dynamical zeta functions: what, why and what are they good for?

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I accept chaos
I am not sure that it accepts me
—Bob Dylan, *Bringing It All Back Home*

**in physics, no problem is tractable**

requires summing up exponentially increasing # of exponentially decreasing terms

yet

**in practice**

every physical problem must be tractable

“can’t do” doesn’t cut it
**dynamical systems**

**state space**

A manifold $\mathcal{M} \subseteq \mathbb{R}^d$: $d$ numbers determine the state of the system.

**representative point**

$x(t) \in \mathcal{M}$

A state of physical system at instant in time.
example of a representative point

\[ x(t) \in \mathcal{M}, \quad d = \infty \]
a state of turbulent pipe flow at instant in time

Stereoscopic Particle Image Velocimetry \( \rightarrow \) 3-d velocity field over the entire pipe

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\(^1\)Casimir W.H. van Doorne (PhD thesis, Delft 2004)
dynamical systems

**dynamics**

map $f^t(x_0) = \text{representative point time } t \text{ later}$

**evolution**

$f^t$ maps a region $\mathcal{M}_i$ of the state space into the region $f^t(\mathcal{M}_i)$. 
dynamical systems

**dynamics defined**

**dynamical system**

the pair \((\mathcal{M}, f)\)

**the problem**

enumerate, classify all solutions of \((\mathcal{M}, f)\)

one needs to enumerate \(\rightarrow\) hence *zeta functions*!
A Brief History of the Periodic Orbit Theory

Number Theory
- Riemann (1850)

Chaotic Dynamics
- Poincaré (1890)
- Smale (1950)
- Ruelle

Quantum Mechanics
- Bohr (1900)
- Einstein
- Gutzwiller
The physicist's life is intractable.

**Dynamical Systems**

Topological trace formula, zeta function

Fokker-Planck evolution

Optimal partition hypothesis

**State Space, Partitioned**

Partition into regions of similar states

**State Space Coarse Partition**

\[ \mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2 \]

ternary alphabet

\[ \mathcal{A} = \{1, 2, 3\} \]

**1-Step Memory Refinement**

Labeled by nine 'words'

\[ \mathcal{M}_i = \mathcal{M}_{i0} \cup \mathcal{M}_{i1} \cup \mathcal{M}_{i2} \]

\[ \mathcal{A} = \{00, 01, 02, \ldots, 21, 22\} \]
state space, partitioned

topological dynamics

one time step
points from $\mathcal{M}_{21}$
reach $\{\mathcal{M}_{10}, \mathcal{M}_{11}, \mathcal{M}_{12}\}$
and no other regions

each region = node
allowed transitions
$T_{10,21} = T_{11,21} = T_{12,21} \neq 0$
directed links
state space, partitioned

topological dynamics

Transition graph $T_{ba}$
regions reached in one time step

element: state space resolved into 7 neighborhoods

$\{M_{00}, M_{011}, M_{010}, M_{110}, M_{111}, M_{101}, M_{100}\}$
how many ways to get there from here?

\[(T^n)_{ij} = \sum_{k_1, k_2, \ldots, k_{n-1}} T_{ik_1} T_{k_1 k_2} \cdots T_{k_{n-1}j}\]

\[\propto \lambda_0^n, \quad \lambda_0 = \text{leading eigenvalue}\]

counts topologically distinct \(n\)-step paths starting in \(M_j\) and ending in partition \(M_i\).

compute eigenvalues by evaluating the determinant of the topological (or Artin-Mazur) zeta function

\[
\frac{1}{\zeta_{\text{top}}} = \det(1 - zT)
\]
**CHAPTER 15. COUNTING**

The determinant can be written down by inspection, as the sum of all possible partitions of the graph into products of non-intersecting loops, with each loop carrying a minus sign:

\[
\det(1 - zT) = \sum \text{[non-(self)-intersecting loops]}
\]

**Figure 15.1:** (a) The region labels in the nodes of transition graph figure 14.3 can be omitted, as the links alone keep track of the symbolic dynamics. (b)-(j) The fundamental cycles (15.23) for the transition graph (a), i.e., the set of its non-self-intersecting loops. Each loop represents a local trace \( t_p \), as in (14.5).

**Example 15.5 Nontrivial pruning:** The non-self-intersecting loops of the transition graph of figure 14.6 (d) are indicated in figure 14.6 (e). The determinant can be written down by inspection, as the sum of all possible partitions of the graph into products of non-intersecting loops.
**loop, periodic orbit, cycle**

walk that ends at the starting node, for example

\[
t_{011} = L_{110,011} L_{011,101} L_{101,110} = 011\rightarrow 101\rightarrow 110\rightarrow 011.
\]
zeta function ("partition function")

\[ \text{det}(1 - zT) \]

can be read off the graph, expanded as a polynomial in \( z \), with coefficients given by products of non-intersecting loops (traces of powers of \( T \))
if there is one idea that one should learn about chaotic dynamics

it is this

there is a fundamental local ↔ global duality which says that

eigenvalue spectrum is dual to periodic orbits spectrum

for dynamics on the circle, this is called Fourier analysis

for dynamics on well-tiled manifolds, Selberg traces and zetas

for generic nonlinear dynamical systems the duality is embodied in the trace formulas and zeta functions
global eigenspectrum ⇔ local periodic orbits

Twenty years of schooling
and they put you on the day shift
Look out kid, they keep it all hid

—Bob Dylan, *Subterranean Homesick Blues*

the eigenspectrum $s_0, s_1, \cdots$ of the classical evolution operator

classical evolution operator

trace formula, infinitely fine partition

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta \cdot A_p - s T_p)}}{|\det(1 - M_p^r)|}.$$ 

the beauty of trace formulas lies in the fact that everything on the right-hand-side

—prime cycles $p$, their periods $T_p$ and the eigenvalues of $M_p$—

is a coordinate independent, invariant property of the flow
deterministic chaos vs. noise

any physical system:
noise limits the resolution that can be attained in partitioning the state space

noisy orbits
probabilistic densities smeared out by the noise: a finite # fits into the attractor

goal: determine the finest attainable partition
A physicist’s life is intractable
dynamical systems
Topological trace formula, zeta function
Fokker-Planck evolution
optimal partition

intuition

deterministic partition

state space coarse partition

1-step memory refinement

\[ M = M_0 \cup M_1 \cup M_2 \]

ternary alphabet
\[ A = \{1, 2, 3\} \]

\[ M_i = M_{i0} \cup M_{i1} \cup M_{i2} \]
labeled by nine ‘words’
\[ \{00, 01, 02, \ldots, 21, 22\} \]
**deterministic vs. noisy partitions**

- Deterministic partition can be refined *ad infinitum*.
- Noise blurs the boundaries when overlapping, no further refinement of partition.
idea #1: partition by periodic points

**periodic points instead of boundaries**

- each partition contains a short periodic point smeared into a ‘cigar’ by noise
idea #1: partition by periodic points

**periodic points instead of boundaries**

- each partition contains a short periodic point smeared into a ‘cigar’ by noise

compute the size of a noisy periodic point neighborhood
idea #1: partition by periodic points

**periodic orbit partition**

deterministic partition

some short periodic points:
fixed point $\bar{1} = \{x_1\}$
two-cycle $\bar{01} = \{x_{01}, x_{10}\}$

noisy partition

periodic points blurred by the Langevin noise into cigar-shaped densities
successive refinements of a deterministic partition: exponentially shrinking neighborhoods

as the periods of periodic orbits increase, the diffusion always wins:

partition stops at the finest attainable partition, beyond which the diffusive smearing exceeds the size of any deterministic subpartition.
idea #1: partition by periodic points

noisy periodic orbit partition

optimal partition hypothesis

optimal partition: the maximal set of resolvable periodic point neighborhoods

why care?

if the high-dimensional flow has only a few unstable directions, the overlapping stochastic ‘cigars’ provide a compact cover of the noisy chaotic attractor, embedded in a state space of arbitrarily high dimension
strategy

- use periodic orbits to partition state space
- compute local eigenfunctions of the Fokker-Planck operator to determine their neighborhoods
- done once neighborhoods overlap
idea #2: evolve densities, not Langevin trajectories

how big is the neighborhood blurred by the Langevin noise?

the (well known) key formula

composition law for the covariance matrix $Q_a$

$$Q_{a+1} = M_a Q_a M_a^T + \Delta_a$$

density covariance matrix at time $a$: $Q_a$
Langevin noise covariance matrix: $\Delta_a$
Jacobian matrix of linearized flow: $M_a$
idea #2: evolve densities, not Langevin trajectories

roll your own cigar

**evolution law for the covariance matrix** $Q_a$

$$Q_{a+1} = M_a Q_a M_a^T + \Delta_a$$

in one time step a Gaussian density distribution with covariance matrix $Q_a$ is smeared into a Gaussian ‘cigar’ whose widths and orientation are given by eigenvalues and eigenvectors of $Q_{a+1}$

(1) deterministically advected and deformed
local density covariance matrix $Q \rightarrow MQM^T$

(2) add noise covariance matrix $\Delta$

add up as sums of squares
idea #2: evolve densities, not Langevin trajectories

**noise along a trajectory**

iterate $Q_{a+1} = M_a Q_a M_a^T + \Delta_a$ along the trajectory

if $M$ is contracting, over time the memory of the covariance $Q_{a-n}$ of the starting density is lost, with iteration leading to the limit distribution

$$Q_a = \Delta_a + M_{a-1} \Delta_{a-1} M_{a-1}^T + M_{a-2}^2 \Delta_{a-2} (M_{a-2}^2)^T + \cdots.$$  

diffusive dynamics of a nonlinear system is fundamentally different from Brownian motion, as the flow induces a history dependent effective noise:

*Always!*
idea #2: evolve densities, not Langevin trajectories

noise and a single attractive fixed point

if all eigenvalues of \( M \) are strictly contracting, any initial compact measure converges to the unique invariant Gaussian measure \( \rho_0(z) \) whose covariance matrix satisfies

**time-invariant measure condition (Lyapunov equation)**

\[
Q = MQM^T + \Delta
\]

[A. M. Lyapunov 1892, doctoral dissertation]
width of the natural measure concentrated at the deterministic fixed point $z = 0$

$$Q = \frac{2D}{1 - |\Lambda|^2}, \quad \rho_0(z) = \frac{1}{\sqrt{2\pi} Q} \exp\left(-\frac{z^2}{2Q}\right),$$

- is balance between contraction by $\Lambda$ and diffusive smearing by $2D$ at each time step
- for strongly contracting $\Lambda$, the width is due to the noise only
- As $|\Lambda| \to 1$ the width diverges: the trajectories are no longer confined, but diffuse by Brownian motion
idea #3: for unstable directions, look back

things fall apart, centre cannot hold

but what if $M$ has *expanding* Floquet multipliers?

both deterministic dynamics and noise tend to smear densities away from the fixed point: no peaked Gaussian in your future
idea #3: for unstable directions, look back

things fall apart, centre cannot hold

but what if $M$ has *expanding* Floquet multipliers?

Fokker-Planck operator is non-selfadjoint

If right eigenvector is peaked (attracting fixed point)  
the left eigenvector is flat (probability conservation)
idea #3: for unstable directions, look back

case of repelling fixed point

if $M$ has only expanding Floquet multipliers, both deterministic dynamics and noise tend to smear densities away from the fixed point.

balance between the two is described by the adjoint Fokker-Planck operator, and the evolution of the covariance matrix $Q$ is now given by

$$Q_a + \Delta = M_a Q_{a+1} M_a^T,$$

[aside to control freaks: reachability and observability Gramians]
optimal partition challenge

finally in position to address our challenge:

determine the finest possible partition for a given noise
As an illustration of the method, consider the chaotic repeller on the unit interval

\[ x_{n+1} = \Lambda_0 x_n(1 - x_n)(1 - bx_n) + \xi_n, \quad \Lambda_0 = 8, \ b = 0.6, \]

with noise strength \( 2D = 0.002 \)
‘the best possible of all partitions’ hypothesis formulated as an algorithm

- calculate the local adjoint Fokker-Planck operator eigenfunction width $Q_a$ for every unstable periodic point $x_a$
- assign one-standard deviation neighborhood $[x_a - Q_a, x_a + Q_a]$ to every unstable periodic point $x_a$
- cover the state space with neighborhoods of orbit points of higher and higher period $n_p$
- stop refining the local resolution whenever the adjacent neighborhoods of $x_a$ and $x_b$ overlap:

$$|x_a - x_b| < Q_a + Q_b$$
optimal partition, 1 dimensional map

\( f_0, f_1 \): branches of deterministic map

Local eigenfunctions \( \tilde{\rho}_a \) partition state space by neighborhoods of periodic points of period 3

Neighborhoods \( \mathcal{M}_{000} \) and \( \mathcal{M}_{001} \) overlap, so \( \mathcal{M}_{00} \) cannot be resolved further
all neighborhoods \( \{ M_{0101}, M_{0100}, \cdots \} \) of period \( n_p = 4 \) cycle points overlap, so

state space can be resolved into 7 neighborhoods

\[
\{ M_{00}, M_{011}, M_{010}, M_{110}, M_{111}, M_{101}, M_{100} \}
\]
Markov partition

-evolution in time maps intervals 
\[ \mathcal{M}_{011} \rightarrow \{\mathcal{M}_{110}, \mathcal{M}_{111}\} \]
\[ \mathcal{M}_{00} \rightarrow \{\mathcal{M}_{00}, \mathcal{M}_{011}, \mathcal{M}_{010}\}, \text{etc..} \]

-summarized by the transition graph (links correspond to elements of transition matrix \( T_{ba} \)): the regions \( b \) that can be reached from the region \( a \) in one time step
transition graph

7 nodes = 7 regions of the optimal partition

dotted links = symbol 0 (next region reached by $f_0$)

full links = symbol 1 (next region reached by $f_1$)

region labels in the nodes can be omitted, with links keeping track of the symbolic dynamics

(1) deterministic dynamics is full binary shift, but
(2) noise dynamics nontrivial and finite
predictions

**escape rate and the Lyapunov exponent of the repeller**
are given by the leading eigenvalue of this $[7 \times 7]$ graph / transition matrix

tests: numerical results are consistent with the full Fokker-Planck PDE simulations
what is novel?

- we have shown how to compute the **locally optimal partition**, for a given dynamical system and given noise, in terms of local eigenfunctions of the forward-backward actions of the Fokker-Planck operator and its adjoint.
what is novel?

- **A handsome reward:** as the optimal partition is always finite, the dynamics on this ‘best possible of all partitions’ is encoded by a finite transition graph of finite memory, and the Fokker-Planck operator can be represented by a finite matrix.
claim:

optimal partition hypothesis

- the best of all possible state space partitions
- optimal for the given noise
claim:

optimal partition hypothesis

- optimal partition replaces stochastic PDEs by finite, low-dimensional Fokker-Planck matrices
claim:

optimal partition hypothesis

- optimal partition replaces stochastic PDEs by finite, low-dimensional Fokker-Planck matrices
- finite matrix calculations, finite cycle expansions $\Rightarrow$ optimal estimates of long-time observables (escape rates, Lyapunov exponents, etc.)
Computation of unstable periodic orbits in high-dimensional state spaces, such as Navier-Stokes, is at the border of what is feasible numerically, and criteria to identify finite sets of the most important solutions are very much needed. Where are we to stop calculating orbits of a given hyperbolic flow?
Intuitively, as we look at longer and longer periodic orbits, their neighborhoods shrink exponentially with time, while the variance of the noise-induced orbit smearing remains bounded; there has to be a turnover time, a time at which the noise-induced width overwhelms the exponentially shrinking deterministic dynamics, so that no better resolution is possible.
We have described here the *optimal partition hypothesis*, a new method for partitioning the state space of a chaotic repeller in presence of weak Gaussian noise, and tested the method in a 1-dimensional setting against direct numerical Fokker-Planck operator calculation.

D. Lippolis and P. Cvitanović, *Optimal resolution of the state space of a chaotic flow in presence of noise* (in preparation)