Reducing continuous symmetries with linear slices

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Abstract

When a dynamical system has a continuous symmetry, it is possible to exploit this symmetry to reduce the system to an equivalent simpler system. One method for doing this is Cartan's method of slices. In this paper we investigate how the method of slices can be applied to linear subspaces. There are two main obstacles to using a subspace for the method of slices: the slice has to intersect every group orbit in order to be valid for the entire state space and the method of slices potentially introduces singularities into the flow. We show that any point in the state space can be rotated into these linear subspaces, guaranteeing they can be used for the entire state space and that singularities introduced into the system by the method of slices correspond to simple jumps in the reduced space and do not cause any actual difficulties. Throughout this paper we focus on SO(2) symmetries, using the complex Lorenz equations as a simple example. In addition we show that if the symmetry is a product of SO(2) symmetries acting on distinct coordinates of the state space, then it is sufficient to consider each SO(2) action independently.

1 Introduction

Many papers have been written about using symmetries to help understand different fluid flows. This has been done for fluid flows such as the Kuramoto-Sivashinsky flow [1, 2], the plane-couette flow [3, 4, 5, 6], and a pressure driven flow through a cylindrical pipe [7, 8]. These are relatively simple systems that experience turbulent flow. The hope is that understanding these simpler systems will provide a better understanding of turbulence.

Rotational and translational symmetries appear in many fluid flows. An example is a cylindrical pipe [7, 8]. If you rotate the pipe around its central axis it does not change the system. Any flow through the pipe can be rotated around the central axis and have the resulting flow still be valid. These rotations of the pipe are an example of a continuous symmetry.

Several methods exist for exploiting a continuous symmetry of a system. The most prominent method is to rewrite the system in terms of a Hilbert basis for the symmetry. The state space coordinates are replaced by polynomials that are invariant under the action of the symmetry group. The polynomials are chosen to form a basis for the space of all invariant polynomials but are related by relations called syzygies [9]. Hilbert bases work very well for low dimensional systems, but the number of basis polynomials and the difficulty to calculate them greatly increase with the dimension of the system, making it infeasible to calculate them for the high dimensional (possibly infinite dimensional) flows encountered in fluid systems. Cartan's method of slices [10] provides a less computationally intensive method for these high dimensional systems. In the method of slices the system is replaced with an equivalent system on a subspace of the state space. This method is already being employed to reduce symmetries in various fluid flows [11, 12, 13].

One simple choice of the subspace is a hyperplane. This paper will investigate the use of hyperplanes for the method of slices with an emphasis on SO(2) symmetries since these appear in many fluid flows (specifically the Kuramoto-Sivashinsky, plane-couette, and pipe flows mentioned earlier). These linear slices have already been used in ref. [11, 12, 13] to reduce the dynamics of fluid flows. There are two main obstacles to being able to use a subspace for the method of slices: every point must be rotatable into the slice and the method of slices introduces singularities into the flow. Locally any point is rotatable into a linear slice, and ref. [12, 13] demonstrate that this is true globally for certain state spaces. This paper provides a more general proof that is applicable to more state spaces than that of ref. [12, 13].

The second difficulty with the method of slices is that it can introduce singularities into the flow. One method of handling this to do as in ref. [11] and choose the subspace so that these singularities do not occur. This can lead to complicated subspaces being used for the symmetry reduction. While linear slices will in general experience these singularities, we show that in the case of a SO(2) symmetry, and in general for 'well-behaved' symmetries, these singularities correspond to nothing more than an instantaneous jump in the trajectory in the linear slice that is easily calculated.

Many fluid flows have symmetry groups which are the product of SO(2) symmetry groups (in particular the Kuramoto-Sivashinsky, plane-couette, and cylindrical pipe flows) acting on distinct coordinates of the state space. We demonstrate that when using linear slices for these symmetry groups it is sufficient to consider each SO(2) action separately.

Throughout this paper the complex Lorenz equations [14] will be used as an example because they posses a simple SO(2) symmetry on which the method of slices can be done. The state space of the complex Lorenz equations is only 5-dimensional, but the method is the same regardless of dimension and to gain an understand of how it works it suffices to use this simpler system.

Working with symmetries requires some background and the beginning part of this paper is devoted to building up the knowledge necessary to understand what a symmetry is and how it can be used. We follow this by proving some important results for linear slices and then close with a short section on how the method of slices can be used to find invariants of a symmetry. This paper draws heavily from the teachings of Chaosbook.org [9] and makes use of its notation.

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2 Dynamical system

We begin with a very brief discussion of dynamical systems.

A deterministic dynamical system is an abstract space $\mathcal{M}$, called the state space, along with a rule for how trajectories in the space progress in time. We are concerned with applications to fluid flows, and as such we will only be concerned with systems where M is a manifold (locally it looks like $\mathbb{R}^d$) and the rule is given by a system of differential equations $\dot{x} = v(x)$.

Equilibria and periodic orbits trajectories are ubiquitous in dynamical system. These types of trajectories are well understood and provide insight into how other trajectories progress through state space [9]. Finding these types of trajectories are at the core of current research in turbulent systems. Many papers, such as [15, 16, 17], are dedicated to finding and exploiting these trajectories to better understand turbulence.

Definition 2.1 ***Equilibria.*** *An equilibrium* E *is a point* x_{E} *for which the velocity field of an ordinary differential equation* $\dot{x} = v(x)$ *is zero,*

$$v(x_{\mathrm{E}}) = 0. \tag{1}$$

The trajectory in the state space will remain in this point indefinitely.

Definition 2.2 ***Periodic orbit.*** *A periodic orbit is a trajectory* $x(\tau)$ *that is a closed curve,*

$$x(0) = x(\tau^*) \textit{ for some time } \tau^*. \tag{2}$$

The smallest τ^* *for which this occurs is known as the period.*

2.1 Linear stability

When studying the trajectories of a flow, it is useful to know how small neighborhoods of points are transported by the flow.

Consider the displacement of an infinitesimally close neighbor $x + \delta x$. Taylor expanding the flow equation $\dot{x} = v(x)$ we find that

$$\dot{x} + \dot{\delta x} = v_i(x + \delta x) \approx v_i(x) + A\delta x \tag{3}$$

where we shall refer to the matrix of velocity gradients

$$A_{ij}(x) = \frac{\partial v_i(x)}{\partial x_j} \tag{4}$$

as the stability matrix.

As explained in ChaosBook.org [9], a stability matrix describes the instantaneous rate of shearing of the infinitesimal neighborhood of $x(t)$ by the flow. That is, it describes how quickly points initially very near to $x(t)$ will diverge away from / converge to it in time. This is generally used in conjunction with equilibrium to study how trajectories close to the equilibrium progress in time.

2.2 Complex Lorenz equations

The complex Lorenz equations were introduced by Gibbon and McGuinness [14] as a low-dimensional model of baroclinic instability in the atmosphere. In the complex form, they are given by

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= (r - z)x - ay \\ \dot{z} &= \frac{1}{2}(xy^* + x^* y) - bz\end{aligned} \tag{5}$$

where x, y, $r = r_1 + i\,r_2$, $a = 1 + i\,e$ are complex and z, b, σ are real. Rewritten in terms of real variables $x = x_1 + i\,x_2$, $y = y_1 + i\,y_2$, the complex Lorenz equations are a 5-dimensional first order ODE system [11]

$$\begin{aligned}\dot{x}_1 &= -\sigma x_1 + \sigma y_1 \\ \dot{x}_2 &= -\sigma x_2 + \sigma y_2 \\ \dot{y}_1 &= (r_1 - z)x_1 - r_2 x_2 - y_1 - e y_2 \\ \dot{y}_2 &= r_2 x_1 + (r_1 - z)x_2 + e y_1 - y_2 \\ \dot{z} &= -bz + x_1 y_1 + x_2 y_2\,.\end{aligned} \tag{6}$$

In all numerical calculations that follow we shall set the parameters to Siminos's values [11],

$$r_1 = 28,\ b = \frac{8}{3},\ \sigma = 10,\ e = \frac{1}{10}, \quad \text{and} \quad r_2 = 0\,. \tag{7}$$

The complex Lorenz equations exhibit an SO(2) symmetry, see example 3.2, of rotations around the z-axis.

The complex Lorenz equations have a low enough dimension that it is possible to calculate a Hilbert basis for the symmetry. Siminos [11] and ChaosBook [9] use the Hilbert basis,

$$\begin{aligned}u_1 &= x_1^2 + x_2^2\,, \\ u_2 &= y_1^2 + y_2^2\,, \\ u_3 &= x_1 y_2 - x_2 y_1\,, \\ u_4 &= x_1 y_1 + x_2 y_2\,, \\ u_5 &= z\,.\end{aligned} \tag{8}$$

Here we are not interested in the physical applications of these equations; rather, we use them as a simple example of a dynamical system with continuous (but no discrete) symmetries. Our goal is to find a computationally straightforward method of reducing the dynamics to a lower-dimensional state space, where each group orbit of the full system (i.e., set of rotationally equivalent states) is represented by a single point. If successful, the methods that we develop might be applicable to very high-dimensional flows, such as translationally equivariant fluid flows bounded by pipes or planes [4, 18].

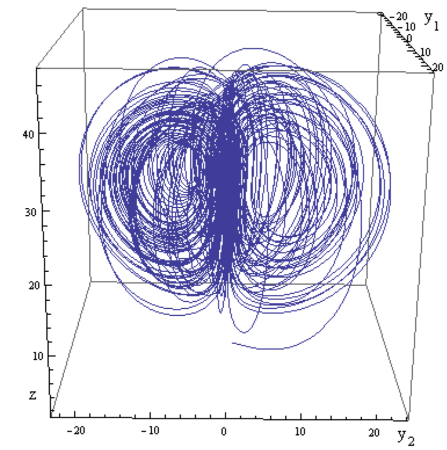

Figure 1: A typical $\{y_2, y_1, z\}$ plot of the complex Lorenz flow strange attractor, with initial point $(x_1, x_2, y_1, y_2, z) = (1, 2, 3, 1, 2)$. Notice how the trajectory appears to rotate around the z-axis.

3 Symmetries of dynamics

Before we can investigate 'symmetries' of a dynamical system, we must first develop a working definition of a what a 'symmetry' is.

We begin by defining the notion of 'equivariance.' A flow $\dot{x} = v(x)$ is *equivariant* under an operation g if

$$g\, v(x) = v(g\, x)\,. \tag{9}$$

If $x(\tau)$ is a solution to the dynamical equations, then this implies $g\, x(\tau))$ is also a solution.

The equivariant operations of dynamical system form a group under composition, and it this group that we call the symmetry of the dynamics.

In many flows (the complex Lorenz equations are a particularly simple example), this symmetry group will form a Lie group. When the symmetry group is a Lie group, Cartan's method of slices (see sect. 4.1) can be applied to the system to replace it with an equivalent lower dimensional system.

3.1 Lie groups

A theory of Lie groups is a vast subject. This report follows the notational conventions of ChaosBook.org [9]. We found Roger Penrose [19] introduction to the subject both enjoyable and understandable.

A Lie group is a group with some additional convenient properties: (1) it is a differential manifold and (2) the composition map $G \times G \to G : (g, h) \to gh^{-1}$ is $\mathbb{C}^{\infty}$.

To avoid unnecessarily complex actions of the Lie group on the state space, we will only be considering Lie groups which are compact. Researchers studying the Kuramoto-Sivashinsky equations [1, 2, 11], plane-couette flow [3, 4, 5, 6], and cylindrical pipe flow [7, 8] all impose periodic boundary conditions to enforce that the symmetry group be compact.

An element of a compact Lie group can be parameterized in exponential form [11, 9]. For example, an element of a compact Lie group that is continuously connected to the identity can be expressed as

$$g(\theta) = e^{\theta \cdot \mathbf{T}}\,, \tag{10}$$

where $\theta \cdot \mathbf{T} = \sum_a \theta_a \mathbf{T}_a$ is a *Lie algebra* element, the θ_a are the parameters of the transformation, and the $\mathbf{T}_a$ are a set of N linearly independent $[d \times d]$ antihermitian matrices acting linearly on the state space [9].

Later on we will be working with trajectories of Lie group elements, making it necessary to have a notion of an infinitesimal rotation. A rotation by an infinitesimal amount, $|\delta\theta| \ll 1$, can be expressed as [9]

$$g(\delta\theta) \simeq 1 + \delta\theta \cdot \mathbf{T}. \tag{11}$$

The $\mathbf{T}_a$ from (10) are called the *generators* of infinitesimal rotations. To see why, define the group action tangent at x,

$$t_a(x) = \mathbf{T}_a x\,, \qquad a = 1, 2, \cdots, N, \tag{12}$$

and consider a transformation induced by an infinitesimal time-dependent variation of group 'phases' $\delta\theta_a = \delta\tau\,\dot{\theta}_a$,

$$\delta x = \delta\tau\,\dot{\theta} \cdot t(x)\,.$$

So $\dot{\theta} \cdot t(x)$ is the velocity of the flow along the group orbit of x. We shall use $t_a(y)$ notation (rather than $\mathbf{T}_a y$) to emphasize that the group action induces a *tangent field* at y.

The statement of equivariance $\dot{x} = g^{-1} v(gx)$ for infinitesimal rotations is:

$$\dot{x} = (1 - \theta \cdot \mathbf{T}) v(x + \theta \cdot \mathbf{T}x) = v(x) - \theta \cdot \left(\mathbf{T} v(x) - \frac{dv}{dx} \mathbf{T} x \right)\,.$$

We can now state the *infinitesimal rotations* version of the equivariance condition (9) as:

$$0 = -t_a(v) + A t_a(x)\,, \tag{13}$$

where A is the stability matrix (4).

When dealing with symmetry groups, certain subsets of the state space play an important role in understanding the action of the group.

Definition 3.1 ***Fixed-point subspace.*** *$\mathcal{M}_H$ or a 'centralizer' of a subgroup $H \subset G$, G a symmetry of dynamics, is the set of all state space points left* H-fixed*,* point-wise *invariant under action of the subgroup*

$$\mathcal{M}_H = \mathrm{Fix}\,(H) = \{x \in \mathcal{M} : h\,x = x \textit{ for all } h \in H\}\,. \tag{14}$$

Points in the fixed-point subspace *$\mathcal{M}_G$ are fixed points of the full group action. They are called* invariant points*,*

$$\mathcal{M}_G = \mathrm{Fix}\,(G) = \{x \in \mathcal{M} : g\,x = x \textit{ for all } g \in G\}\,. \tag{15}$$

If a point is an invariant point of the symmetry group, the definition of equivariance (9) tells us that the velocity at that point is also in $\mathcal{M}_G$, so the trajectory through that point will remain in $\mathcal{M}_G$. $\mathcal{M}_G$ is disjoint from the rest of the state space since no trajectory can ever enter or leave $\mathcal{M}_G$. The fixed-point subspace of the SO(2) symmetry group of the complex Lorenz equations is the z-axis (see example 3.2). The velocity(6) at a point in the state space points only in the z-direction and so the trajectory remains on the z-axis for all time as expected.

Definition 3.2 ***Group orbit.*** *The orbit of a point x under the group G is the set of all points that x are mapped to under the groups elements*

$$\mathcal{M}_x = \{g\,x : g \in G\}. \tag{16}$$

The points in $\mathcal{M}_G$ are exactly those points whose group orbits consist of only itself ($\mathcal{M}_x = \{x\}$).

Definition 3.3 ***SO*(2).** *The special orthogonal group, $SO(2)$, is a group of length-preserving rotations of the state space. 'Special' refers to requirement that $\det g = 1$, in contradistinction to the orthogonal group $O(n)$ which allows for $\det g = \pm 1$. $SO(2)$ is a compact Lie group with a single infinitesimal generator.*

SO(2) symmetries are common amongst fluid flows. The Kuramoto-Sivashinsky flow [1, 2], plane-couette flow [3, 4, 5, 6], and flow through a cylindrical pipe [7, 8] all have symmetry groups which are products (see definition 5.1) of SO(2) symmetries along with a discrete symmetry.

Example 3.1 ***SO*(2) *irreducible representations.*** *Expand a smooth periodic function $u(\theta + 2\pi) = u(\theta)$ as a Fourier series*

$$u(\theta) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos m\theta + b_m \sin m\theta)\,. \tag{17}$$

The matrix representation of the $SO(2)$ action on the mth Fourier coefficient pair (a_m, b_m) is

$$g^{(m)}(\theta') = \begin{pmatrix} \cos m\theta' & \sin m\theta' \\ -\sin m\theta' & \cos m\theta' \end{pmatrix} = \cos m\theta'\, \mathbb{1}^{(m)} + \sin m\theta'\, \frac{1}{m}\mathbf{T}^{(m)}\,, \tag{18}$$

with the Lie group generator

$$\mathbf{T}^{(m)} = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}. \tag{19}$$

$\mathbf{T}^{(m)}$ is the Lie algebra generator and $\mathbb{1}^{(m)}$ is the identity on the m-irreducible subspace, 0 elsewhere. The $SO(2)$ group tangent $t(u)$ to state space point $u(\theta)$ is the sum over invariant subspace contributions

$$t(u) = \sum_{m=1}^{\infty} t^{(m)}(u)\,, \qquad t^{(m)}(u) = m \begin{pmatrix} b_m \\ -a_m \end{pmatrix}. \tag{20}$$

The L^2 norm of $t(u)$ is weighted by the $SO(2)$ quadratic Casimir, $C_2^{(m)} = m^2$,

$$\oint \frac{d\theta}{2\pi} \left(\mathbf{T}u(\theta)\right)^T \mathbf{T}u(2\pi - \theta) = \sum_{m=1}^{\infty} m^2 \left(a_m^2 + b_m^2\right)\,, \tag{21}$$

and converges only for sufficiently smooth $u(\theta)$. What does that mean? $\mathbf{T}$ generates translations, and by (19) the velocity of the mth Fourier mode is m times higher than for the $m = 1$ component. If $|u^{(m)}|$ does not fall off faster the $1/m$, the action of $SO(2)$ is overwhelmed by the high Fourier modes.

Example 3.2 ***SO*(2) rotations for complex Lorenz equations.** *The $SO(2)$ symmetry group of complex Lorenz equations acts on the 5-dimensional space (6) by a finite angle $SO(2)$ rotation:*

$$g(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (22)$$

The corresponding Lie algebra generator is

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (23)$$

The action of $SO(2)$ on the complex Lorenz equations state space thus decomposes into $m = 0$ G-invariant subspace (z-axis) and $m = 1$ subspace with multiplicity 2.

The generator $\mathbf{T}$ is indeed anti-hermitian, $\mathbf{T}^\dagger = -\mathbf{T}$, and the group is compact, its elements parameterized by θ mod 2π. Locally, at $x \in \mathcal{M}$, the infinitesimal action of the group is given by the group tangent field $t(x) = \mathbf{T}x = (x_2, -x_1, y_2, -y_1, 0)$. In other words, the flow induced by the group action is normal to the radial direction in the (x_1, x_2) and (y_1, y_2) planes, while the z-axis is left invariant.

3.2 Relative equilibria and relative periodic orbits

The existence of a continuous symmetry provides for the existence of trajectories analogous to equilibria 2.1 and periodic orbits 2.2.

Definition 3.4 ***Relative equilibria.*** *A relative equilibrium [9] is a trajectory which stays in a single group orbit,*

$$x(\tau) = g(\tau)x(0). \quad (24)$$

The velocity field at every point along a relative equilibrium must point in the same direction as group tangent of the flow. It does so with a constant 'angular' velocity c [9], i.e. $v(x) = c \cdot t(x)$ everywhere along the relative equilibrium. Up to the action of the group, a relative equilibrium is the same as an equilibrium.

Definition 3.5 ***Relative periodic orbit.*** *A relative periodic orbit is trajectory that periodically returns to a point on the group orbit of its initial point*

$$x(\tau*) = g\,x(0)\ for\ some\ g \in G\ and\ time\ \tau*. \quad (25)$$

Up to the action of the group, a relative periodic orbit is the same as a periodic orbit.

The method of slices is used to produce a system equivalent to the original of lower dimension. The relative equilibria and relative periodic orbits of the original system are mapped to equilibria and periodic orbits in the reduced system. We can then apply the same wealth of knowledge to studying these trajectories in the reduced state space that we have for equilibria and periodic orbits [9].

4 Reduced state space

Now that we know what a symmetry of a dynamical system is, how can it actually be used?

The basic idea behind symmetry reduction is to define an equivalence relation on the state space where two points are equivalent if they are in the same group orbit of the symmetry group. With any trajectory in the full state space $x(\tau)$, we can associate with it a new 'reduced state space' trajectory, $\bar{y}(\tau)$, which is the equivalence class of the full trajectory at any time (i.e. $x(\tau) \in \bar{y}(\tau)$ for each τ). Knowing only the reduced state space trajectory, it is not possible to reconstruct the full space trajectory; additional information is needed to determine which point of the group orbit is in the original trajectory.

If a unique representative from an equivalence class is chosen, then every point in the equivalence class is expressible as a group element acting on this point. If a unique representative, denoted y, is chosen from each equivalence class, then to recover the full space trajectory all that is needed is $g(\tau)$, the group element needed to rotate the representative $y(\tau)$ of the equivalence class to $x(\tau)$ (i.e. $x(\tau) = g(\tau)y(\tau)$).

4.1 Method of slices

Here we describe symmetry reduction by the *method of slices* of Cartan [10, 20, 21, 22].

The method of slices is when the reduced state space representative of a group orbit equivalence class is chosen to be the point where the group orbit intersects a fixed hypersurface (the representatives are where the hypersurface 'slices' the group orbits). We begin by describing how the method works near a point in the state space.

Definition 4.1 ***Slice.*** *If G acts 'regularly' on a d-dimensional manifold $\mathcal{M}$, i.e. all its group orbits are N dimensional, then we define a* slice *[10, 20, 21, 22] through a point y' to be a $(d-N)$ dimensional submanifold $\bar{\mathcal{M}}$ such that all group orbits in an open neighborhood of y' intersect $\bar{\mathcal{M}}$ transversally and only once.*

If we are working in a subspace of $\mathbb{R}^d$, then the simplest hypersurface through y' one can construct is a hyperplane. The restriction that the slice intersect the group orbits in an open neighborhood of y' transversally guarantees locally that there is a unique representative of a group orbit in the slice. The easiest way to have the hyperplane be transverse to the group orbits is to require the hyperplane be normal to the group action. Being able to choose a hyperplane normal to the group action requires that the state space have an inner product. For the rest of this paper we will assume that the state space has an inner product denoted by $\langle x|y\rangle$ (for $\mathbb{R}^d$ the inner product is the usual Euclidean product, i.e. the dot product $\langle x|y\rangle = \sum_{i=1}^{d} x_i y_i$). This leads us to the definition of a linear slice.

Definition 4.2 ***Linear slice.*** *Pick a non-zero* slice-fixing point *$y' \in \mathcal{M}$. We call the $(d-N)$-dimensional hyperplane $y \in \bar{\mathcal{M}}$ a* linear slice, *where $\langle y - y'|t'_a\rangle = 0$ is normal to all group tangents $t'_a = \mathbf{T}_a y'$ at y'. The slice-fixing point should lie outside the invariant subspace $\mathcal{M}_G$ or any of the invariant subspaces $\mathcal{M}_H$ defined in (14). Were y' invariant under the group, then $t'_a = 0$ and the 'slice' so defined would be the entire space.*

As $\langle y'|t'_a\rangle = \langle y'|\mathbf{T}_a y'\rangle = 0$ by the antihermiticity of $\mathbf{T}$, the condition that a point y lies in the slice $\bar{\mathcal{M}}$ is

$$\langle y|t'_a\rangle = 0\,, \qquad t'_a = \mathbf{T}_a y'\,. \tag{26}$$

While in general a slice need not be a hyperplane, we find the linear slice condition (26) easiest to implement. For the linear case, the same slice is fixed by any point on the ray *const* y' through the point y'.

In this paper, whenever we use the term slice it will refer to a hyperplane as in the above definition. The main focus of this paper is these linear slices.

As we shall show in sect. 4.2, a given group orbit intersects a slice at least twice, and potentially arbitrarily many times, so we need a prescription for how to pick a unique reduced state space point as the representative of the entire group orbit. We shall eliminate half of the slice crossings as we do for Poincaré sections, by fixing the orientation of the crossing. The choice of which of the remaining crossings is the representative is more arbitrary.

Definition 4.3 ***Moving frame.*** *For any x, the slice condition $y = g(\theta)x$ determines the group action $g(\theta)$ that brings x into the slice. We begin to fix the unique reduced state space by requiring that the crossing is oriented:*

$$\langle t(x)|t'\rangle > 0. \tag{27}$$

In general this will not be sufficient to select unique representative and additional constraints are required. Such a map from a point in the full state space to the group action θ is called a moving frame *[20, 21, 22].*

Example 4.1 ***Complex Lorenz equations rotation angle.*** *To show how the rotation into the slice is computed, consider first the complex Lorenz equations. There is only one infinitesimal generator for the $SO(2)$ symmetry group, so the reduced state space trajectory is given by $y = g(\theta)x$ where θ is such that $\langle y|t'\rangle = 0$. Substituting the $SO(2)$ Lie algebra generator (23) and a finite angle $SO(2)$ rotation (22) acting on a 5-dimensional space (6) into the slice condition (26) yields the explicit formula for θ:*

$$\langle x|t'\rangle \cos\theta + \langle t(x)|t'\rangle \sin\theta = 0 \tag{28}$$

$$\tan\theta = -\frac{\langle x|t'\rangle}{\langle t(x)|t'\rangle}$$

$$(x_1x_2' - x_2x_1' + y_1y_2' - y_2y_1')\cos\theta + (x_1x_1' + x_2x_2' + y_1y_1' + y_2y_2')\sin\theta = 0\,.$$

Note that if θ is a solution, so is $\theta + \pi$. If either of the inner products in (28) is nonzero then there are exactly two θ. This formula is particularly simple, as in this example the group acts only through $m = 0$ and $m = 1$ representations; in general the 'phases' θ_a have to be computed numerically.

4.2 Linear slices

For this slice to be operationally useful, we first must show that a slice (26) cuts the group orbit of every point in the full state space.

Let $x \in \mathcal{M}$ be a point in the state space, and G be Lie group with group elements represented by $g = e^{\theta\cdot\mathbf{T}}$, as in (10).

Consider $f(\theta) = \left\langle e^{\theta\cdot\mathbf{T}}x|y'\right\rangle$, the projection of the group orbit of x onto the slice-fixing ray through y' (If the state space is real, this differs by a constant from $\|e^{\theta\cdot\mathbf{T}}x - y'\|$, the function used in [12, 13] to find a point in a slice). f is a continuous and differentiable function of θ. If x is the invariant subspace (3.1), its group orbit is itself and $f(\theta)$ takes a constant value.

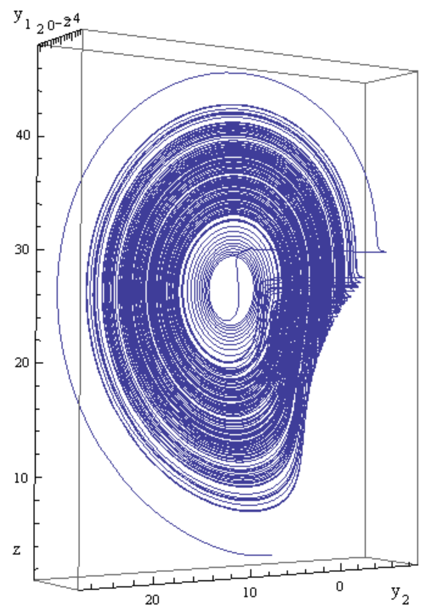

Figure 2: A typical $\{y_1, y_2, z\}$ plot of the reduced complex Lorenz flow strange attractor with initial point $(x_1, x_2, y_1, y_2, z) = (1, 2, 3, 1, 2)$ in the slice normal to $t' = (1, 0, 0, 0, 0)$. Notice that figure 1 appears to be this trajectory rotating around the z-axis.

If θ_E is an extremum of f then all of the first order partial derivatives of f vanish at θ_E, $\frac{\partial f(\theta_E)}{\partial \theta_a} = \left\langle \mathbf{T}_a e^{\theta_E \cdot \mathbf{T}} x | y' \right\rangle = 0$. $\mathbf{T}_a$ is antihermitian, so $0 = \left\langle \mathbf{T}_a e^{\theta_E \cdot \mathbf{T}} x | y' \right\rangle = -\left\langle g_E x | \mathbf{T}_a y' \right\rangle$, and

$$\langle g_E x | t'_a \rangle = 0 \,.$$

We therefore have that $y = g_E x$ is normal to all the group tangents of x so it is in $\bar{\mathcal{M}}$. This means that the slice condition is satisfied by $y_E = g_E x$ corresponding to an extremum of f, and thus y_E is in the slice (for $\mathbb{R}^n$ this is equivalent to looking for the extrema of $\|e^{\theta \cdot \mathbf{T}} x - y'\|$ [12, 13], so this says that any points that are locally the closest or the farthest from a our slice fixing point will be in the slice). All that is left to do then is to show that f has extrema. But as the group is compact, the group orbit of every point in $\mathcal{M}$ is compact, so its projection on the slice-point y' has at least two extremal points, and thus every group orbit intersects the slice. For example, group orbits of SO(2) are topologically circles, and their projections have maxima, minima and inflection points as extrema.

4.3 Dynamics in the slice

Now that we have shown these slices can be used for the entire state space, our next objective is to investigate what the equations of motion look like for the reduced state space trajectories.

The reduced state space trajectory is given by $x(\tau) = g(\tau) y(\tau)$. Differentiating both sides with respect to time and setting $u = \frac{dy}{d\tau}$ we find that,

$$v(x) = \dot{g}\, y + g\, u(y)$$

$$v(g\, y) = \dot{g}\, y + g\, u(y)$$

$$v = g^{-1}\,\dot{g}\,y + u.$$

The product of the inverse of the Lie group element and its time derivative give the group tangent at the point: $g^{-1}\dot{g} = e^{-\theta\cdot\mathbf{T}}\,\frac{d\,e^{\theta\cdot\mathbf{T}}}{d\,\tau} = \dot{\theta}\cdot\mathbf{T}$, leaving us with the equation for the velocity of the reduced flow in the slice:

$$u(y) = u(y) - \theta(y)\cdot t(y). \tag{29}$$

Equation (29) tells us that the velocity in the slice is the difference between the velocity in the full space and the portion in the direction of the group tangent. The component of the velocity in the direction of the group orbit is removed to leave only that part which is in the slice.

This equation is true for any slice, not just linear, and provides no information to calculate θ, which depends on the choice of the slice. Let t'_a be the group tangents 4.2 at the slice fixing point. When we add on the restrictions of our linear slices 4.2, we find that θ must satisfy the system of equations:

$$\langle u(y)|t'_a\rangle = v(y) - \left\langle \dot{\theta}\cdot t(y)|t'_a \right\rangle = 0 \tag{30}$$

for each group tangent t'_a at the slice fixing point. SO(2) has a single group tangent resulting in the more explicit system of equations:

$$\begin{aligned} \dot{\theta}(y) &= \frac{\langle v(y)|t'\rangle}{\langle t(y)|t'\rangle} \\ u(y) &= v(y) - \dot{\theta}(y)\cdot t(y). \end{aligned} \tag{31}$$

5 Slice singularities

Looking back at equation (31), we see that the method of slices potentially introduces singularities to the flow. When the group tangent of a point is in the slice, $\langle t(y)|t'\rangle$ is zero. Hence $\dot{\theta}(\tau)$ is not defined and as a consequence neither is the velocity in the slice. These singularities do not exist in the full space and are entirely artifacts of the method of slices. When the full space trajectory passes through one of these points the trajectory just passes through unhindered. The hope is then that these singularities do not greatly affect the trajectory in the reduced space.

5.1 Passing through a singularity

The behavior of the reduced state space trajectory is entirely determined by the θ used to rotate the full space trajectory into the slice. If we can find an expression for θ as the trajectory approaches a singularity then we know the behavior of the reduced state space trajectory as it approaches a singularity.

Suppose the trajectory passes through a singularity at time $\tau = \tau_0$. $x(\tau)$ is C^∞ since we can explicitly calculate any order derivative using $\dot{x} = v(x)$. This allows us to use Taylor expansions to approximate the trajectory at any time τ_0; $x(\tau) = x(\tau_0) + v(x(\tau_0))(\tau-\tau_0) + O((\tau-\tau_0)^2)$. Being in the linear slice imposes the condition (26) on $\theta(\tau)$ for each infinitesimal generator t'_a of the Lie group at y'. Using the results of sect. 4.2 we can rotate the trajectory in the full state space so that $x(\tau_0)$ is in the slice and the equivariance of the flow (9) allows us to work with this equivalent trajectory. This gives us the condition for θ (30) in terms of the Taylor expansion:

$$\left\langle e^{\theta\cdot\mathbf{T}}(x(\tau_0)+v(x(\tau_0))(\tau-\tau_0)+O(\tau^2))|t'_a\right\rangle$$

If in addition we know that all the higher order terms in the inner product are negligible compared to the linear term near τ_0,

$$\left\langle e^{\theta\cdot\mathbf{T}}O((\tau-\tau_0)^2)|t'_a\right\rangle << \left\langle e^{\theta\cdot\mathbf{T}}v(x(\tau))(\tau-\tau_0)|t'_a\right\rangle, \tag{32}$$

then we can take the limit of this expression as $\tau\to\tau_0$:

$$\begin{aligned}
& \left\langle e^{\theta\cdot\mathbf{T}}(x(\tau_0)+v(x(\tau_0))(\tau-\tau_0)+O(\tau^2))|t'_a\right\rangle \\
\approx\ & \left\langle e^{\theta\cdot\mathbf{T}}x(\tau_0)+e^{\theta\cdot\mathbf{T}}v(x(\tau_0))(\tau-\tau_0)|t'_a\right\rangle \\
\approx\ & \left\langle e^{\theta\cdot\mathbf{T}}v(x(\tau_0))(\tau-\tau_0)|t'_a\right\rangle \\
\approx\ & (\tau-\tau_0)\left\langle e^{\theta\cdot\mathbf{T}}v(x(\tau_0))|t'_a\right\rangle \approx 0\,.
\end{aligned} \tag{33}$$

θ approaches θ^* such that $\left\langle e^{\theta^*\cdot\mathbf{T}}v(x(\tau_0))|t(y')\right\rangle=0$ so θ approaches an angle that rotates $v(x(\tau_0))$ into the slice. Which angle it approaches before and after the singularity depend on the restrictions 4.3 put on the moving frame, and the trajectory can jump from one θ to another depending on this choice. The only affect passing through a singularity will have on the reduced state space trajectory then is to cause it jump from one point on a group orbit to another at the singularity, and the trajectory will be smooth otherwise.

Example 5.1 ***Complex Lorenz equations singularities.*** *Suppose the singularity occurs at $\tau=0$. From example 4.1 we have the equation for θ*

$$\tan(\theta) \quad = \quad -\frac{\langle x|t'\rangle}{\langle t(x)|t'\rangle}. \tag{34}$$

Rotate the trajectory so that $x(\tau_0)$ is in the slice. Using the Taylor expansion for the trajectory and letting $\tau\to 0$ we find that:

$$\begin{aligned}
\tan(\theta) &= -\frac{\langle x_0+v(x_0)\tau+O(\tau^2)|t'\rangle}{\langle t(x_0+v(x_0)\tau+O(\tau^2))|t'\rangle} \\
\tan(\theta) &\to -\frac{\langle x_0|t'\rangle+\tau\langle v(x_0)|t'\rangle+O(\tau^2)}{\langle t(x_0)|t'\rangle+\tau\langle v(x_0)|t'\rangle+O(\tau^2)} \\
\tan(\theta) &\to -\frac{\langle v(x_0)|t'\rangle}{\langle t(v(x_0))|t'\rangle}\,.
\end{aligned} \tag{35}$$

As predicted, θ approaches an angle that rotates $v(x_0)$ into the slice. Next add on the restriction $\langle t(x)|t'\rangle$ 4.3 be nonnegative, then as the trajectory approaches the singularity $\langle t(x)|t'\rangle\approx\tau\langle\mathbf{T}v^*|t'\rangle$. As τ goes from negative to positive, this expression changes sign. Hence we must rotate the trajectory by π to satisfy the condition.

Example 5.2 ***$SO(2)$ singularities.*** *Consider now the general from of an $SO(2)$ symmetry from example 3.1. Plugging this into the slice condition (26) we find that*

$$\begin{aligned}\left\langle e^{\theta\mathbf{T}}x|t(y')\right\rangle &= \left\langle x|(\sum_m(\cos(-m\theta)\,\mathbb{1}^{(m)}+\sin(-m\theta)\frac{1}{m}\mathbf{T}^{(m)}))t'\right\rangle \\ &= \sum_m(\cos(m\theta)\left\langle x|\mathbf{T}^{(m)}y'\right\rangle - \sin(m\theta)\left\langle x|\,\mathbb{1}^{(m)}y'\right\rangle). \end{aligned} \tag{36}$$

Both $\cos(m\theta)$ and $\sin(m\theta)$ are expressible as polynomials of degree m in $\sin(\theta)$ and $\cos(\theta)$, so (36) is expressible of a polynomial whose coefficients are determined by $\left\langle x|\mathbf{T}^{(m)}y'\right\rangle$ and $\left\langle x|\,\mathbb{1}^{(m)}\right\rangle$. θ corresponds to a root of this polynomial. The coefficients of the polynomial vary smoothly with x, so its roots vary smoothly too. This means (32) is satisfied for these $SO(2)$ symmetries and we can use the result of (35).

5.2 Singularities of $\mathbf{SO}(2)\times\mathbf{SO}(2)$

While most systems in fluid dynamics do not exhibit a single SO(2) symmetry, many do have continuous symmetries which are the products of SO(2) groups, each of which act on different coordinates of the state space. The Kuramoto-Sivashinsky equations [1, 2], plane-couette flow [3, 4, 5, 6], and pipe flow [7, 8] all have continuous symmetries of this form.

Definition 5.1 ***Product group.*** *If G and H are any two groups, then they can be combined in what is called the* product group *$G\times H$, whose elements are pairs (g,h), where g belongs to G, and h belongs to H, with the group multiplication rule*

$$(g_1,h_1)(g_2,h_2)=(g_1g_2,h_1h_2)\,.$$

Suppose that the state space is the product of two spaces, $\mathbb{X}=\mathbb{A}\times\mathbb{B}$. Let G be a Lie group with two infinitesimal generators, $\mathbf{T}_1$ and $\mathbf{T}_2$, such that the $e^{\theta_1\mathbf{T}_1}$ action on $\mathbb{X}$ fixes the $\mathbb{B}$ coordinates ($e^{\theta_1\mathbf{T}_1}(a,b)=(a',b')$ where $b'=b$) and $e^{\theta_2\mathbf{T}_2}$ fixes the $\mathbb{A}$ coordinates (G is the product 5.1 of two SO(2) groups which act on different coordinates of $\mathbb{X}$).

We begin by looking at the action of infinitesimal rotations of $\mathbf{T}_1$. $e^{\delta\theta_1\mathbf{T}_1}(a,b)=(1+\delta\theta_1\mathbf{T}_1)(a,b)$(11). $e^{\delta\theta\mathbf{T}_1}$ fixes the $\mathbb{B}$ coordinates, so $e^{\delta\theta\mathbf{T}_1}=(a',b)$ for some a'. This results in $(a',b)=(a,b)+\delta\theta_1\mathbf{T}_1(a,b)$ so $\delta\theta_1\mathbf{T}_1(a,b)=(a'-a,0)$. This is true for any $\delta\theta_1$, so $\mathbf{T}_1$ maps the $\mathbb{B}$ coordinates to 0. The same argument gives us that $\mathbf{T}_2$ maps the $\mathbb{A}$ coordinates to 0. Looking at $\mathbf{T}_1\mathbf{T}_2$ we find $\mathbf{T}_1\mathbf{T}_2(a,b)=\mathbf{T}_1(0,b')=(0,0)$ for any $(a,b)\in\mathbb{X}$, which is only possible if $\mathbf{T}_1\mathbf{T}_2=0$. The same argument tells us that $\mathbf{T}_2\mathbf{T}_1=0$

We can now use $\mathbf{T}_1\mathbf{T}_2=\mathbf{T}_2\mathbf{T}_1=0$ to demonstrate a significant result about the singularities of this symmetry group.

Suppose we are rotating a trajectory $x(\tau)$ into the slice normal to the group tangents at y'. Using $t'_a=t_1(y')$ for equation (30) tells us that $\langle v(y)|t_1(y')\rangle-\dot{\theta}_1\langle t_1(y)|t_1(y')\rangle-\dot{\theta}_2\langle t_2(y)|t_1(y')\rangle=0$.

We have $\langle t_2(y)|t_1(y')\rangle=0$ since $\mathbf{T}_1\mathbf{T}_2=0$, leaving us with $\langle v(y)|t_1(y')\rangle-\dot{\theta}_1\langle t_1(y)|t_1(y')\rangle=0$. This gives us the equation for $\dot{\theta}_1$,

$$\dot{\theta}_1=\frac{\langle v(y)|t_1(y')\rangle}{\langle t_1(y)|t_1(y')\rangle}. \tag{37}$$

This is the same as equation (31) for the rotation group consisting of only the rotations generated by $\mathbf{T}_1$. This equation is independent of the action of $\mathbf{T}_2$. This means a point being singular depends only on whether or not it is singular in either of the slices normal to only one of the group tangents. This permits us to break up the problem of determining if a point is singular and how this affects the reduced state space trajectory for the entire group into the same problem for each of the SO(2) groups generated by one of the infinitesimal generators individually, a situation we are comfortable with.

This makes sense since if a point y is in the slice normal to $t_2(y')$ then any point in the $\mathbf{T}_1$ orbit is in the slice since $\left\langle e^{\theta_1 \mathbf{T}_1} y | t_2(y') \right\rangle = \langle (1 + \theta_1 \mathbf{T}_1 + \ldots) y | t_2(y') \rangle = \langle y | t_2(y') \rangle + \langle \theta_1 \mathbf{T}_1 y | t_2(y') \rangle + \ldots = 0$. The first inner product is zero because we assumed that y was in the slice and the rest of the terms are zero because $\mathbf{T}_1 \mathbf{T}_2 = 0$. So the only restriction for the values of θ_1 that rotate the point into the slice are the restrictions it gets from the $t_1(y')$ condition.

This argument is easily extended to a product of arbitrarily many SO(2) acting on different coordinates of the state space (any two of the SO(2) act on distinct coordinates so the singularities are pairwise independent). In example 5.2 we found a simple description for what happens to the reduced state space trajectory as it passes through singularity of a single SO(2) symmetry group (it is rotated by a finite amount), so using this result we can handle the singularities for the product of arbitrarily many SO(2) groups which pairwise act on different coordinates of the state space.

6 Constructing invariants of the symmetry

When dealing with a system with symmetry, knowing invariants of the symmetry group can be very useful [22]. Reducing the symmetry using a slice provides a simple means of calculating invariants of a symmetry. The representatives in the slice are for entire group orbits, so if you find a formula for the representative in terms of the initial point, this equation is necessarily left invariant under the group action. Ref. [20, 21] provide a detailed explanation of how this can be done efficiently for more general symmetry groups.

Example 6.1 ***Invariants for complex Lorenz equations.*** *Consider the slice normal to the vector* $(1, 0, 0, 0, 0)$ *for the* $SO(2)$ *symmetry of the complex Lorenz equations. Using equation (28) we find that a point* (x_1, x_2, y_1, y_2, z) *will be rotated to the point*

$$(0,\, r_1,\, \frac{x_2 y_1 - x_1 y_2}{r_1},\, \frac{x_1 y_1 + x_2 y_2}{r_1},\, z)$$

where $r_1 = \sqrt{x_1^2 + x_2^2}$*. The equations for each of these coordinates are invariants of the flow. These differ only by a factor of* r_1 *from the Hilbert basis (8) of [11, 9].*

7 Conclusion

Many systems in fluid dynamics exhibit a continuous symmetry. Systems such as the Kuramoto-Sivashinsky flow [1, 2], plane-couette flow [3, 4, 5, 6], and flow through an cylindrical pipe [7, 8] demonstrate a simple product of SO(2) symmetries.

In this paper we have investigated using linear subspaces in Cartan's method of slices [10] to replace a dynamical system with an equivalent lower dimensional system. The two main obstacles to using the method of slices are that every point must be rotatable into the subspace and it can introduce singularities into the flow. Locally linear slices are guaranteed to intersect each group orbit only once, but it was shown that linear slices will intersect every group orbit of a compact Lie group in the state space, though it will do so multiple times. The method of slices can introduce singularities into the flow that did not exist in the full space. We demonstrated that as long as the group action is well behaved (which it is for any general SO(2) symmetry) then a trajectory passing through a singularity corresponds to a simple shift in the trajectory and does not cause any difficulties. In addition we demonstrated that the problem of dealing with singularities of a product of SO(2) groups acting on different coordinates of the state space (as is the case for the Kuramoto-Sivashinsky [1, 2], plane-coutte [3, 4, 5, 6], and pipe [7, 8] flows) is equivalent to dealing with the symmetries of each SO(2) symmetry independently.

Linear slices are a very simplistic condition to use for a subspace and are practical to implement. They provide a practical alternative to Hilbert bases for high dimensional flows. They also provide a computationally simple method for finding invariants of a symmetry group; what the Hilbert bases strive to do but are very costly to implement for high dimensional flows.

While we have demonstrated that linear slices can be implemented in the method of slices, work remains to be done before they can be used in more general systems. We were able to impose a simple restriction for the complex Lorenz equations [14] to choose a unique representative from the hyperplane, but unfortunately it does not generalize to more advanced systems. In addition, fluid flows, such as the plane-couette and cylindrical pipe flow, can also exhibit a discrete symmetry that the method of slices does not handle. More has to be done to reduce a system with a discrete symmetry before the method of slices can be used for the continuous symmetry.

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