ChaosBook.org chapter
noise

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Outline

1. what this chapter is about
   - knowing when to stop

2. deterministic partitions
   - idea #1: partition by periodic points

3. dynamicist’s view of noise
   - idea #2: evolve densities, not noisy trajectories
   - idea #3: for unstable directions, look back

4. optimal partition hypothesis
dynamical theory of turbulence?

dynamics of high-dimensional flows - open questions
is the dynamics like what we know from low dimensional systems?

describe the attracting ‘inertial manifold’ for Navier-Stokes?
knowing when to stop

computation of unstable periodic orbits in high-dimensional state spaces, such as Navier-Stokes,

is at the border of what is feasible numerically, and criteria to identify finite sets of the most important solutions are very much needed.

when are we to stop calculating these solutions?
knowing when to stop

need the 3D velocity field at every $(x, y, z)$!

motions of fluids: require $\infty$ bits?

numerical simulations track $10^2$ - $10^6$ of computational degrees of freedom; terabytes of data, but how much information is there in all of this?
knowing when to stop

motions of fluids: require $\infty$ bits??

that cannot be right...
knowing when to stop

Science originates from curiosity and bad eyesight.
— Bernard de Fontenelle,
Entretiens sur la Pluralité des Mondes Habités

in practice

every physical problem is coarse partitioned and finite
noise rules the state space

- any physical system experiences (some kind of) noise
- any numerical computation is ‘noisy’
- any prediction only needs a desired finite accuracy
deterministic partition

state space coarse partition

$\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2$

ternary alphabet
$A = \{1, 2, 3\}$.

$\mathcal{M}_i = \mathcal{M}_{i0} \cup \mathcal{M}_{i1} \cup \mathcal{M}_{i2}$
labeled by nine ‘words’
$\{00, 01, 02, \ldots, 21, 22\}$. 
noise limited state space partitions

noise limited cell

a resolvable neighborhood is no smaller than a ball whose radius is the noise amplitude

noise limited partition grid

state space noise-partitioned into neighborhoods indicated by their centers
**deterministic, idealized state space**

a manifold \( \mathcal{M} \in \mathbb{R}^d \): \( d \) real numbers determine the state of the system \( x \in \mathcal{M} \)

**noise-limited state space**

a ‘grid’ \( \mathcal{M}' \): \( N \) discrete states of the system \( a \in \mathcal{M}' \), one for each noise covariance ellipsoid \( \Delta_a \)
dynamics + noise: unique coarse-grained partition

reasonable to assume that the noise limits the resolution that can be attained in partitioning the state space
dynamics + noise: unique coarse-grained partition

reasonable to assume that the noise
is uniform,
leading to a uniform grid partition of the state space
reasonable to assume that the noise is uniform, leading to a uniform grid partition of the state space.

in dynamics, this is wrong!

noise has memory
dynamics + noise: unique coarse-grained partition

noise memory
accumulated noise along dynamical trajectories
always coarsens the partition nonuniformly
dynamics + noise: unique coarse-grained partition

noise memory
accumulated noise along dynamical trajectories always coarsens the partition nonuniformly

that is good, because

dynamics + noise determine the finest attainable partition
the challenge

turbulence.zip

or ‘equation assisted’ data compression:
replace the $\infty$ of turbulent videos by the best possible

small finite set

of videos encoding all physically distinct motions of the
turbulent fluid
dynamical system

state space

A manifold $\mathcal{M} \in \mathbb{R}^d$ : $d$ numbers determine the state of the system

representative point

$x(t) \in \mathcal{M}$
a state of physical system at instant in time
dynamics

map $f^t(x_0) = \text{representative point time } t \text{ later}$

evolution in time

$f^t$ maps a region $\mathcal{M}_i$ of the state space into the region $f^t(\mathcal{M}_i)$
deterministic dynamics

dynamical system

the pair $(\mathcal{M}, f)$

the problem

enumerate, classify all solutions of $(\mathcal{M}, f)$
deterministic partition into regions of similar states

1-step memory partition

\[ \mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2 \]

ternary alphabet

\[ \mathcal{A} = \{1, 2, 3\} . \]

2-step memory refinement

\[ \mathcal{M}_i = \mathcal{M}_{i0} \cup \mathcal{M}_{i1} \cup \mathcal{M}_{i2} \]
labeled by nine ‘words’

\[ \{00, 01, 02, \ldots , 21, 22\} . \]
deterministic partitions are no good

deterministic dynamics: partitioning can be arbitrarily fine
requires exponential # of exponentially small regions
Deterministic partitions are no good.

**Deterministic dynamics**: partitioning can be arbitrarily fine requires exponential # of exponentially small regions.

Yet

In practice
every physical problem must be coarse partitioned.
deterministic vs. noisy partitions

deterministic partition

can be refined

*ad infinitum*

noise blurs the boundaries

when overlapping, no further refinement of partition
periodic points instead of boundaries

- mhm, do not know how to compute boundaries...
- however, each partition contains a short periodic point smeared into a ‘cigar’ by noise
periodic points instead of boundaries

- each partition contains a short periodic point smeared into a ‘cigar’ by noise

compute the size of a noisy periodic point neighborhood!
periodic orbit partition

deterministic partition

some short periodic points:
- fixed point $\bar{1} = \{x_1\}$
- two-cycle $01 = \{x_{01}, x_1\}$

noisy partition

periodic points blurred by the noise into cigar-shaped densities
successive refinements of a deterministic partition: exponentially shrinking neighborhoods

as the periods of periodic orbits increase, the diffusion always wins:

partition stops at the finest attainable partition, beyond which the diffusive smearing exceeds the size of any deterministic subpartition.
the local diffusion rate differs from a trajectory to a trajectory, as different neighborhoods merge at different times, so

there is *no one single time* beyond which noise takes over
noisy dynamics

stochastic dynamical system

the triple \((\mathcal{M}, f, \Delta)\)

where \(\Delta(x)\) is the noise covariance matrix

the problem

enumerate, classify all solutions of \((\mathcal{M}, f, \Delta)\)

i.e., partition \(\mathcal{M} \simeq \bigcup Q_j\)

where \(Q(x_j)\) is the density covariance matrix
strategy

- use periodic orbits to partition state space
- compute local eigenfunctions of the Fokker-Planck operator to determine their neighborhoods
- done once neighborhoods overlap
some short periodic points:
fixed point $\bar{1} = \{ x_1 \}$
two-cycle $01 = \{ x_{01}, x_{10} \}$

periodic points blurred by noise into cigar-shaped densities
periodic points and their cigars

- each partition contains a short periodic point smeared into a ‘cigar’ by noise
periodic points and their cigars

- each partition contains a short periodic point smeared into a ‘cigar’ by noise
- compute the size of a noisy periodic point neighborhood!
how big is the neighborhood blurred by the accumulated noise?

the (well known) **key formula** that we now derive:

\[ Q_{n+1} = M_n Q_n M_n^T + \Delta_n \]

density covariance matrix at time \( n \): \( Q_n \)
noise covariance matrix: \( \Delta_n \)
Jacobian matrix of linearized flow: \( M_n \)

Lyapunov equation, doctoral dissertation 1892
Ornstein-Uhlenbeck 1930
Kalman filter ‘prediction’ 1960
derivation

keep things simple: illustrate by

\textit{d}-dimensional \textit{discrete time} stochastic flow

\[ x_{a+1} = f(x_a) + \xi_a \]

uncorrelated in time

\[ \langle \xi_a \rangle = 0 , \quad \langle \xi_a \cdot \xi_b \rangle = 2 d D \delta_{ab} \]

[all results apply both to the continuous and discrete time flows]
linearized deterministic flow

\[ x_{n+1} + z_{n+1} = f(x_n) + M_n z_n, \quad M_{ij} = \frac{\partial f_i}{\partial x_j} \]

In one time step, a linearized neighborhood of \( x_n \) is

(1) advected by the flow

(2) transported by the Jacobian matrix \( M_n \) into a neighborhood given by the \( M \) eigenvalues and eigenvectors
let the initial density of deviations $z$ from the deterministic center be a Gaussian whose covariance matrix is

$$Q_{jk} = \langle z_j z_k^T \rangle$$

a step later the Gaussian is advected to

$$\langle z_j z_k^T \rangle \rightarrow \langle (M z)_j (M z)_k^T \rangle$$

$$Q \rightarrow M Q M^T$$

next: add noise
in one time step
a Gaussian density distribution with covariance matrix $Q_n$ is
(1) advected by the flow
(2) smeared with additive noise
into a Gaussian ‘cigar’ whose widths and orientation are given by the singular values and vectors of $Q_{n+1}$
covariance evolution

\[ Q_{n+1} = M_n Q_n M_n^T + \Delta_n \]

(1) advect deterministically
   local density covariance matrix \( Q \rightarrow MQM^T \)

(2) add noise covariance matrix \( \Delta \)

covariances add up as sums of squares
noisy periodic orbit partition

**optimal partition hypothesis**

optimal partition: the maximal set of resolvable periodic point neighborhoods

**why care?**

if the high-dimensional flow has only a few unstable directions, the overlapping stochastic ‘cigars’ provide a *compact cover* of the noisy chaotic attractor, embedded in a state space of arbitrarily high dimension
standard normal (Gaussian) probability distribution

d-dimensional discrete time stochastic flow

\[ x' = f(x) + \xi_a \]

1-time step evolution = probability of reaching \( x' \) given random kick, Gaussian distributed \( \xi_a = x' - f(x) \)

\[
\frac{1}{\sqrt{4\pi D}} \exp \left( -\frac{\xi_a^2}{4D} \right)
\]

variance \( 2D \), standard deviation \( \sqrt{2D} \)
local Fokker-Planck operator

let

\{ \ldots, x_{-1}, x_0, x_1, x_2, \ldots \}

be a deterministic trajectory

\[ x_{a+1} = f(x_a) \]

noisy trajectory is centered on the deterministic trajectory

\[ x = x_a + z_a, \quad f_a(z_a) = f(x_a + z_a) - x_{a+1} \]

local Fokker-Planck operator

\[ \mathcal{L}_{FPA}(z_{a+1}, z_a) = \frac{1}{\sqrt{4\pi D}} \exp \left[ -\frac{(z_{a+1} - f_a(z_a))^2}{4D} \right] \]
Fokker-Planck formulation replaces individual noisy trajectories by evolution of their densities

\[
\mathcal{L}_{FP}^k(z_k, z_0) = \int [dz] e^{-\frac{1}{2} \sum_a (z_{a+1} - f_a(z_a))^T \frac{1}{\Delta} (z_{a+1} - f_a(z_a))}
\]
evolution to time $k$ is given by the $d$-dimensional path integral over the $k - 1$ intermediate noisy trajectory points

$$\mathcal{L}_{FP}^k(z_k, z_0) = \int [dz] e^{-\frac{1}{2} \sum_{a} (z_{a+1} - f_a(z_a))^T \frac{1}{\Delta} (z_{a+1} - f_a(z_a))}$$

zero mean; covariance matrix / diffusion tensor $\Delta$

$$\langle \xi_j(t_a) \rangle = 0, \quad \langle \xi_{a,i} \xi_{a,j}^T \rangle = \Delta_{ij},$$

where $\langle \cdots \rangle$ stands for ensemble average over many realizations of the noise
map $f(x_a)$ is nonlinear. Taylor expand

$$f_a(z_a) = M_a z_a + \cdots$$

approximate the noisy map by its linearized action,

$$z_{a+1} = M_a z_a + \xi_a,$$

where $M_a$ is the Jacobian matrix, $(M_a)_{ij} = \partial f(x_a)_i / \partial x_j$
$M_a$ is the Jacobian matrix, $(M_a)_{ij} = \partial f(x_a)_i / \partial x_j$

**linearized Fokker-Planck operator**

$$\mathcal{L}_{FPA}(z_{a+1}, z_a) = \frac{1}{N} e^{-\frac{1}{2}(z_{a+1} - M_za)^T \frac{1}{\Delta} (z_{a+1} - M_za)}$$

[Kalman filter ‘prediction’, WKB, semiclassical, saddlepoint, ... approximation]
linearized evolution operator maps a cigar-shaped Gaussian density distribution with covariance matrix $Q_a$ at time $a$

$$\rho_a(z_a) = \frac{1}{C_a} e^{-\frac{1}{2} z_a^T \frac{1}{Q_a} z_a}$$

into cigar

$$\rho_{a+1}(z_{a+1}) = \int dz_a \mathcal{L}_{FPa}(z_{a+1}, z_a) \rho_a(z_a)$$

one time step later
convolution of a Gaussian with a Gaussian is again a Gaussian. Integrate, obtain that

the covariance of the transported packet is given by

**evolution law for the covariance matrix** $Q_a$

$$Q_{a+1} = M_a Q_a M_a^T + \Delta_a$$
evolution law for the covariance matrix $Q_a$

$$Q_{a+1} = M_a Q_a M_a^T + \Delta_a$$

in one time step a Gaussian density distribution with covariance matrix $Q_a$ is smeared into a Gaussian ‘cigar’ whose widths and orientation are given by eigenvalues and eigenvectors of $Q_{a+1}$

(1) deterministically transported and deformed
local density covariance matrix $Q \rightarrow MQM^T$, and
(2) and noise covariance matrix $\Delta$

add up as sums of squares
iterate $Q_{a+1} = M_a Q_a M_a^T + \Delta_a$ along the trajectory

if $M$ is contracting, over time the memory of the covariance $Q_{a-n}$ of the starting density is lost, with iteration leading to the limit distribution

$$Q_a = \Delta_a + M_{a-1} \Delta_{a-1} M_{a-1}^T + M_{a-2}^2 \Delta_{a-2} (M_{a-2}^2)^T + \cdots.$$ 

diffusive dynamics of a nonlinear system is fundamentally different from Brownian motion, as the flow induces a history dependent effective noise. Always
example: noise and a single attractive fixed point

if all eigenvalues of $M$ are strictly contracting, all $|\Lambda_j| < 1$,
any initial compact measure converges to the unique invariant Gaussian measure $\rho_0(z)$ whose covariance matrix satisfies

**Lyapunov equation: time-invariant measure condition**

$$Q = MQM^T + \Delta$$

[A. M. Lyapunov doctoral dissertation 1892]
assume that \([d \times d]\) matrix \(M\) has only nonzero eigenvalues \(\{\Lambda_j\}\) and \(d\) linearly independent right and left eigenvectors (\(M\) is not defective)

\[ M \mathbf{e}^{(j)} = \Lambda_j \mathbf{e}^{(j)}, \quad \mathbf{e}^{(j)} M = \Lambda_j \mathbf{e}^{(j)} \]

eigenvectors can always be rescaled so that they are mutually orthogonal

\[ \mathbf{e}^{(j)} \cdot \mathbf{e}^{(k)} = \delta_{jk} \]
form from the $d$ column eigenvectors a $[d \times d]$ matrix

$$S = \begin{bmatrix} e^{(1)}, e^{(2)}, \cdots, e^{(d)} \end{bmatrix}, \quad MS = \Lambda S$$

by $e^{(j)} \cdot e^{(k)} = \delta_{jk}$, the matrix whose rows are left eigenvectors is then the inverse

$$S^{-1} = \begin{bmatrix} e^{(1)}, e^{(2)}, \cdots, e^{(d)} \end{bmatrix}^T$$

$S$ diagonalizes $M$ and its transpose $M^T$ by

**similarity transformation**

$$S^{-1}MS = \Lambda, \quad S^T M^T (S^{-1})^T = \Lambda$$
define $\hat{Q} = S^{-1} Q (S^{-1})^T$ and $\hat{\Delta} = S^{-1} \Delta (S^{-1})^T$

time-invariant measure condition $Q = MQM^T + \Delta$ now takes form

$$\hat{Q} - \Lambda \hat{Q} \Lambda = \hat{\Delta}$$

matrix elements are $\hat{Q}_{ij} (1 - \Lambda_i \Lambda_j) = \hat{\Delta}_{ij}$, so

$$\hat{Q}_{ij} = \frac{\hat{\Delta}_{ij}}{1 - \Lambda_i \Lambda_j}$$

and the attracting fixed point covariance matrix is given by

$$Q = S \hat{Q} S^T$$
note!
covariance matrix

\[
\hat{Q}_{ij} = \frac{\hat{\Delta}_{ij}}{1 - \Lambda_i \Lambda_j}
\]

elements must be strictly positive

true only if all Floquet multipliers (Jacobian matrix $M$ eigenvalues) are contracting, $|\Lambda_j| < 1$
determine the Jacobian matrix $M$ eigenvalues and eigenvectors

$$M e^{(j)} = \Lambda_j e^{(j)}$$

go to coordinate frame where $M$ is diagonal,

$$S^{-1}MS = \Lambda, \quad \hat{Q} = S^{-1}Q(S^{-1})^T, \quad \hat{\Delta} = S^{-1}\Delta(S^{-1})^T$$

evaluate

$$\hat{Q}_{ij} = \frac{\hat{\Delta}_{ij}}{1 - \Lambda_i \Lambda_j}$$

go back to the original coordinates

$$Q = S\hat{Q}S^T$$
a numerical diagonalization of the covariance matrix
$Q = S \hat{Q} S^T$ yields the principal axis of the equilibrium Gaussian ‘cigar’
eigenvectors of $Q$ (it is a symmetric matrix) are orthogonal and have orientations distinct from the left/right eigenvectors of the non-normal Jacobian matrix $M$
example: Ornstein-Uhlenbeck process

contracting noisy 1-dimensional map

\[ z_{n+1} = \Lambda z_n + \xi_n, \quad |\Lambda| < 1 \]

width of the natural measure concentrated at the deterministic fixed point \( z = 0 \)

\[ Q = \frac{2D}{1 - |\Lambda|^2}, \quad \rho_0(z) = \frac{1}{\sqrt{2\pi} Q} \exp \left( -\frac{z^2}{2 Q} \right), \]
example: Ornstein-Uhlenbeck process

width of the natural measure concentrated at the deterministic fixed point $z = 0$

$$Q = \frac{2D}{1 - |\Lambda|^2}, \quad \rho_0(z) = \frac{1}{\sqrt{2\pi} Q} \exp \left( -\frac{z^2}{2Q} \right)$$

- is balance between contraction by $\Lambda$ and diffusive smearing by $2D$ at each time step
- for strongly contracting $\Lambda$, the width is due to the noise only
- As $|\Lambda| \to 1$ the width diverges: the trajectories are no longer confined, but diffuse by Brownian motion
example: 2D Brusselator limit cycle

FIG. 2. Time development of distribution for Brusselator. 10000 samples of Monte Carlo simulations are plotted by the red dots along with the covariance matrix $\hat{M}$ estimated by Eq. (E7). $\hat{M}$'s are represented by the green ellipses given by $\delta x^T \hat{M}^{-1} \delta x = 4/\Omega$, where $\delta x^T = (x - x^*(t), y - y^*(t))$. The percentages of the samples that fall within the ellipses are shown in each panel. The gray curves represent the trajectory by the rate equation starting from the initial point marked by the blue circles. The system parameters are $k_1 = 0.5, k_2 = 1.5, k_3 = 1.0, k_4 = 1.0, and \Omega = 10^4$. The initial point is $(x_0^*, y_0^*) = (0.8, 2.6)$. 

Nakanishi, Sakaue, and Wakou
noisy dynamics of a nonlinear system is fundamentally different from Brownian motion, as the flow **ALWAYS** induces a local, history dependent effective noise
things fall apart, centre cannot hold

but what if $M$ has expanding Floquet multipliers?
both deterministic dynamics and noise tend to smear densities away from the fixed point: no peaked Gaussian in your future
things fall apart, centre cannot hold

but what if $M$ has \textit{expanding} Floquet multipliers?

Fokker-Planck operator is non-selfadjoint

If right eigenvector is peaked (attracting fixed point) the left eigenvector is flat (probability conservation)
to estimate the size of a noisy neighborhood of a trajectory point \( x_a \) along its *unstable* directions, we need to determine the effect of noise on the points *preceding* \( x_a \).

this is described by the *adjoint Fokker-Planck operator*

\[
\tilde{\rho}(y, k - 1) = \mathcal{L}_{\text{FP}}^\dagger \circ \tilde{\rho}(y, k) \\
= \int [dy] \exp \left\{ -\frac{1}{2} (y - f(x))^T \frac{1}{\Delta} (y - f(x)) \right\} \tilde{\rho}(y, k),
\]

carries a density concentrated around the previous point \( x_{n-1} \) to a density concentrated around \( x_n \).
things fall apart, centre cannot hold

but what if $M$ has expanding eigenvalues?

both deterministic dynamics and noise tend to smear densities away from the fixed point: no peaked Gaussian in your future
things fall apart, centre cannot hold

but what if $M$ has *expanding* eigenvalues?

look into the past, for initial peaked distribution that spreads to the present state
for unstable directions, look back

if $M$ has only *expanding* eigenvalues, balance between the two is attained by iteration from the past, and the evolution of the covariance matrix $\tilde{Q}$ is now given by

$$
\tilde{Q}_{n+1} + \Delta_n = M_n \tilde{Q}_n M_n^T,
$$

[aside to control theorists: reachability and observability Gramians]
solving the Lyapunov equation

iterate \( Q_{n+1} = M_n Q_n M_n^T + \Delta_n \)

attractive fixed point, \( Q = Q_\infty, M = M_n, Q = Q_n: \)

\[
Q = \Delta + M \Delta M^T + M^2 \Delta (M^T)^2 + \cdots = \sum_{m,n=0}^{\infty} \delta_{mn} M^n \Delta (M^T)^m
\]

bring to resolvent form, \( \delta_{mn} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(m-n)} \)

for \( M \) contracting, expanding, or hyperbolic (!)

\[
Q = \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{1 - e^{-i\theta} M} \Delta \frac{1}{1 - e^{i\theta} M^T}
\]
Cauchy magic

A similarity transformation $S$ separates the expanding and contracting subspaces

$$\Lambda \equiv S^{-1}MS = \begin{bmatrix} \Lambda_e & 0 \\ 0 & \Lambda_c \end{bmatrix}$$

transformed noise covariance matrix

$$\hat{\Delta} \equiv S^{-1}\Delta(S^{-1})^\top = \begin{bmatrix} \Delta_{ee} & \Delta_{ec} \\ \Delta_{ce} & \Delta_{cc} \end{bmatrix}$$
Cauchy magic

Contour integral representation

\[ Q = \oint \frac{ds}{2\pi i} (1 - s^{-1} M)^{-1} \Delta (1 - sM)^{-1} \]

separates \( Q \) into expanding and contracting covariances:

\[ \tilde{Q}_e \equiv S \begin{bmatrix} Q_e & 0 \\ 0 & 0 \end{bmatrix} S^\top, \quad Q_c \equiv S \begin{bmatrix} 0 & 0 \\ 0 & Q_c \end{bmatrix} S^\top \]

two stationary ‘cigars’, one in the expanding manifold and the other in the contracting manifold (not orthogonal to each other!)
local problem solved: can compute every cigar
a periodic point of period $n$ is a fixed point of $n$th iterate of dynamics

global problem solved: can compute all cigars
more algebra: can compute the noisy neighborhoods of all periodic points
optimal partition challenge

finally in position to address our challenge:

determine the finest possible partition for a given noise
evaluation of these Gaussian densities requires no Fokker-Planck PDE formalism

width of a Gaussian packet centered on a trajectory is fully specified by a deterministic computation that is already a pre-computed byproduct of the periodic orbit computations: the deterministic orbit and its linear stability
resolution of a one-dimensional chaotic repeller

As an illustration of the method, consider the chaotic repeller on the unit interval

\[ x_{n+1} = \Lambda_0 x_n (1 - x_n)(1 - bx_n) + \xi_n, \quad \Lambda_0 = 8, \ b = 0.6, \]

with noise strength \( 2D = 0.002 \)
$f_0, f_1$: branches of deterministic map

A deterministic orbit itinerary is given by the $\{f_0, f_1\}$ branches visitation sequence

[symbolic dynamics, however, is not a prerequisite for implementing the method]
‘the best possible of all partitions’ hypothesis formulated as an algorithm

- calculate the local adjoint Fokker-Planck operator eigenfunction width $Q_a$ for every unstable periodic point $x_a$
- assign one-standard deviation neighborhood $[x_a - Q_a, x_a + Q_a]$ to every unstable periodic point $x_a$
- cover the state space with neighborhoods of orbit points of higher and higher period $n_p$
- stop refining the local resolution whenever the adjacent neighborhoods of $x_a$ and $x_b$ overlap:

$$|x_a - x_b| < Q_a + Q_b$$
optimal partition, 1 dimensional map

$f_0, f_1$: branches of deterministic map

local eigenfunctions $\tilde{\rho}_a$ partition state space by neighborhoods of periodic points of period 3

neighborhoods $\mathcal{M}_{000}$ and $\mathcal{M}_{001}$ overlap, so $\mathcal{M}_{00}$ cannot be resolved further
all neighborhoods \( \{ M_{0101}, M_{0100}, \cdots \} \) of period \( n_p = 4 \) cycle points overlap, so

state space can be resolved into 7 neighborhoods

\[
\{ M_{00}, M_{011}, M_{010}, M_{110}, M_{111}, M_{101}, M_{100} \}
\]
Markov partition

evolution in time maps intervals
\( \mathcal{M}_{011} \rightarrow \{ \mathcal{M}_{110}, \mathcal{M}_{111} \} \)
\( \mathcal{M}_{00} \rightarrow \{ \mathcal{M}_{00}, \mathcal{M}_{011}, \mathcal{M}_{010} \} \), etc..

summarized by the transition graph (links correspond to elements of transition matrix \( T_{ba} \)): the regions \( b \) that can be reached from the region \( a \) in one time step
transition graph

7 nodes = 7 regions of the optimal partition

dotted links = symbol 0 (next region reached by $f_0$)

full links = symbol 1 (next region reached by $f_1$)

region labels in the nodes can be omitted, with links keeping track of the symbolic dynamics

(1) deterministic dynamics is full binary shift, but
(2) noise dynamics nontrivial and finite
predictions

escape rate and the Lyapunov exponent of the repeller are given by the leading eigenvalue of this $[7 \times 7]$ graph / transition matrix.

tests: numerical results are consistent with the full Fokker-Planck PDE simulations.
what is novel?

- we have shown how to compute the **locally optimal partition**, for a given dynamical system and given noise, in terms of local eigenfunctions of the forward-backward actions of the Fokker-Planck operator and its adjoint
what is novel?

- **A handsome reward**: as the optimal partition is always finite, the dynamics on this ‘best possible of all partitions’ is encoded by a finite transition graph of finite memory, and the Fokker-Planck operator can be represented by a finite matrix.
claim:

optimal partition hypothesis

- the best of all possible state space partitions
- optimal for the given noise
claim:

optimal partition hypothesis

- optimal partition replaces stochastic PDEs by finite, low-dimensional Fokker-Planck matrices
claim:

optimal partition hypothesis

- optimal partition replaces stochastic PDEs by finite, low-dimensional Fokker-Planck matrices
- finite matrix calculations, finite cycle expansions $\Rightarrow$ optimal estimates of long-time observables (escape rates, Lyapunov exponents, etc.)
how to combine Fokker-Planck and adjoint Fokker-Planck operators to describe hyperbolic periodic points (saddles)?
how to combine Fokker-Planck and adjoint Fokker-Planck operators to describe hyperbolic periodic points (saddles)?

Hint: H. H. Rugh (1992) combined deterministic evolution operator and adjoint operators to describe hyperbolic periodic points (saddles)
apply to Navier-Stokes turbulence?

computation of unstable periodic orbits in high-dimensional state spaces, such as Navier-Stokes, is at the border of what is feasible numerically, and criteria to identify finite sets of the most important solutions are very much needed. Where are we to stop calculating orbits of a given hyperbolic flow?
the rest is noise
brief history of noise

literature on stochastic dynamical systems is vast, starts with the Laplace 1810 memoir

all of this literature assumes uniform / bounded hyperbolicity and seeks to define a single, globally averaged diffusion induced average resolution (Heisenberg time, in the context of semi-classical quantization).
appears to have been first introduced by Wiener as the exact solution for a purely diffusive Wiener-Lévy process in one dimension. Onsager and Machlup use it in their variational principle to study thermodynamic fluctuations in a neighborhood of single, linearly attractive equilibrium point (i.e., without any dynamics).
brief history of noise

dynamical ‘action’ Lagrangian, and symplectic noise Hamiltonian were first written down by Freidlin and Wentzell (1970’s), whose formulation of the ‘large deviation principle’ was inspired by the Feynman quantum path integral (1940’s). Feynman, in turn, followed Dirac (1933’s) who was the first to discover that in the short-time limit the quantum propagator (imaginary time, quantum sibling of the Wiener stochastic distribution) is exact. Gaspard: ‘pseudo-energy of the Onsager-Machlup-Freidlin-Wentzell scheme.’ Roncadelli: the ‘Wiener-Onsager-Machlup Lagrangian.’
noisy flow

here we briefly repeat the derivation of local Fokker-Planck operator for a continuous time flow

d-dimensional stochastic flow

\[
\frac{dx}{dt} = v(x) + \xi(t),
\]

deterministic velocity field \( v(x) \), called ‘drift’ in the stochastic literature
in time $\delta \tau$ the deterministic trajectory advances by $v(x_n) \delta \tau$. The probability that the trajectory reaches $x_{n+1}$

$$\mathcal{L}_{FP}^\delta(x_{n+1}, x_n) = \frac{1}{N} \exp \left[ -\frac{1}{2 \delta \tau} (\xi^T_n \frac{1}{\Delta} \xi_n) \right].$$
in time $\delta \tau$ the deterministic trajectory advances by $v(x_n) \delta \tau$. 

the probability that the trajectory reaches $x_{n+1}$

$$\mathcal{L}_{FP}^{\delta \tau}(x_{n+1}, x_n) = \frac{1}{N} \exp \left[ -\frac{1}{2 \delta \tau} \left( \xi_n^T \frac{1}{\Delta} \xi_n \right) \right].$$

$\xi_n$ is the deviation of the noisy trajectory from the deterministic one,

$$\xi_n = \delta x_n - v(x_n) \delta \tau,$$
density evolution

the probability that the trajectory reaches $x_{n+1}$

$$\mathcal{L}^{\delta \tau}_{FP}(x_{n+1}, x_n) = \frac{1}{N} \exp \left[ -\frac{1}{2} \frac{\delta \tau}{\Delta} (\xi_n^T \xi_n) \right] .$$

$\xi_n$ is the deviation of the noisy trajectory from the deterministic one,

$$\xi_n = \delta x_n - \nu(x_n) \delta \tau ,$$

$$\delta x_n = x_{n+1} - x_n \simeq \dot{x}_n \delta \tau , \quad f^{\delta \tau}(x_n) - x_n \simeq \nu(x_n) \delta \tau ,$$
the probability that the trajectory reaches $x_{n+1}$

$$\mathcal{L}_{FP}^{\delta_T}(x_{n+1}, x_n) = \frac{1}{N} \exp \left[ -\frac{1}{2 \delta_T} (\xi_n^T \frac{1}{\Delta} \xi_n) \right].$$

where

$$\{x_0, x_1, \cdots, x_n, \cdots, x_k\} = \{x(0), x(\delta_T), \cdots, x(n\delta_T), \cdots, x(t)\}$$

is a sequence of $k + 1$ points $x_n = x(t_n)$ along the noisy trajectory, separated by time increments $\delta_T = t/k$. 
density evolution

finite time Fokker-Planck evolution $\rho(x, t) = \mathcal{L}_FP^t \circ \rho(x, 0)$ of an initial density $\rho(x_0, 0)$ is obtained by a sequence of consecutive short-time steps

$$
\mathcal{L}_FP^t(x_k, x_0) = \int [dx] \exp \left\{ -\frac{1}{4D\delta\tau} \sum_{n=1}^{k-1} [x_{n+1} - f^{\delta\tau}(x_n)]^2 \right\}
$$
(Gaussian) probability distribution function,

\[ \mathcal{L}_{FP}(x, x_0) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp \left[ -\frac{(x - x_0)^2}{2\sigma^2 t} \right] \]

variance \( \sigma^2 t = 2Dt \), standard deviation \( \sqrt{2Dt} \)
uncorrelated in time

\[ \langle x_{n+1} - x_n \rangle = 0, \quad \langle (x_{m+1} - x_m)(x_{n+1} - x_n) \rangle = 2D\delta_{mn} \]
in time $\delta \tau$ the deterministic trajectory advances by $v(x_n) \delta \tau$. The probability that the trajectory reaches $x_{n+1}$ is given by:

$$\mathcal{L}^{\delta \tau}_{FP}(x_{n+1}, x_n) = \frac{1}{N} \exp \left[ -\frac{1}{2 \delta \tau} (\xi_n^T \frac{1}{\Delta} \xi_n) \right].$$
in time $\delta \tau$ the deterministic trajectory advances by $v(x_n) \delta \tau$. The probability that the trajectory reaches $x_{n+1}$

$$L^{\delta \tau}_{FP}(x_{n+1}, x_n) = \frac{1}{N} \exp \left[ -\frac{1}{2 \delta \tau} (\xi_n^T \frac{1}{\Delta} \xi_n) \right].$$

$\xi_n$ is the deviation of the noisy trajectory from the deterministic one,

$$\xi_n = \delta x_n - v(x_n) \delta \tau,$$
the probability that the trajectory reaches $x_{n+1}$

$$\mathcal{L}_{FP}^{\delta\tau}(x_{n+1}, x_n) = \frac{1}{N} \exp \left[ -\frac{1}{2\delta\tau}(\xi_n^T \frac{1}{\Delta} \xi_n) \right].$$

$\xi_n$ is the deviation of the noisy trajectory from the deterministic one,

$$\xi_n = \delta x_n - v(x_n) \delta\tau,$$

$$\delta x_n = x_{n+1} - x_n \simeq \dot{x}_n \delta\tau,$$  

$$f^{\delta\tau}(x_n) - x_n \simeq v(x_n) \delta\tau,$$
the probability that the trajectory reaches $x_{n+1}$

$$L_{FP}^{\delta\tau}(x_{n+1}, x_n) = \frac{1}{N} \exp \left[ -\frac{1}{2\delta\tau} (\xi_n^T \frac{1}{\Delta} \xi_n) \right].$$

where

$$\{x_0, x_1, \ldots, x_n, \ldots, x_k\} = \{x(0), x(\delta\tau), \ldots, x(n\delta\tau), \ldots, x(t)\}$$

is a sequence of $k+1$ points $x_n = x(t_n)$ along the noisy trajectory, separated by time increments $\delta\tau = t/k$
zero mean and covariance matrix (diffusion tensor)

\[
\langle \xi_j(t_n) \rangle = 0, \quad \langle \xi_i(t_m) \xi_j^T(t_n) \rangle = \Delta_{ij} \delta_{nm},
\]

where \( \langle \cdots \rangle \) stands for ensemble average over many realizations of the noise.
Fokker-Planck formulation replaces individual noisy trajectories by the evolution of their density. Finite time Fokker-Planck evolution $\rho(x, t) = L_{FP}^t \circ \rho(x, 0)$ of an initial density $\rho(x_0, 0)$ is obtained by a sequence of consecutive short-time steps

$$L_{FP}^t(x_k, x_0) = \int [dx] \exp \left\{ -\frac{1}{4D\delta_T} \sum_{n=1}^{k-1} [x_{n+1} - f^{\delta_T}(x_n)]^2 \right\},$$
Fokker-Planck formulation replaces individual noisy trajectories by the evolution of their density. finite time Fokker-Planck evolution $\rho(x, t) = \mathcal{L}_{FP}^t \circ \rho(x, 0)$ of an initial density $\rho(x_0, 0)$ is obtained by a sequence of consecutive short-time steps

$$
\mathcal{L}_{FP}^t(x_k, x_0) = \int [dx] \exp \left\{ -\frac{1}{4D\delta \tau} \sum_{n=1}^{k-1} [x_{n+1} - f^\delta \tau (x_n)]^2 \right\},
$$
Continuous time limit, $\delta_\tau = t/k \to 0$, defines the Fokker-Planck operator

$$\mathcal{L}^t_{\text{FP}}(x, x_0) = \int [dx] \exp \left\{ -\frac{1}{4D} \int_0^t [\dot{x}(\tau) - v(x(\tau))]^2 d\tau \right\}$$

as a stochastic path (Wiener) integral

Associated continuous time Fokker-Planck equation for the time evolution of a density of noisy trajectories is

$$\partial_t \rho(x, t) + \nabla \cdot (v(x)\rho(x, t)) = D \nabla^2 \rho(x, t).$$
continuous time limit, \( \delta \tau = t/k \to 0 \), defines the Fokker-Planck operator

\[
\mathcal{L}_{FP}^t(x, x_0) = \int [dx] \exp \left\{ -\frac{1}{4D} \int_0^t [\dot{x}(\tau) - v(x(\tau))]^2 d\tau \right\}
\]
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\[
\partial_t \rho(x, t) + \nabla \cdot (v(x)\rho(x, t)) = D \nabla^2 \rho(x, t).
\]
predictions

- finite partition $\Rightarrow$ finite Fokker-Planck matrix
- its determinant yields time averages of dynamical observables
Computation of unstable periodic orbits in high-dimensional state spaces, such as Navier-Stokes, is at the border of what is feasible numerically, and criteria to identify finite sets of the most important solutions are very much needed. Where are we to stop calculating orbits of a given hyperbolic flow?
Intuitively, as we look at longer and longer periodic orbits, their neighborhoods shrink exponentially with time, while the variance of the noise-induced orbit smearing remains bounded; there has to be a *turnover time*, a time at which the noise-induced width overwhelms the exponentially shrinking deterministic dynamics, so that no better resolution is possible. Given a specified (possibly state space dependent) noise, we need to find, periodic orbit by periodic orbit, whether a further sub-partitioning is possible.
We have described here the *optimal partition hypothesis*, a new method for partitioning the state space of a chaotic repeller in presence of weak Gaussian noise, and tested the method in a 1-dimensional setting against direct numerical Fokker-Planck operator calculation.