ChaosBook.org chapter

local stability

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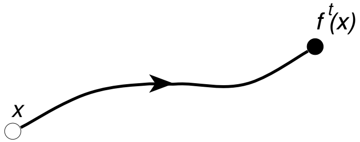
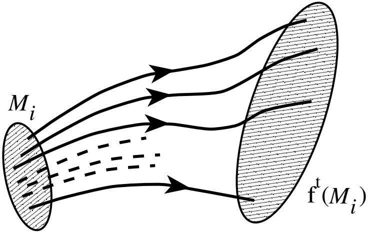
flows transport neighborhoods





so far

trajectory of a single initial point

next

transport a neighborhood

matrix of velocity gradients

flow transports displacement $x(t) + \delta x(t)$ along trajectory $x(t)$
an infinitesimal neighborhood evolves by

$$\dot{x}_i + \dot{\delta x}_i = v_i(x + \delta x) \approx v_i(x) + \sum_j \frac{\partial v_i}{\partial x_j} \delta x_j$$

together with equations of motion this yields:

equations of variations

$$\dot{x}_i = v_i(x)\,, \quad \dot{\delta x}_i = \sum_j A_{ij}(x) \delta x_j$$

stability matrix

$$A_{ij}(x) = \frac{\partial v_i(x)}{\partial x_j}$$

is the instantaneous rate of shearing of $x(t)$ neighborhood

Jacobian matrix

infinitesimal neighborhood after a finite time:

$$f_i^t(x_0 + \delta x) = f_i^t(x_0) + \sum_j \frac{\partial f_i^t(x_0)}{\partial x_{0j}} \delta x_j + \cdots \,,$$

linearized neighborhood is transported by

Jacobian matrix

$$\delta x(t) = J^t(x_0)\delta x(0)\,, \qquad J_{ij}^t(x_0) = \frac{\partial x_i(t)}{\partial x_j(0)}$$

stability of trajectories

exponential of a constant matrix

$$e^{tA} = \lim_{m\to\infty}\left(\mathbf{1}+\frac{t}{m}A\right)^m .$$

tax-accountant's discrete step definition of an exponential

local rate of neighborhood distortion $A(x)$ depends on $x(t)$

$$\begin{aligned} J^t &= \lim_{m\to\infty}\prod_{n=m}^{1}\left(\mathbf{1}+\delta t A(x_n)\right) \\ &= \lim_{m\to\infty} e^{\delta t\, A(x_n)} e^{\delta t\, A(x_{m-1})} \cdots e^{\delta t\, A(x_2)} e^{\delta t\, A(x_1)} , \end{aligned}$$

$$\delta t = (t-t_0)/m , \qquad x_n = x(t_0 + n\delta t)$$

take the $\delta t \to 0$ limit:

Jacobian matrix is the integral of stability matrix

finite time Jacobian matrix

$$J^t_{ij}(x_0) = \left[\mathbf{T} e^{\int_0^t d\tau A(x(\tau))} \right]_{ij} ,$$

where **T** stands for time-ordered integration

Jacobian matrices are multiplicative along the flow,

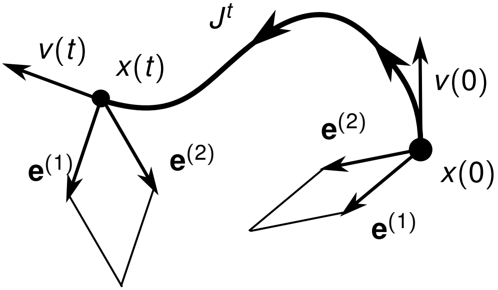
$$J^{t+t'}(x) = J^{t'}(x')J^t(x), \qquad \text{where } x' = f^t(x)$$

stability multiplier, exponent

Λ_k = kth stability multiplier, finite time Jacobian matrix M^t
λ_k = kth stability exponent

$$\Lambda_k = e^{t\lambda^{(k)}} = e^{t(\mu^{(k)}+i\omega^{(k)})} , \qquad \Lambda_k = \Lambda_k(x_0, t) , \; \lambda_k = \lambda_k(x_0, t)$$

Jacobian matrix transports local coordinate frames



computation of Jacobian matrix

d^2 matrix elements of Jacobian matrix satisfy

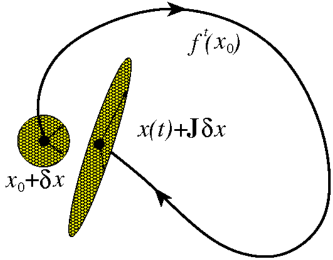
$$\frac{d}{dt} J^t(x_0) = A(x)\, J^t(x_0)\,, \quad \text{initial condition } \; J^0(x_0) = \mathbf{1}$$

evaluation requires minimal additional programming effort

extend the d-dimensional integration routine, integrate concurrently with $f^t(x)$ the d^2 elements of $J^t(x_0)$

will work for short finite times, but for exponentially unstable flows one quickly runs into numerical over- and/or underflow problems...

Jacobian matrix



Jacobian matrix maps a spherical neighborhood of x_0 into an ellipsoidal neighborhood time t later

Neighbors separate along unstable directions,
approach each other along stable directions,
creep along the marginal directions

stability of equilibria

stability matrix $A = A(x_q)$ evaluated at an equilibrium point x_q is constant

$$f^t(x) = x_q + e^{At}(x - x_q) + \cdots ,$$

$$J^t(x_q) = e^{At} \qquad A = A(x_q)$$

for a constant A the Jacobian matrix

$$x(t) = e^{tA}x(0)$$

is the solution of the linear equation

$$\dot{x} = Ax$$

so study linear flows first:

linear flows

stability multipliers, diagonal case:
if $A =$ diagonal matrix A_D with eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_d)$

$$J^t = e^{tA_D} = \begin{pmatrix} e^{t\lambda_1} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & e^{t\lambda_d} \end{pmatrix}$$

Λ_k = kth stability multiplier of the finite time Jacobian matrix J^t
λ_k = kth stability exponent

$$\Lambda_k = e^{t\lambda^{(k)}} = e^{t(\mu^{(k)} + i\omega^{(k)})}$$

complex stability multipliers

diagonal example:

Jacobian matrix J

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{t\mu} \begin{pmatrix} e^{it\omega} & 0 \\ 0 & e^{-it\omega} \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

exponent $\mu > 0$: trajectory $x(t)$ spirals out
exponent $\mu < 0$: it spirals in

frequency ω: rate of rotation

two-dimensional flows

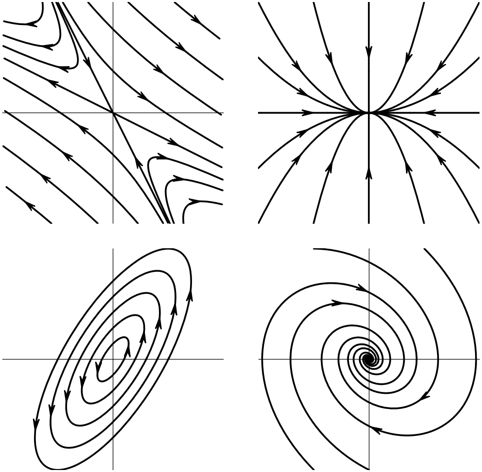
streamlines for typical 2-dimensional flows:

saddle (hyperbolic)

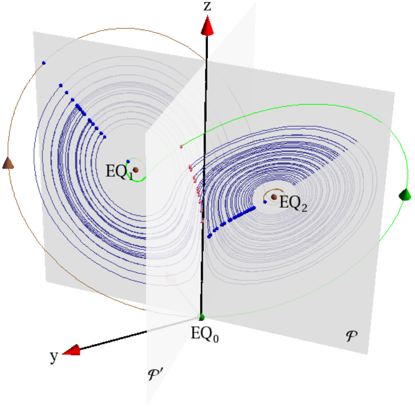
in-node (attracting)

center (elliptic)

in-spiral



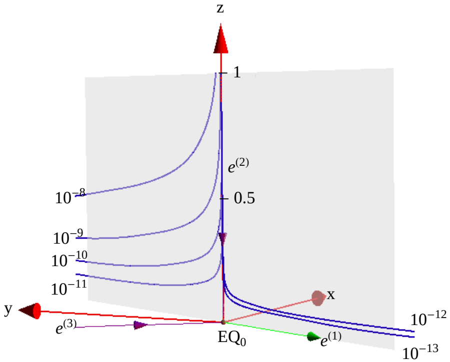
example : stability of Lorenz flow equilibria



Lorenz flow is organized by its 3 unstable equilibria

- hyperbolically unstable origin EQ_0 equilibrium
- unstable pair EQ_1 and EQ_1
  with complex spiral-out stability exponents

example : stability of hyperbolic equilibrium EQ_0



flow near the EQ_0:

unstable eigenvector $\mathbf{e}^{(1)}$,

stable eigenvectors $\mathbf{e}^{(2)}$, $\mathbf{e}^{(3)}$

note the strong $\lambda^{(1)}$ expansion: the EQ_0 equilibrium is unreachable, and the repelling $EQ_1 \to EQ_0$ heteroclinic connection never observed in simulations

complex stability multipliers

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

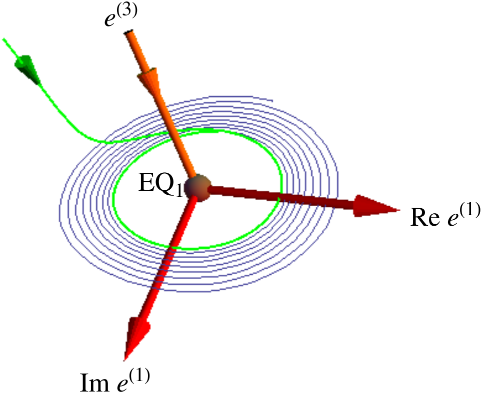
eigenvalues λ_1, λ_2 of A

$$\lambda_{1,2} = \frac{1}{2}\left(\mathrm{tr}\, A \pm \sqrt{(\mathrm{tr}\, A)^2 - 4 \det A}\right)$$

can form a complex conjugate pair

$$\lambda_1 = \mu + i\omega \,, \quad \lambda_2 = \lambda_1^* = \mu - i\omega$$

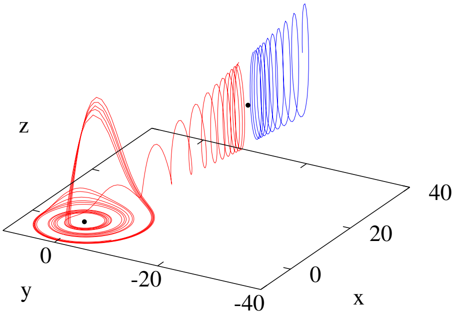
example : stability of Lorenz equilibrium EQ_1



unstable eigenplane spanned by $\mathrm{Re}\,\mathbf{e}^{(1)}$ and $\mathrm{Im}\,\mathbf{e}^{(1)}$,

stable eigenvector $\mathbf{e}^{(3)}$

example : Rössler flow equilibria



two equilibrium points
(x^-, y^-, z^-)
(x^+, y^+, z^+)

stable manifold of “$+$” equilibrium point = attraction basin boundary:

right of the “$+$” equilibrium trajectories escape,

left of the “$+$” spiral toward the “$-$” equilibrium point
$\rightarrow$ seem to wander chaotically for all times

stability of Rössler flow equilibria

linearized stability exponents

$$
\begin{aligned}
(\lambda_1^-, \mu_2^- \pm i\,\omega_2^-) &= \ (-5.686, \qquad 0.0970 \pm i\,0.9951\,) \\
(\lambda_1^+, \mu_2^+ \pm i\,\omega_2^+) &= \ (\,0.1929, \quad -4.596 \times 10^{-6} \pm i\,5.428\,)
\end{aligned}
$$

$\mu_2^- \pm i\,\omega_2^-$ eigenvectors span a plane

this plane rotates with angular period

$$T_- \approx \left|2\pi/\omega_2^-\right| = 6.313$$

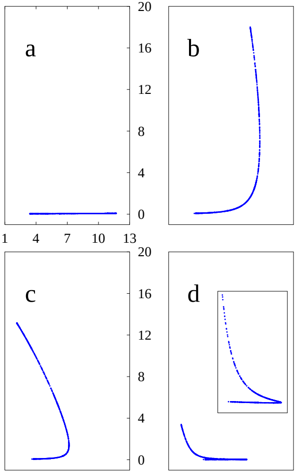
a trajectory that starts near the "−" equilibrium point spirals away per one rotation with multiplier

$$\Lambda_{\text{radial}} \approx \exp(\lambda_2^- T_-) = 1.84$$

each Poincaré section return, contracted into the stable manifold by amazing factor of $\Lambda_1 \approx \exp(\lambda_1^- T_-) = 10^{-15.6}$ (!)

start with a 1 mm interval pointing in the contracting Λ_1 eigendirection

After one Poincaré return the interval is of order of 10^{-4} fermi



Rössler Poincaré return map is in practice 1 – *dimensional*

Résumé

a neighborhood of $x(t)$ is determined by the flow linearized around $x(t)$. Nearby points aligned along the stable (contracting) directions remain in the neighborhood of the trajectory $x(t) = f^t(x_0)$;

the ones to keep an eye on are the points which leave the neighborhood along the unstable directions. The repercussion are far-reaching:

as long as the number of unstable directions is finite, the same theory applies to finite-dimensional ODEs, phase-space volume preserving Hamiltonian flows, and dissipative, volume contracting infinite-dimensional PDEs

▸ Link to full text