is space time?

a spatio-temporal theory of transitional turbulence

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overview

1. what this talk is about
2. “turbulence” in small domains
3. coupled cat maps lattice
4. space is time
5. bye bye, dynamics

R. I. P.

January 24, 2016 at 6 AM PST, Santa Barbara, CA

why did it have to die?

do clouds solve PDEs?

do clouds integrate Navier-Stokes equations?

NO!

 $\Longrightarrow$ other swirls $\Longrightarrow$ 

do clouds satisfy Navier-Stokes equations?

yes!

they satisfy them locally, everywhere and at all times

part 1

1. “turbulence” in small domains
2. coupled cat maps lattice
3. space is time
4. bye bye, dynamics

goal : go from equations to turbulence

Navier-Stokes equations

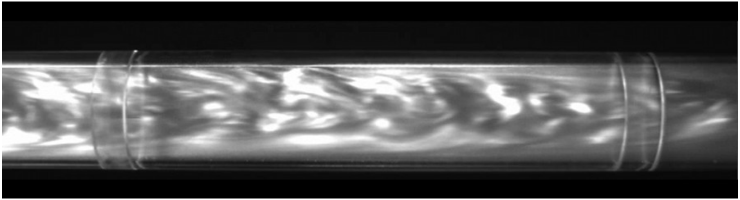
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} - \nabla p + \mathbf{f}, \qquad \nabla \cdot \mathbf{v} = 0,$$

velocity field $\mathbf{v} \in \mathbb{R}^3$; pressure field p ; driving force $\mathbf{f}$

describe turbulence

starting from the equations (no statistical assumptions)

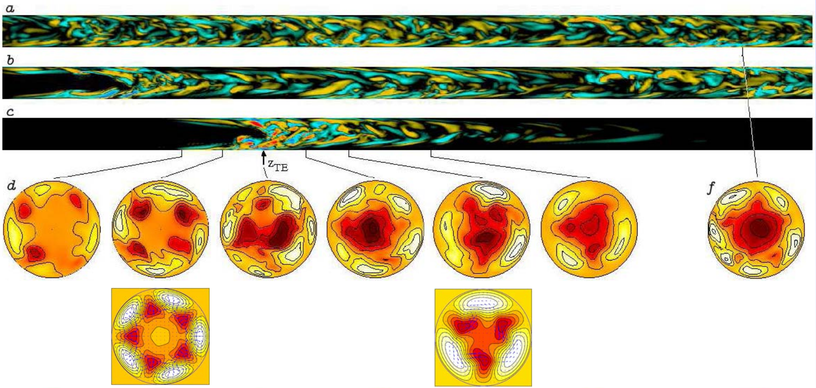
pipe experiment



T. Mullin lab

example : pipe flow

amazing data! amazing numerics!



dynamical description of turbulence

state space

a manifold $\mathcal{M} \in \mathbb{R}^d$: d numbers determine the state of the system

representative point

$x(t) \in \mathcal{M}$
a state of physical system at instant in time

integrate the equations

trajectory $x(t) = f^t(x_0)$ = representative point time t later

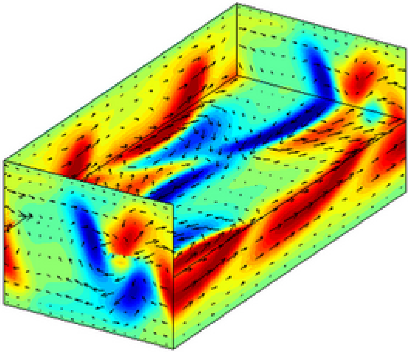
charting the state space of a turbulent flow

A long long time ago
I can still remember how
That dynamics used to make me smile
And I knew if I had my chance
That I could make those coherent structures dance

And maybe they'd be happy for a while

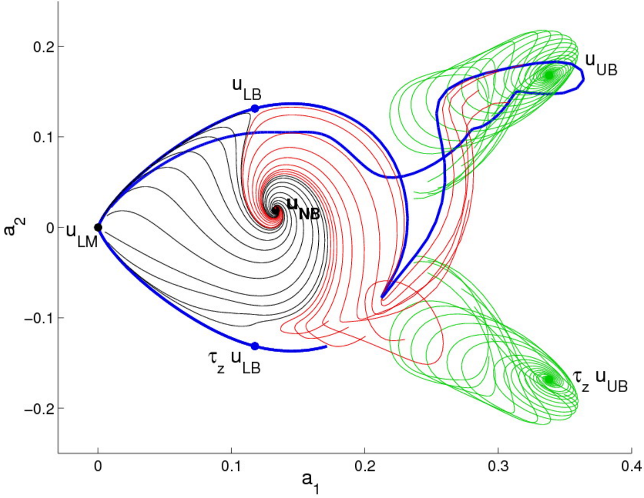
John F Gibson (U New Hampshire)
Jonathan Halcrow (Google)
P. C. (Georgia Tech)

plane Couette : so far, small computational cells



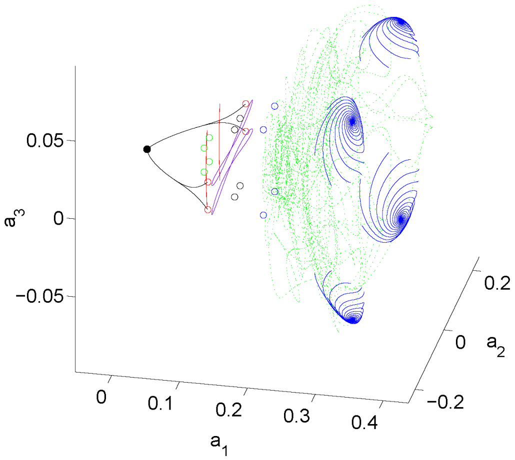
velocity field visualization

can visualize 61,506 dimensional state space of turbulent flow



equilibria of turbulent plane Couette flow,
their unstable manifolds, and
myriad of turbulent videos mapped out as one happy family

plane Couette state space $10^5 \to 3D$



equilibria, periodic orbits, their (un)stable manifolds

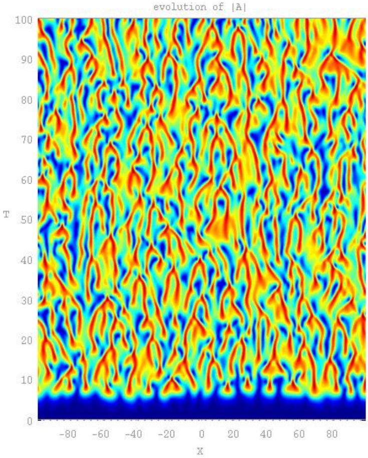
shape the turbulence

part 2

1. “turbulence” in small domains
2. coupled cat maps lattice
3. space is time
4. bye bye, dynamics

next: large space-time domains

example : complex Ginzburg-Landau on a large domain



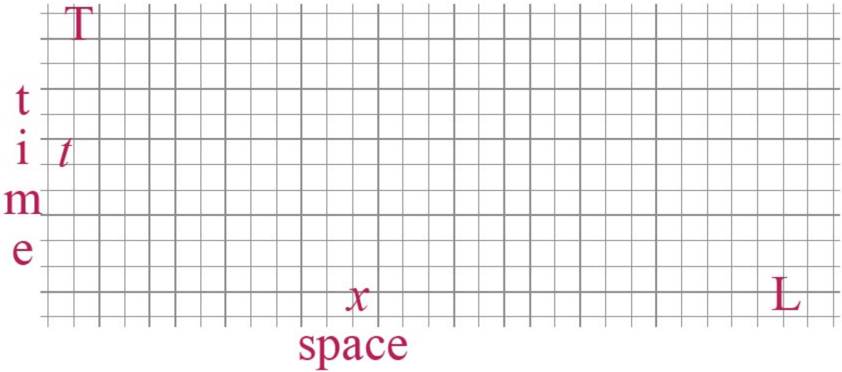
[horizontal] space $x \in [-L/2, L/2]$ [up] time evolution

codeinthehole.com/static/tutorial/coherent.html

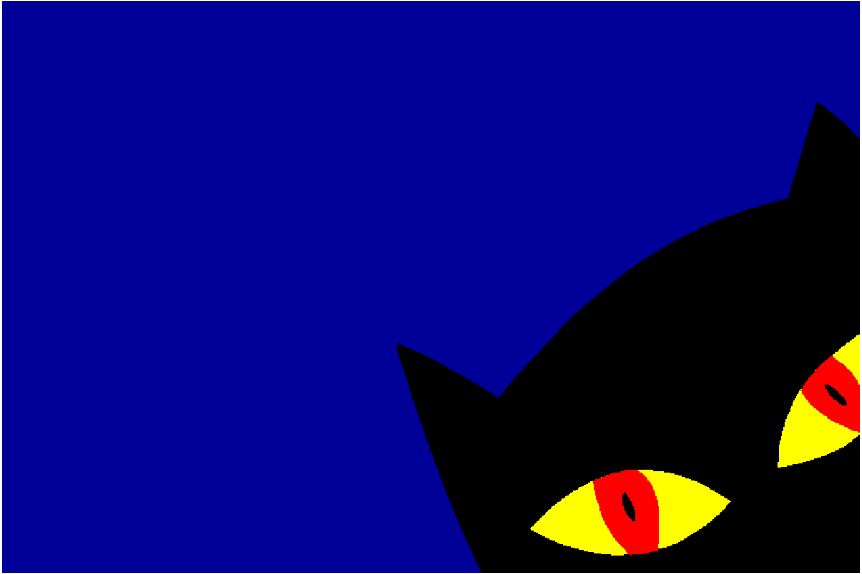
describe $(x, t) \in (-\infty, \infty) \times (-\infty, \infty)$

continuous symmetries : space, time translations

spacetime discretization

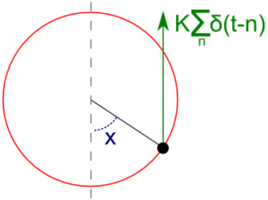


1) chaos and a single kitten



example of a "small domain dynamics" : kicked rotor

an electron circling an atom, subject to
a discrete time sequence of angle-dependent kicks $F(x_t)$



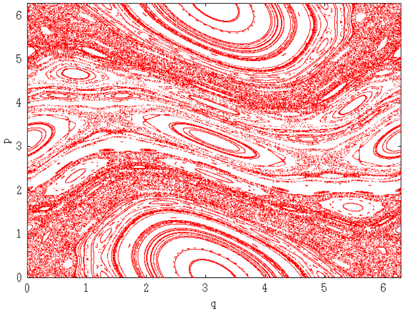
Taylor, Chirikov and Greene standard map

$$
\begin{aligned}
x_{t+1} &= x_t + p_{t+1} \qquad \text{mod } 1, \\
p_{t+1} &= p_t + F(x_t)
\end{aligned}
$$

$\rightarrow$ chaos in Hamiltonian systems

standard map

example of chaos in a Hamiltonian system



the simplest example : a single kitten in time

force $F(x) = Kx$ linear in the displacement x , $K \in \mathbb{Z}$

$$
\begin{aligned}
x_{t+1} &= x_t + p_{t+1} \qquad \mod 1 \\
p_{t+1} &= p_t + Kx_t \qquad \mod 1
\end{aligned}
$$

Continuous Automorphism of the Torus, or

Hamiltonian cat map

a linear, area preserving map of a 2-torus onto itself

$$
\begin{pmatrix} x_{t+1} \\ p_{t+1} \end{pmatrix} = A \begin{pmatrix} x_t \\ p_t \end{pmatrix} \qquad \mod 1\,, \qquad A = \begin{pmatrix} s-1 & 1 \\ s-2 & 1 \end{pmatrix}
$$

for integer $s = \operatorname{tr} A > 2$ the map is hyperbolic $\rightarrow$ a fully chaotic Hamiltonian dynamical system

cat map in Lagrangian form

replace momentum by velocity

$$p_{t+1} = (x_{t+1} - x_t)/\Delta t$$

dynamics in (x_t, x_{t-1}) state space is particularly simple

2-step difference equation

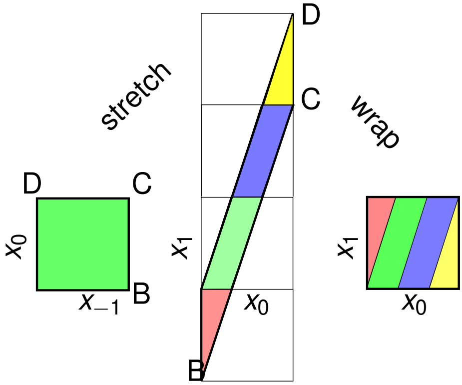
$$x_{t+1} - s\,x_t + x_{t-1} = -m_t$$

unique integer m_t ensures that
x_t lands in the unit interval at every time step t

nonlinearity : mod 1 operation, encoded in

$$m_t \in \mathcal{A}\,, \quad \mathcal{A} \;=\; \text{finite alphabet of possible values for } m_t$$

example : $s = 3$ cat map symbolic dynamics



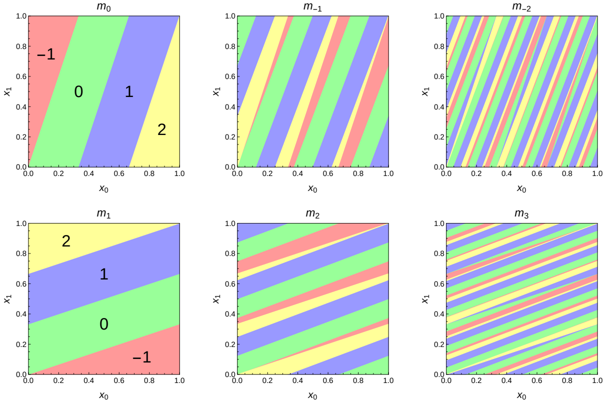
cat map stretches the unit square

translations by

$$m_0 \in \mathcal{A} = \{\underline{1}, 0, 1, 2\} = \{\text{red}, \text{green}, \text{blue}, \text{yellow}\}$$

return stray kittens back to the torus

cat map (x_0, x_1) state space partition



(a) 4 regions labeled by $m_0.$, obtained from (x_{-1}, x_0) state space by one iteration
(b) 14 regions, 2-steps past $m_{-1}m_0$. (c) 44 regions, 3-steps past $m_{-2}m_{-1}m_0$.

(d) 4 regions labeled by future $.m_1$

(e) 14 regions, 2-steps future $.m_1 m_2$ (f) 44 regions, 3-steps future block $m_3 m_2 m_1$.

2) chaos and the spatiotemporally infinite cat

spatiotemporal cat map

Consider a 1-dimensional spatial lattice, with field $x_{n,t}$ (the angle of a kicked rotor "particle" at instant t) at site n.

require
(0) each site couples to its nearest neighbors $x_{n\pm1,t}$
(1) invariance under spatial translations
(2) invariance under spatial reflections
(3) invariance under the space-time exchange

obtain

2-dimensional coupled cat map lattice

$$x_{n,t+1} + x_{n,t-1} - s\, x_{n,t} + x_{n+1,t} + x_{n-1,t} = -m_{n,t}$$

herding cats : a Euclidean field theory

convert the spatial-temporal differences to discrete derivatives

discrete d-dimensional Euclidean space-time Laplacian in $d = 1$ and $d = 2$ dimensions

$$
\begin{aligned}
\Box x_t &= x_{t+1} - 2x_t + x_{t-1} \\
\Box x_{n,t} &= x_{n,t+1} + x_{n,t-1} - 4\, x_{n,t} + x_{n+1,t} + x_{n-1,t}
\end{aligned}
$$

$\to$ the cat map equations generalized to

d-dimensional spatiotemporal cat map

$$(\Box - s + 2d)x_z = m_z$$

where $x_z \in \mathbb{T}^1\,, \quad m_z \in \mathcal{A}$ and $z \in \mathbb{Z}^d$ = lattice site label

deep insight, derived from observing kittens

an insight that applies to all coupled-map lattices, and all PDEs with translational symmetries

a *d*-dimensional spatiotemporal pattern
$\{x_z\} = \{x_z, z \in \mathbb{Z}^d\}$

is labelled by a *d-dimensional* spatiotemporal block of symbols
$\{m_z\} = \{m_z, z \in \mathbb{Z}^d\}$,

rather than a single temporal symbol sequence

(as is done when describing a small coupled few-“particle” system, or a small computational domain).

“periodic orbits” are now invariant d-tori

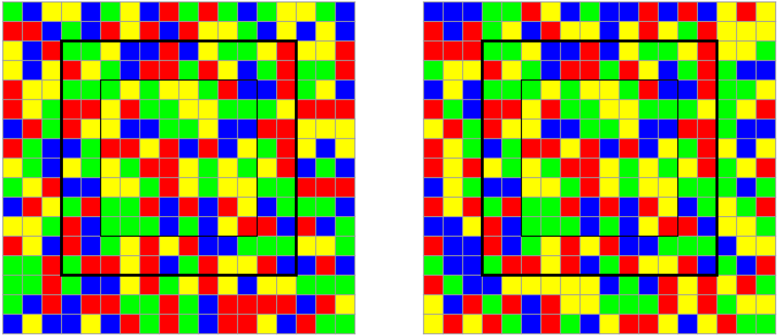
1 time, 0 space dimensions

a state space point is *periodic* if its orbit returns to it after a finite time T; in time direction such orbit tiles the time axis by infinitely many repeats

1 time, d-1 space dimensions

a state space point is *spatiotemporally periodic* if it belongs to an invariant d-torus $\mathcal{R}$, i.e., a block $\mathsf{M}_{\mathcal{R}}$ that tiles the lattice state M periodically, with period ℓ_j in jth lattice direction

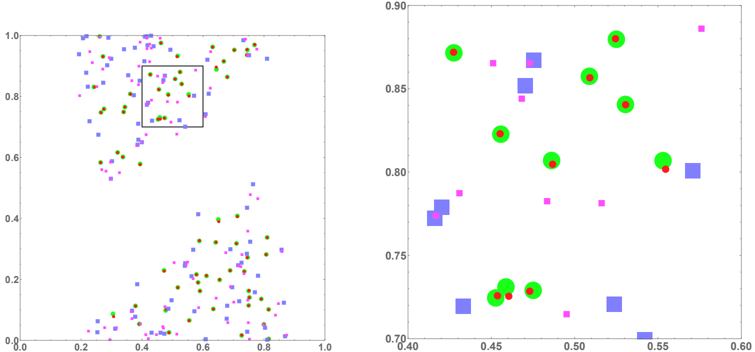
an example of invariant 2-tori : shadowing, symbolic dynamics space



2d symbolic representation of two invariant 2-tori shadowing each other within the shared block $\mathsf{M}_{\mathcal{R}} = \mathsf{M}_{\mathcal{R}_0} \cup \mathsf{M}_{\mathcal{R}_1}$ (blue)

- border $\mathcal{R}_1$ (thick black), interior $\mathcal{R}_0$ (thin black)
- symbols outside $\mathcal{R}$ differ

shadowing, state space



(left) state space points $(x_{0,t}, x_{0,t-1})$ of the two invariant 2-tori
(right) zoom into the small rectangular area
interior points $\in \mathcal{R}_0$ (large green), (small red) circles respectively
border points $\in \mathcal{R}_1$ (large violet), (small magenta) squares respectively
within the interior of the shared block, the shadowing is exponentially small

part 3

1. “turbulence” in small domains
2. coupled cat maps lattice
3. space is time
4. bye bye, dynamics

yes, lattice schmatiz, but

does it work for PDEs?

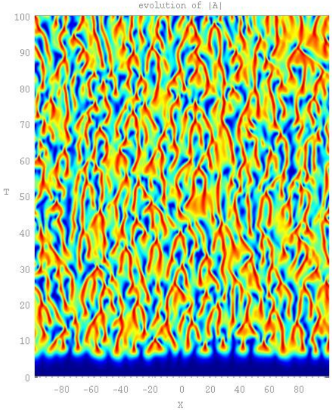
chronotope

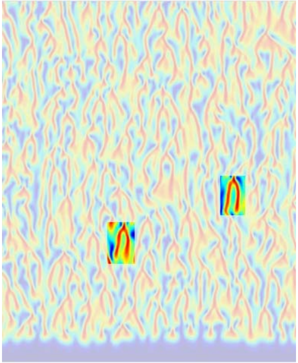
> In literary theory and philosophy of language, the chronotope is how configurations of time and space are represented in language and discourse.
>
> — Wikipedia : Chronotope

- Mikhail Mikhailovich Bakhtin (1937)
- Politi, Giacomelli, Lepri, Torcini (1996)

space-time complex Ginzburg-Landau on a large domain

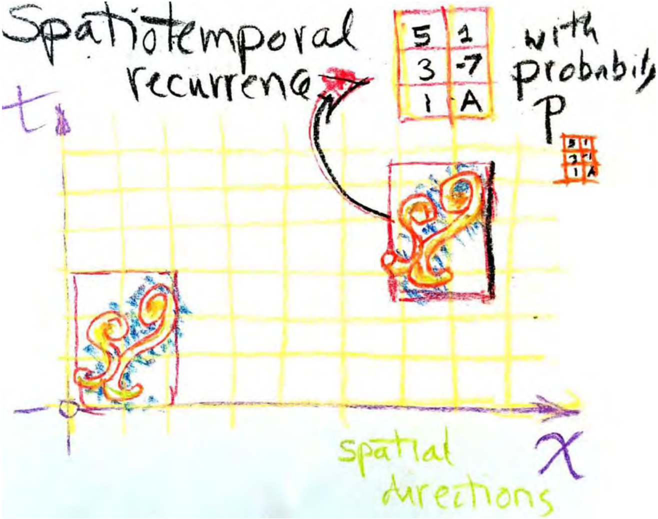
a nearly recurrent chronotope





[horizontal] space $x \in [-L/2, L/2]$ [up] time evolution

must have : 2D symbolic dynamics $\in (-\infty, \infty) \times (-\infty, \infty)$



(1+1) space-time dimensional "Navier-Stokes"

computationally not ready yet to explore the inertial manifold of $(1+3)$-dimensional turbulence - start instead with $(1+1)$-dimensional

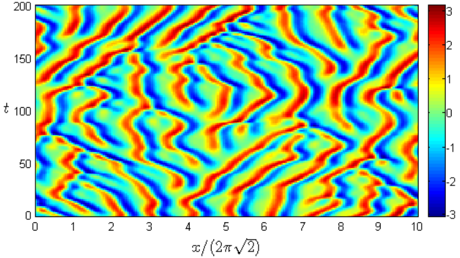
Kuramoto-Sivashinsky time evolution equation

$$u_t + u\nabla u = -\nabla^2 u - \nabla^4 u\,, \qquad x \in [-L/2, L/2]\,,$$

describes spatially extended systems such as

- flame fronts in combustion
- reaction-diffusion systems
- ...

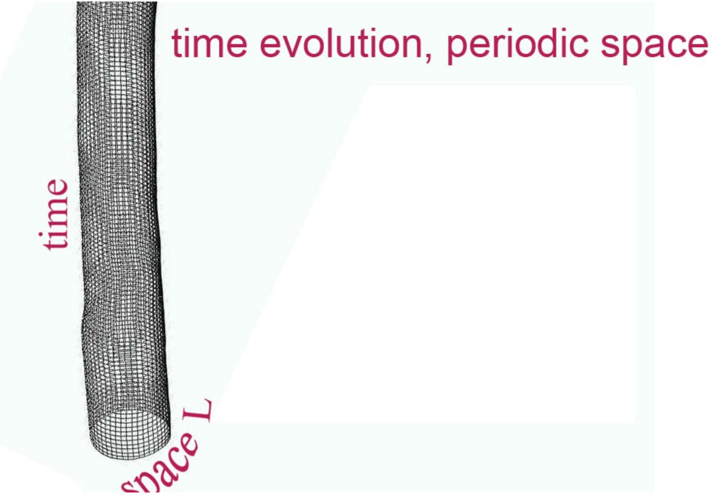
a test bed : Kuramoto-Sivashinsky on a large domain



[horizontal] space $x \in [0, L]$ [up] time evolution

- turbulent behavior
- simpler physical, mathematical and computational setting than Navier-Stokes

compact space, infinite time cylinder



so far : Navier-Stokes on compact spatial domains, all times

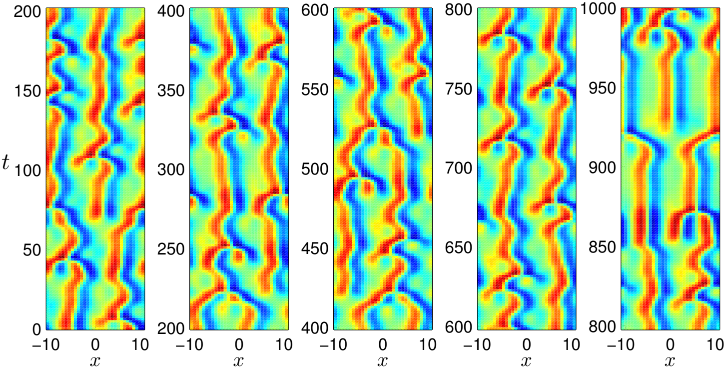
compact space, infinite time Kuramoto-Sivashinsky

in terms of discrete spatial Fourier modes

N ordinary differential equations (ODEs) in time

$$\dot{\tilde{u}}_k(t) = (q_k^2 - q_k^4)\,\tilde{u}_k(t) - i\frac{q_k}{2}\sum_{k'=0}^{N-1}\tilde{u}_{k'}(t)\tilde{u}_{k-k'}(t)\,.$$

evolution of Kuramoto-Sivashinsky on small $L = 22$ cell

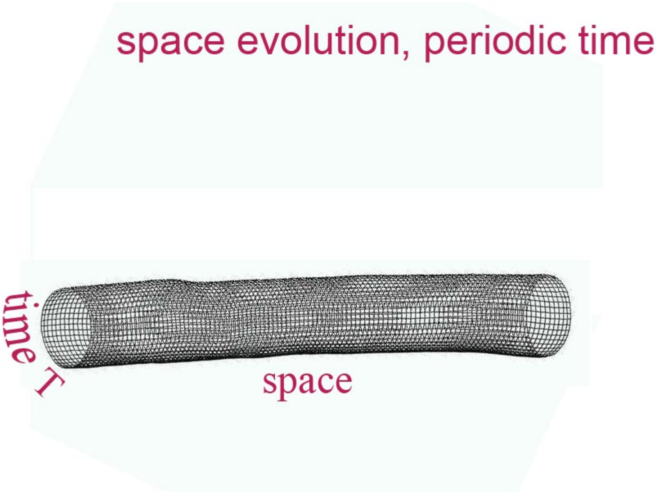


horizontal: $x \in [-11, 11]$
vertical: time
color: magnitude of $u(x, t)$

yes, but

is space time?

compact time, infinite space cylinder



compact time, infinite space Kuramoto-Sivashinsky

$$
\begin{aligned}
u_t &= -uu_x - u_{xx} - u_{xxxx}\,, \\
u^{(0)} &\equiv u\,, \quad u^{(1)} \equiv u_x\,, \quad u^{(2)} \equiv u_{xx}\,, \quad u^{(3)} \equiv u_{xxx}
\end{aligned}
$$

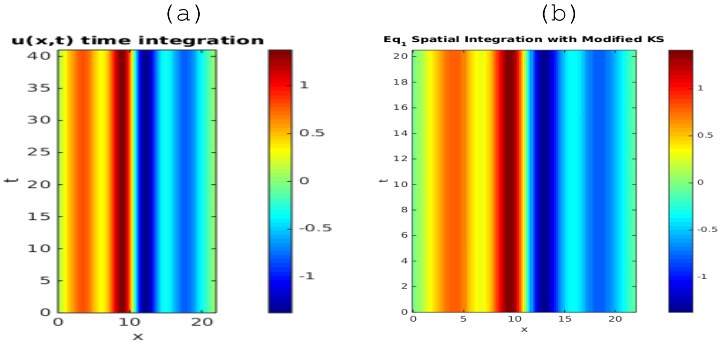
periodic boundary condition in time $u(x,t) = u(x,t+T)$

evolve $u(t,x)$ in x, 4 equations, 1st order in spatial derivatives

$$
\begin{aligned}
u_x^{(0)} &= u^{(1)}\,, \quad u_x^{(1)} = u^{(2)}\,, \quad u_x^{(2)} = u^{(3)} \\
u_x^{(3)} &= -u_t^{(0)} - u^{(2)} - u^{(0)}u^{(1)}
\end{aligned}
$$

initial values $u(x_0,t)$, $u_x(x_0,t)$, $u_{xx}(x_0,t)$, $u_{xxx}(x_0,t)$,
for all $t \in [0,T)$ at a space point x_0

a time-invariant equilibrium, spatial periodic orbit

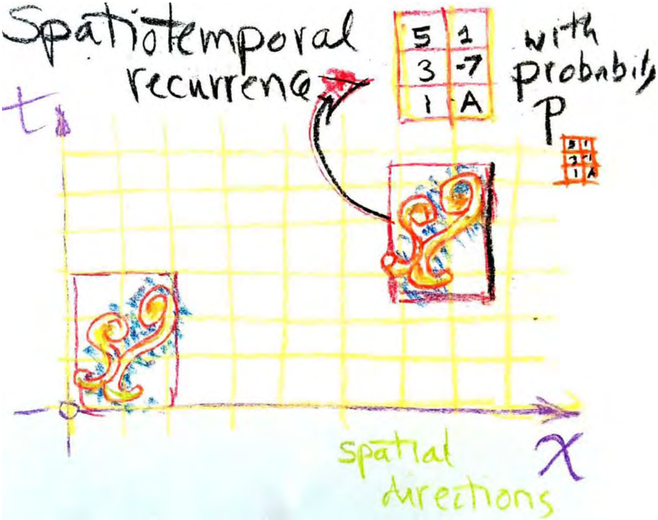


evolution of EQ_1 : (a) in time, (b) in space
initial condition for the spatial integration is the time strip $u(x_0, t)$, $t = [0, T)$, where time period $T = 0$, spatial x period is $L = 22$.

Michelson 1986

chronotope :

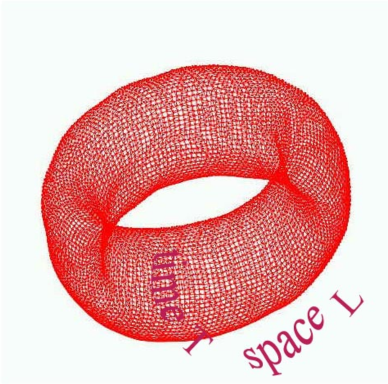
a finite $(1+D)$-dimensional symbolic dynamics rectangle



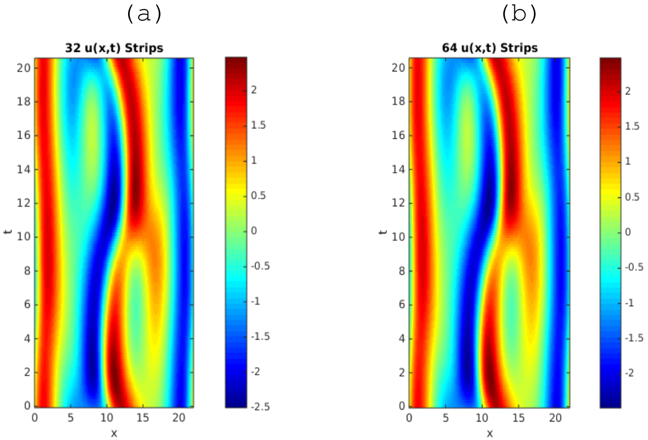
make it doubly periodic

compact space and time chronotope

periodic spacetime : 2-torus



a spacetime invariant 2-torus



(a) old : time evolution. (b) new : space evolution
$x = [0, L]$ initial condition : time periodic line $t = [0, T]$

Gudorf 2016

zeta function for a field theory ? much like Ising model

"periodic orbits" are now spacetime tilings

$$Z(s) \approx \sum_p \frac{e^{-A_p s}}{|\det(1 - J_p)|}$$

tori / spacetime tilings : each of area $A_p = L_p T_p$

symbolic dynamics : $(1 + D)$-dimensional

essential to encoded shadowing

at this time : this zeta is still but a dream

1. “turbulence” in small domains
2. coupled cat maps lattice
3. space is time
4. **bye bye, dynamics**

computing spacetime solutions

ARRIVAL

kiss your DNS codes

goodbye

for long time and/or space integrations

they never worked and could never work

life outside of time

the trouble:
forward time-integration codes too unstable

multishooting inspiration: replace a guess that a point is on the periodic orbit by a guess of the entire orbit.

an example is “Newton descent” : a variational method to drive the initial guess toward the exact solution.

$\rightarrow$

a variational method for finding spatio-temporally periodic solutions of classical field theories

compute locally, adjust globally

Computing literature : parallelizing spatiotemporal computation is FLOPs intensive, but more robust than integration forward in time

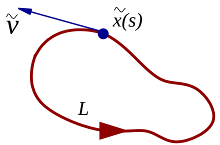
$1d$ example : variational principle for any periodic orbit

N guess points $\to \infty$ points along a smooth loop (snapshots of the pattern at successive time instants)[1]

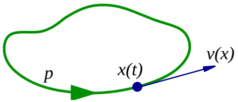
[1] Y. Lan and P. Cvitanovic', "Variational method for finding periodic orbits in a general flow," Phys. Rev. **E 69**, 016217 (2004); `nlin.CD/0308008`.

a guess loop vs. the desired solution

loop defines tangent vector $\tilde{v}$



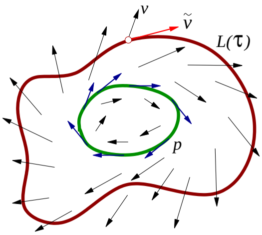
periodic orbit defined by
velocity field $v(x)$



extremal principle for a general flow

loop tangent $\tilde{v}(\tilde{x}) \neq v(\tilde{x})$

periodic orbit $\tilde{v}(\tilde{x})$, $v(\tilde{x})$ aligned



cost function

$$F^2[\tilde{x}] = \oint_L ds\,(\tilde{v} - v)^2\,; \quad \tilde{v} = \tilde{v}(\tilde{x}(s,\tau))\,,\; v = v(\tilde{x}(s,\tau))\,,$$

penalizes misorientation of the loop tangent $\tilde{v}(\tilde{x})$ relative to the true dynamical flow $v(\tilde{x})$

Newton descent

cost minimization

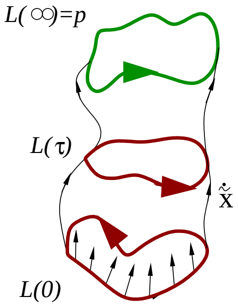
drives

initial guess $L(0)$

$\rightarrow$

cycle $p = L(\infty)$

as fictitious time $\tau \rightarrow \infty$



clouds do not solve PDEs

do clouds integrate Navier-Stokes equations?

NO!

$\Longrightarrow$ other swirls $\Longrightarrow$

at any spacetime point Navier-Stokes equations describe the local tangent space

they satisfy them locally, everywhere and at all times

summary

1. small computational domains reduce “turbulence” to “single particle” chaos
2. consider instead turbulence in infinite spatiatemporal domains
3. theory : classify all spatiotemporal tilings
4. numerics : parallelize spatiotemporal computations

there is no more time

there is only enumeration of spacetime solutions

Arrival of spacetime kitten

single kitten bonus slides

each chronotope is a fixed point

discretize $u_{n,m} = u(x_n, t_m)$ over NM points of spatiotemporal periodic lattice $x_n = nT/N$, $t_m = mT/M$, Fourier transform :

$$\tilde{u}_{k,\ell} = \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} u_{n,m} \, e^{-i(q_k x_n + \omega_\ell t_m)} \,, \quad q_k = \frac{2\pi k}{L} \,, \; \omega_\ell = \frac{2\pi \ell}{T}$$

Kuramoto-Sivashinsky is no more a PDE / ODE, but a fixed point problem of determining all invariant unstable 2-tori

$$\left[-i\omega_\ell - (q_k^2 - q_k^4) \right] \tilde{u}_{k,\ell} + i \frac{q_k}{2} \sum_{k'=0}^{N-1} \sum_{m'=0}^{M-1} \tilde{u}_{k',m'} \tilde{u}_{k-k',m-m'} = 0$$

Newton method for a NM-dimensional fixed point :
invert $1 - J$,
where J is the 2-torus Jacobian matrix, yet to be elucidated

dynamical Zeta function for a field theory

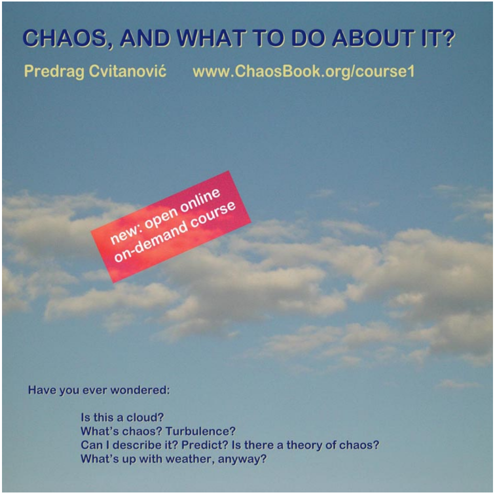
∞ of spacetime tilings

$$Z(s) \approx \sum_p \frac{e^{-A_p s}}{|\det(1 - J_p)|}$$

tori / plane tilings
each of area $A_p = L_p T_p$

trace formula for a field theory

what is next for the students of Landau's Theoretical Minimum? take the course!



student raves :

...10^6 times harder than any other online course...