

# Control of a Chaotic Double Pendulum Using the OGY Method

Matthew Cammack\*

*School of Physics*

*Georgia Institute of Technology,*

*Atlanta, GA 30332-0430, U.S.A*

(Dated: December 11, 2003)

## Abstract

This paper discusses the implementation of the OGY method of control. A double pendulum is simulated, then a fixed point of the system is found and stabilized, resulting in controlled periodic motion.

PACS numbers:

Keywords:

---

\*Electronic address: [gtg535j@prism.gatech.edu](mailto:gtg535j@prism.gatech.edu)

**Georgia Tech PHYS 7123:**  
**CHAOS, AND WHAT TO DO ABOUT IT**  
**course project, fall semester 2003**

## **I. INTRODUCTION**

The goal of this project was to take the chaotic motion of a double pendulum and control it, resulting in periodic motion. This system has points which are periodic, but they are unstable, which prevents sustained periodic motion. To allow this motion, the periodic orbit needs to be stabilized.

A method for stabilizing unstable period orbits (UPOs) is the OGY method, named for Ott, Gebogi and Yorke [3]. This method uses a small perturbation in a control parameter to move a trajectory to the UPO, if the trajectory is already sufficiently close.

This project closely followed the principles described in ref. [2]. However, ref. [2] used an experimental double pendulum, and found values experimentally. This paper discusses a simulated double pendulum, and the data needed for control are found numerically.

In II, the OGY method is discussed in more detail, and the form the control parameter should take is derived. In III, double pendulum is discussed and the equations of motion are derived. In IV, the method for finding a fixed point of the system is described. In V, the methods for determining the effect of applying the control parameter are described. In VI, the methods for determining the unstable directions of a fixed point are described. In VII, the above sections are combined to implement a control algorithm. In VIII, the results of this control algorithm are discussed.

## **II. OGY METHOD**

As described in ref. [3], the OGY Method is an approach to controlling a chaotic system that involves stabilizing a particular UPO. When a trajectory is near the UPO, the OGY method uses a perturbation to move the trajectory off the unstable manifold of the UPO and onto the stable manifold.

Very close to the UPO, one can consider a hyper-sphere around the UPO. The dynamics of the system will map this hyper-sphere into a hyper-ellipsoid. The axes of the hyper-ellipsoid

which are larger than the radius of the original hyper-sphere are the unstable directions, while those which are smaller are the stable directions. So in the immediate neighborhood of the UPO, trajectories will either be pushed away or pulled into the UPO. The goal of the OGY method is to disturb the trajectory such that it only has components in the stable directions.

This perturbation is applied by adjusting a control parameter. Changing this parameter will have the effect of shifting the UPO. The goal is to shift the UPO such that its unstable direction will push the trajectory on to a stable direction of the unshifted UPO.

In the neighborhood of the UPO, the next point of the system's Poincaré section can be approximated using the Jacobian [4],  $J$ , evaluated at the UPO.

$$X_{n+1} - X_{F(n+1)}(p_n) = J \cdot (X_n - X_{F_n}(p_n)), \quad (1)$$

where  $X_{F_n}(p_n)$  is the  $n^{\text{th}}$  point along the UPO shifted by control parameter  $p_n$  and  $p_n = p_0 + (\delta p)_n$ . The shifting of the UPO can be determined by a linear approximation

$$X_{F_n}(p_n) - X_{F_n}(p_0) = (\delta p)_n g_n, \quad (2)$$

where  $g_n = dX_{F_n}/dp_n$ .

Combining these two equations results in

$$X_{n+1} - X_{F(n+1)}(p_0) - (\delta p)_n g_{n+1} = J * (X_n - X_{F_n}(p_0) - (\delta p)_n g_n). \quad (3)$$

Then solving for  $(\delta p)_n$  gives

$$(\delta p)_n = \frac{-(X_{n+1} - X_{F(n+1)}(p_0)) + J \cdot (X_n - X_{F_n}(p_0))}{J * g_n - g_{n+1}} \quad (4)$$

This equation involves  $X_{n+1}$ , which is a future point. The goal of this control method is to reduce the projection of  $X_n$  along the unstable direction(s). So [2]

$$(X_{n+1} - X_{F(n+1)}(p_0)) \cdot f_u = (1 - \rho)(X_n - X_{F_n}(p_0)) \cdot f_u, \quad (5)$$

where  $f_u$  is a unit vector in the unstable direction (or in this project, the dominant unstable direction). Dotting (4) by  $f_u$  and substituting (5) gives the equation for the control perturbation,

$$(\delta p)_n = \frac{(-(1 - \rho)(X_n - X_{F(n)}(p_0)) + J \cdot (X_n - X_{F_n}(p_0))) \cdot f_u}{(J \cdot g_n - g_{n+1}) \cdot f_u} \quad (6)$$

### III. DOUBLE PENDULUM

The system for which the OGY method was implemented was a driven double pendulum. The double pendulum is simply one pendulum fixed to the end of another pendulum. For a driven double pendulum, a sinusoidal torque is added to the inner pendulum. In this case, the inner pendulum is modeled as a massless rod of length  $l_1$  with a mass,  $m_1$ , on the end. The outer pendulum is attached to the mass of the inner pendulum and is similarly a massless rod of length  $l_2$  and a mass of  $m_2$ .

The result is a system with 5 dimensions:  $\theta_1$  and  $p_{\theta_1}$ , the angle and angular momentum for the inner pendulum,  $\theta_2$  and  $p_{\theta_2}$ , the angle and angular momentum for the outer pendulum and  $\phi$ , the drive phase.

The non-driven double pendulum is a Hamiltonian system, with a Hamiltonian,

$$H = \frac{1}{1 - \frac{m_2}{M} \cos^2 \Delta\theta} \left( \frac{p_{\theta_1}^2}{2Ml_1^2} + \frac{p_{\theta_2}^2}{2m_2l_2^2} - \frac{p_{\theta_1}p_{\theta_2} \cos \Delta\theta}{Ml_1l_2} \right) - Mgl_1 \cos \theta_1 - m_2gl_2 \cos \theta_2, \quad (7)$$

where  $M = m_1 + m_2$  and  $\Delta\theta = \theta_1 - \theta_2$ .

The equations of motion for the driven double pendulum are those of the non-driven double pendulum with an additional term for the driving torque and terms for dissipation. Then  $\delta p$  is added to the inner pendulum's momentum equation of motion, meaning that the control parameter is small addition to the driving torque. Using  $c = \frac{1}{1 - \frac{m_2}{M} \cos^2(\Delta\theta)}$ , the equations of motion are

$$\begin{aligned} \dot{\theta}_1 &= c \left( \frac{p_{\theta_1}}{Ml_1^2} - \frac{p_{\theta_2} \cos \Delta\theta}{Ml_1l_2} \right) \\ \dot{\theta}_2 &= c \left( \frac{p_{\theta_2}}{m_2l_2^2} - \frac{p_{\theta_1} \cos \Delta\theta}{Ml_1l_2} \right) \\ \dot{p}_{\theta_1} &= \frac{\partial c}{\partial \theta_1} \left( \frac{p_{\theta_1}^2}{2Ml_1^2} + \frac{p_{\theta_2}^2}{2m_2l_2^2} - \frac{p_{\theta_1}p_{\theta_2} \cos \Delta\theta}{Ml_1l_2} \right) \\ &\quad + c \left( \frac{p_{\theta_1}p_{\theta_2} \sin \Delta\theta}{Ml_1l_2} \right) - Mgl_1 \sin \theta_1 - p_{\theta_1}/Q \\ &\quad + A \cos \phi + \delta p \\ \dot{p}_{\theta_2} &= \frac{\partial c}{\partial \theta_2} \left( \frac{p_{\theta_1}^2}{2Ml_1^2} + \frac{p_{\theta_2}^2}{2m_2l_2^2} - \frac{p_{\theta_1}p_{\theta_2} \cos \Delta\theta}{Ml_1l_2} \right) \\ &\quad - c \left( \frac{p_{\theta_1}p_{\theta_2} \sin \Delta\theta}{Ml_1l_2} \right) - m_2gl_2 \sin \theta_2 - p_{\theta_2}/Q \\ \dot{\phi} &= \omega. \end{aligned} \quad (8)$$

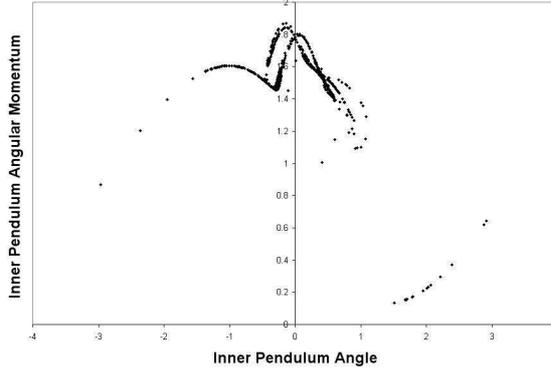


FIG. 1: Stroboscopic Poincaré section for double pendulum, projected onto  $\theta_1$  and  $p_{\theta_1}$ .

It is convenient to take  $\phi = 0$  as the condition for the Poincaré section shown in Fig. 1. This makes a stroboscopic Poincaré section, with the period of the driving torque.

#### IV. FINDING A FIXED POINT

In order to implement the OGY method, the first important step is to find a UPO to stabilize. For this case, a UPO of period 1 was wanted. Finding a fixed point is based on finding  $x$  such that

$$f^T(x) - x = 0 \quad (9)$$

with the period  $T$  being the period of the driving torque. Finding  $x$  in this case is done using the Newton-Raphson method as described in ref. [1]. This method requires the Jacobian  $J$  which can be found by integrating,

$$dJ/dt = A \cdot J, \quad J(x, 0) = \mathbf{1}, \quad (10)$$

where  $A_{ij} = \frac{\partial \dot{x}_i}{\partial x_j}$ .

With the Newton-Raphson method, for an initial guess  $x$ , a correction can be found by solving the matrix equation

$$\begin{pmatrix} 1 - J & v(x) \\ a & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta T \end{pmatrix} = \begin{pmatrix} f(x) - x \\ 0 \end{pmatrix}. \quad (11)$$

That correction is then applied to the guess, to become the new guess and so on. For a good enough initial guess, this should converge to a fixed point of the system.

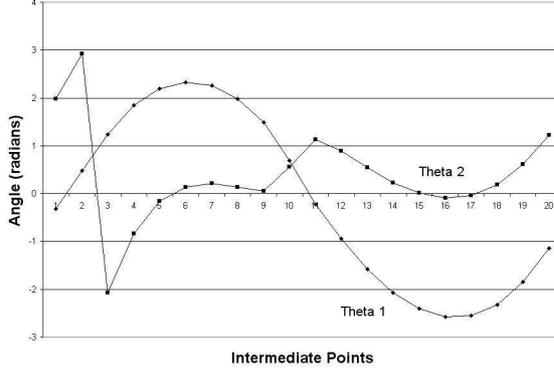


FIG. 2: This is the periodic orbit to be stabilized. This figure shows  $\theta_1$  and  $\theta_2$  over the course of one period.

In this particular system, applying the full correction led to overshooting which prevented the method from converging. The solution was to apply half the correction,  $x' = x + \delta x/2$  instead of  $x' = x + \delta x$ , basically taking smaller steps.

The fixed point found by this method, and used from here on, corresponds to fairly traditional pendulum motion for the inner pendulum, though not for the outer pendulum as in Fig. 2.

## V. EFFECT OF CONTROL PARAMETER

The previous step allows one to find both a fixed point to stabilize, and the Jacobian at that fixed point, two of the four missing pieces of information. Next is the vector  $g$  which describes how the system is affected by the control parameter.

There seem to be two ways to consider the effect of the control parameter. First, as described earlier, the control parameter can be considered to move the fixed point. In that case, finding the vector  $g$  is just a modification of the previous step. Basically apply a constant control parameter, and find the fixed point using the Newton-Raphson method. For a small control parameter, the shifted fixed point should be near the original. After finding the shifts for several values of the control parameter, the vector  $g$  can be found by a linear fit with the data.

A similar concept treats  $g$  as a vector describing how the trajectory is affected by the control parameter [2]. Then the same principle applies, as the parameter is picked to shift

the trajectory onto the stable manifold. From this point of view, finding  $g$  is based on integrating the system from the fixed point with different values for the control parameter, and then fitting them.

For this project, initially the 1st approach was taken. But when the control parameter was changed several times per period and it became necessary to have a  $g$  vector for several intermediate points, it was easier to calculate the variation of the trajectory.

The equation for the control parameter under the 2nd version is very similar [2],

$$(\delta p)_n = \frac{((1 - \rho)(X_n - X_{F(n)}) - J \cdot (X_n - X_{F(n)})) \cdot f_u}{-g_n \cdot f_u}. \quad (12)$$

## VI. UNSTABLE DIRECTIONS

The last remaining piece is to find the unstable directions of the fixed point, or at least the most unstable direction. These directions can be found from looking at the Jacobian.

In some cases, the unstable directions correspond to the eigenvectors of the Jacobian with eigenvalues greater than one. But this isn't true for this particular case.

To get the leading unstable direction, one can take a unit vector, and apply the Jacobian of the fixed point to it [4]. The resulting vector will stretch along the unstable direction. Then the vector is renormalized, which makes the result like rotation towards the unstable direction. When this is done repeatedly, the result is a good approximation to the greatest unstable direction.

Alternately, a singular value decomposition can be done to the Jacobian [2]. The column vectors of  $U$  that correspond to singular values greater than 1 are unstable direction, and the greatest unstable direction is that with the highest singular value.

Both of these methods were tried, but the singular value method was easier to use when many points were taken.

## VII. CONTROL

Rather than attempt to control the system with one control signal per period, the period was divided into intermediate sections. Specifically, the proper control signal was recalculated every 20th of a period [2]. At each interval, the control signal was chosen to reduce the unstable component of the next point in the trajectory.

To accomplish this, the system was integrated through one period, starting from the fixed point. At each interval, all of the required data was collected. The intermediate points and Jacobians were determined simply by integrating to that point in time.

The  $g$  vectors were determined by rerunning the integration of each interval with different control parameter settings. This integration was started from each intermediate point with two different control settings, and the difference was taken and scaled. This gave a rough approximation to the derivative of the flow with respect to the control parameter at each intermediate point.

The unstable direction was determined using the singular value decomposition technique previously discussed. This method was applied to each intermediate Jacobian to get the intermediate unstable directions.

With this data available, the control parameter could be updated 20 times per period. Correcting the parameter more often helps reduce the effect of the inaccuracies of this process by not allowing these errors to accumulate through a full period.

## VIII. RESULTS

The control scheme described was successfully implemented with good results. For variety of initial conditions, the control algorithm managed to capture the trajectory and hold it near the chosen fixed point. When the control was switched off, the system resumed chaotic behavior as can be seen in Fig. 3.

One major issue with this type of control is that the trajectory must pass close to the fixed point to be stabilized before the control algorithm can capture it. All of the data used to calculate the control parameter is based on local linear approximations which don't hold outside the immediate neighborhood of the fixed point in question. So unless the system is started at the fixed point, there is a delay before the trajectory passes close enough to be captured.

Another interesting effect was that for some parameters, the system settled into 2 or even 3-cycles around the intended fixed point. It's unclear if these are real UPOs of length 2 or 3 that were stabilized instead of the fixed point, or if this is an artificial side effect of the control parameter shifting the system around.

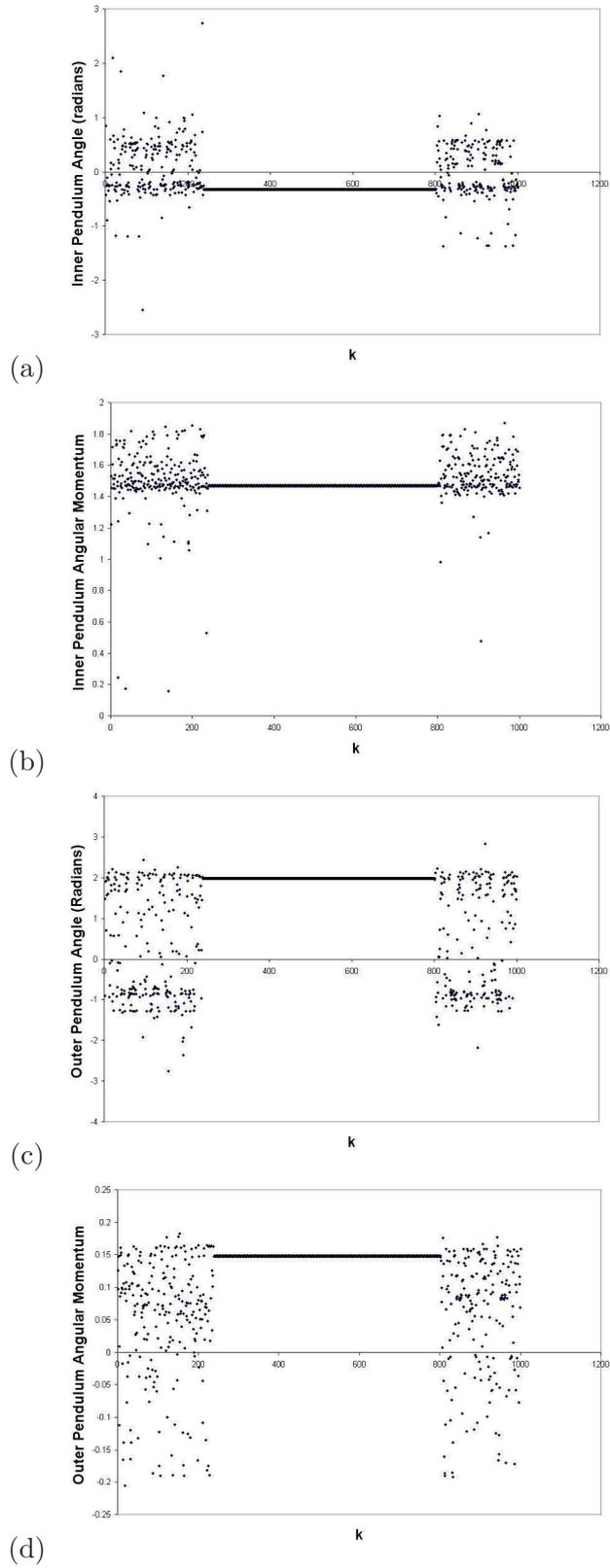


FIG. 3: a) $\theta_1$ , b) $p_{\theta_1}$ , c) $\theta_2$ , and d) $p_{\theta_2}$ . Control ends on the 800th cycle

- 
- [1] P. Cvitanović *et al.*, *Chaos: Classical and Quantum*, advanced graduate e-textbook, available online at [ChaosBook.org](http://ChaosBook.org) (Niels Bohr Institute, Copenhagen 2003).
- [2] D. Christini, J. Collins, and P. Linsay, “Experimental control of high-dimensional chaos: The driven double pendulum,” *Phys. Rev. E* **54**, 5 (1996).
- [3] E. Ott, C. Grebogi, and J.A. Yorke, “Controlling Chaos,” *Phys. Rev. Lett.* **64**, 1196 (1990).
- [4] Y.C. Lai, M. Ding and C. Grebogi, “Controlling Hamiltonian Chaos,” *Phys. Rev. E* **47**, 86 (1993).