nabla \times \left(-\nabla p + \frac{1}{Re}\nabla^2 \mathbf{v} + \mathbf{F}\right)\right) \tag{14}$$

This allows us to eliminate the pressure and force terms. Rearranging the derivatives gives:

$$(\partial_t - \frac{1}{Re}\nabla^2)\nabla^2 v + \mathbf{P}_v \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) = 0 \tag{15}$$

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Define $\nabla \times \mathbf{v} \equiv \eta$ and apply $\mathbf{P}_\eta$ and using the identity $\nabla \times (\nabla^2 \mathbf{v}) = \nabla^2(\nabla \times \mathbf{v})$

$$\left(\frac{\partial}{\partial t} - \frac{1}{Re}\nabla^2\right)\eta + \mathbf{P}_\eta \cdot (\mathbf{v}\cdot\nabla)\mathbf{v} = 0 \tag{16}$$

Now, we apply periodic boundary conditions to the x and z directions. Taking the x component of Navier-Stokes and averaging over x and z gives:

$$\frac{1}{L_x L_z}\int_0^{L_x}\int_0^{L_z} dxdz\left(\frac{\partial u}{\partial t} + \hat{\mathbf{x}}\cdot(\mathbf{v}\cdot\nabla)\mathbf{v}\right)$$
$$= \frac{1}{L_x L_z}\int_0^{L_x}\int_0^{L_z} dxdz\left(-\frac{\partial p}{\partial x} + \frac{1}{Re}\hat{\mathbf{x}}\cdot\nabla^{\mathbf{2}}\mathbf{v} + \hat{\mathbf{x}}\cdot\mathbf{F}\right) \tag{17}$$
$$\frac{\partial\overline{U}}{\partial t} - \frac{1}{Re}\overline{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - \frac{\partial p}{\partial x}} + \overline{\hat{\mathbf{x}}\cdot(\mathbf{v}\cdot\nabla)\mathbf{v} - \hat{\mathbf{x}}\cdot\mathbf{F}} = 0 \tag{18}$$

Intergrating wrt x and z and applying the periodic boundary condition to p, $\frac{\partial u}{\partial x}$, and $\frac{\partial u}{\partial z}$ gives:

$$(\frac{\partial}{\partial t} - \frac{1}{Re}\frac{\partial^2}{\partial y^2})\overline{U} + \overline{\hat{\mathbf{x}}\cdot(\mathbf{v}\cdot\nabla)\mathbf{v} - \hat{\mathbf{x}}\cdot\mathbf{F}} = 0 \tag{19}$$

$\overline{W}$ must be zero according to symmetry (consider rotation by π about the z axis). In summary up to this point, we have:

$$\left(\partial_t - \frac{1}{Re}\nabla^2\right)\nabla^2 v + \mathbf{P}_v\cdot(\mathbf{v}\cdot\nabla\mathbf{v}) = 0 \tag{20}$$
$$\left(\frac{\partial}{\partial t} - \frac{1}{Re}\nabla^2\right)\eta + \mathbf{P}_\eta\cdot(\mathbf{v}\cdot\nabla)\mathbf{v} = 0 \tag{21}$$
$$\left(\frac{\partial}{\partial t} - \frac{1}{Re}\frac{\partial^2}{\partial y^2}\right)\overline{U} + \overline{\hat{\mathbf{x}}\cdot(\mathbf{v}\cdot\nabla)\mathbf{v} - \hat{\mathbf{x}}\cdot\mathbf{F}} = 0 \tag{22}$$

Next, we impose the condition that the velocity distribution may be viewed as a traveling wave perturbation. That is $\mathbf{v} = U_L^{(C)}\hat{\mathbf{x}} + \mathbf{u}$, where $\mathbf{u}(x,y,z,t) = (u,v,w) = \mathbf{u}(x - Ct, y, z, 0)$. Applying this constraint to our equations eliminates time as a degree of freedom. To embed this constraint we set $\partial_t = -C\partial_x$ Applying this to our set of equations:

$$(C\partial_x + \frac{1}{Re}\nabla^2)\nabla^2 v - \mathbf{P}_v\cdot(\mathbf{v}\cdot\nabla\mathbf{v}) = 0 \tag{23}$$
$$(C\frac{\partial}{\partial x} + \frac{1}{Re}\nabla^2)\eta - \mathbf{P}_\eta\cdot(\mathbf{v}\cdot\nabla)\mathbf{v} = 0 \tag{24}$$
$$\frac{1}{Re}\frac{d^2\overline{u}}{dy^2} - \overline{\hat{\mathbf{x}}\cdot(\mathbf{v}\cdot\nabla)\mathbf{v} + \hat{\mathbf{x}}\cdot\mathbf{F}} = 0 \tag{25}$$

Note that in the third equation $\partial_x\overline{U} = 0$ since it is a function of y. Similarly the partial derivative wrt y becomes a full derivative since u is averaged over x, eliminating t-dependance. $\overline{U}$ becomes $\overline{u}$, since the laminar background has $\frac{d^2 U_L}{dy^2} = 0$. Fixing the phase of this wave, $\eta \sin\frac{2\pi x}{L_x} = 0$, will yield a unique solution.

To integrate this we expand in Fourier modes in the x,z directions and Chebyshev-based modes in the y:

$$v = \sum_{l=-L_T}^{L_T} \sum_{m=0}^{M_T} \sum_{n=-N_T}^{N_T} A_{lmn} e^{il\alpha x} e^{in\gamma z} \phi_m(y) \tag{26}$$

$$\eta = \sum_{l=-L_T}^{L_T} \sum_{m=0}^{M_T} \sum_{n=-N_T}^{N_T} B_{lmn} e^{il\alpha x} e^{in\gamma z} \psi_m(y) \tag{27}$$

$$\overline{u} = \sum_{m=0}^{M_T} \hat{u}_m \psi_m(y) \tag{28}$$

$$\hat{u}_m = \frac{1}{c_k} \int_{-1}^{1} u(y) T_k(y) (1-y^2)^{-1/2} dy \tag{29}$$

$$c_k = \begin{cases} \pi & \text{if } k = 0, \\ \pi/2 & \text{if } k \neq 0 \end{cases} \tag{30}$$

Where $D^4\phi_m(y) = T_m(y)$, $D^2\psi_m(y) = T_m(y)$ with $D \equiv d/dy$ and $T_m(y) = \cos m \arccos y$, the mth degree Chebyshev polynomial. The purpose of doing it this way as opposed to using the usual Chebyshev expansion is to allow matching of boundary conditions.

[1] Fabian Waleffe. Homotopy of exact coherent structures in plane shear flows. *Physics of Fluids*, 15(6):1517, 2003.