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dimension of turbulence

July 1, 2011

physical dimension of a ‘turbulent’ flow

question

what is the dimensionality of an attractor of a dissipative flow?

proposition : Foias *et al* (1985)

dimension of ‘inertial manifold’ is finite

physical dimension of a ‘turbulent’ flow

question

what is the dimensionality of an attractor of a dissipative flow?

proposition : Ginelli, Chaté, Radons, *et al* (2009)

‘Lyapunov covariant vectors’ split in

(a) finite number of ‘physical,’ entangled directions, in the tangent space of the attractor

(b) infinitely many hyperbolically decaying directions that are isolated and do not mix and

the ‘physical’ ones are pointing along the ‘entangled,’ physical tangent directions

physical dimension of a ‘turbulent’ flow

question

what is the dimensionality of an attractor of a dissipative flow?

proposition

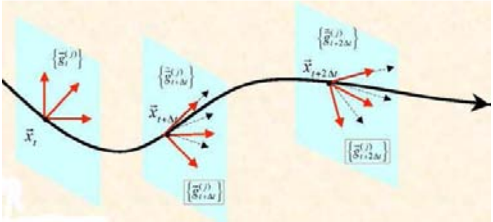
Floquet vectors of unstable periodic orbits, or ‘covariant Lyapunov vectors,’ identify the *local* number of degrees of freedom that captures the physics of a ‘turbulent’ PDE on a compact spatial domain

that number is proportional to the size L^D of the D-dimensional PDE system

characterizing dynamics with covariant Lyapunov vectors

Ginelli *et al.* use the QR algorithm for computing Gram-Schmidt vectors (GSV) and the covariant Lyapunov vectors (CLV) to show that the chaotic solutions for a variety of spatially-coupled maps, evolve within a manifold spanned by a finite number of physical eigen-directions

Jacobian matrix transports covariant frames



Jacobian matrix transports initial orthogonal frame into a non-orthogonal one. This frame is then QR decomposed into an *R*-matrix and the next Gram-Schmidt frame, which is then transported by the next Jacobian matrix, and so on. A vector that starts within the subspace spanned by such finite Gram-Schmidt basis stays within the subspace spanned by the successive orthogonal frames

Benettin *et al.* Lyapunov exponents algorithm relies on construction of orthogonal sets of Gram-Schmidt vectors. They are not covariant, i.e., the Gram-Schmidt vectors at a given state space point are not mapped by the linearized dynamics into the Gram-Schmidt vectors of the forward images of this point

in contrast, the Jacobian matrix J_n eigenvectors $\mathbf{e}^{(i)}$ also span the d-dimensional tangent space, but are generically not normal recent methods enable computation of *all* eigenvectors of J_n. The key ides are:

order the Jacobian matrix J_n eigenvectors $\mathbf{e}^{(\ell)}$ by the real parts of their eigen-exponents, and let $T\mathcal{M}^{[j]}$ be the tangent subspace spanned by the leading j eigenvectors a vector δx that starts within the tangent subspace $\delta x \in T\mathcal{M}_x^{[j]}$ spanned by the first j Gram-Schmidt basis vectors, *stays* within that subspace spanned by a comoving frame under subsequent evolution and re-orthogonalizations

once a set of $\{\mathbf{R}_1, \mathbf{R}_2, \cdots, \mathbf{R}_n\}$ along trajectory $\{x_1, x_2, \cdots, x_n\}$ has been computed and stored in the memory, it can be used to describe the action of the the linearized flow Jacobian matrix J_n both *forward* and *backward* in time strongly contracting $\Lambda^{(j)}$ multiplier forward in time, becomes the leading $1/\Lambda^{(j)}$ multiplier backward in time. Matrix power method then pulls out this eigenvalue as the leading one within the subspace $T\mathcal{M}^{[j]}$

increase the dimension of the subspace by one, and you get the next $\Lambda^{(j+1)}$

repeat, and you get all eigenvalues and eigenvectors, even those insanely contacting ones, like $\Lambda^{(j)} \approx 10^{-137}$

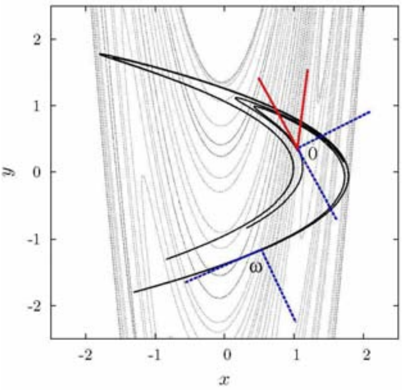
‘physical’ tangent space is transported across the whole curved strange manifold ergodically, and nonhyperbolicity of the attractor is used as a test that trajectory initiated along a given direction stays within the attractor

probability density of angles between adjacent physical eigenvectors computed over long time is flat, not peaked at 90^o

‘trivial,’ hyperbolically decaying eigen-directions that are isolated exhibit no such small inter-angles anywhere along the ergodic trajectory

angle between stable / unstable eigendirections

Hénon attractor



non-hyperbolicity

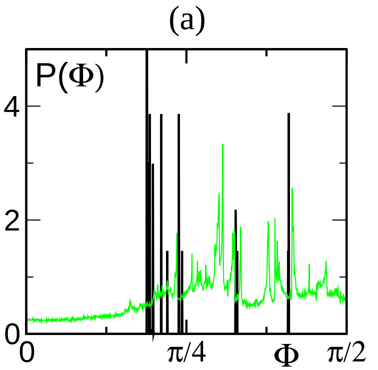
attractor (black line) and a finite-length approximation of its stable manifold (dotted line)

red vectors: the covariant vectors at the state space point

blue vectors: Gram-Schmidt vectors

separate the women from girls : hyperbolicity

Hénon attractor



Hénon map (green)

$x_{n+1} = 1 - 1.4\, x_n^2 + 0.3 x_{n-1}$

Lozi map (black)

$x_{n+1} = 1 - 1.4\, |x_n| + 0.3 x_{n-1}$

probability distribution of the angle between stable and unstable eigenvector the leading Lyapunov vectors are tangent to the attractor. Perturbations that are on the attractor can be found in the subspace of the leading Lyapunov vectors

the main advance of using Lyapunov vectors instead of eigenvalues alone is that the approximate orthogonality of the ‘isolated’ ones provides a clear threshold between the ‘physical’ and the rest

Ginelli *et al.* they show that the chaotic solutions of spatially extended dissipative systems, Kuramoto-Sivashinsky and complex Landau-Ginzburg, evolve within a manifold spanned by a finite number of physical modes hyperbolically isolated from a set of residual degrees of freedom, themselves individually isolated from each other

open problem : Navier-Stokes turbulence

Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u}$$

requires at least 61,506 dimensional direct numerical simulation

‘physical’ dimension unknown: at least 100?

1-dimensional "Navier-Stokes"

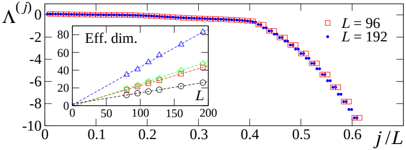
Kuramoto-Sivashinsky equation

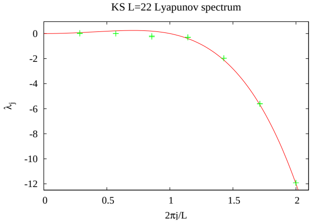
$$u_t + u\, u_x = -u_{xx} - \nu\, u_{xxxx}$$

- "inertial" term $u \partial_x u$; nonlinear
- "anti-diffusive" term $-\partial_x^2 u$,
- "viscosity" $\nu \partial_x^4 u$ - suppresses fast spatial variations

only parameter: dimensionless length $\tilde{L} = \frac{L}{2\pi\sqrt{\nu}}$

example : Kuramoto-Sivashinsky flow





(left) Inset: number of non-negative exponents (circles), Kaplan-Yorke dimension (squares), metric entropy (diamonds, multiplied by 50), and number of physical Lyapunov vectors (triangles).

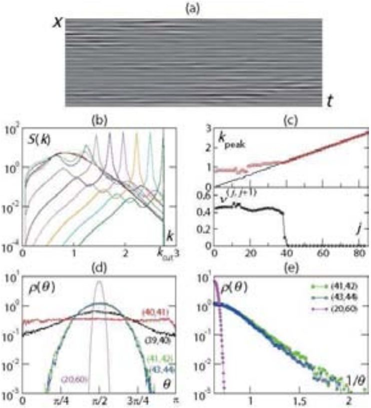
(right) First 14 Lyapunov exponents λ_j for the full state space, periodic b.c. KS for $L = 22$, from a 62 real Fourier modes long-time simulation

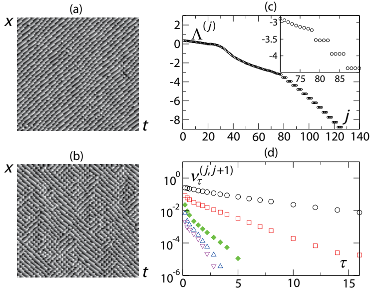
Full line corresponds to the stability eigenvalues of the $u(x, t) = 0$ stationary solution $(j/\tilde{L})^2 - (j/\tilde{L})^4$, for arbitrary system size L.

example : Kuramoto-Sivashinsky flow

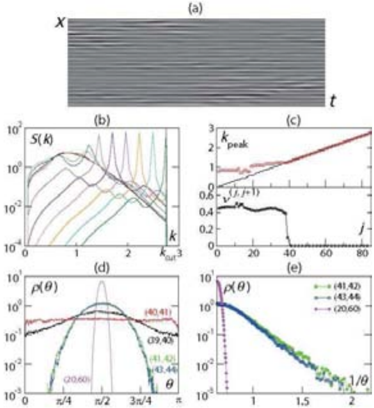
dimension of the Kuramoto-Sivashinsky flow is roughly four times the number of positive/marginal Floquet (or Lyapunov) exponents, and twice its Kaplan-Yorke estimate one might be unimpressed by the KS example, as the $-k^4$ hyper-diffusion term kills all higher Fourier modes very effectively

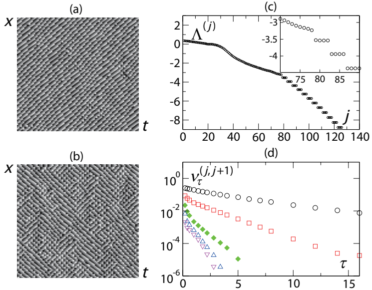
complex Ginzburg-Landau equation is persuasive; the nonlinearity is of $u(x)^3$ variety (instead of $u\partial u$ of Navier-Stokes and KS), but there is only a $-k^2$ diffusive term, and nevertheless there is a clear threshold for the ‘isolated’ Lyapunov vectors





(left panel) KS system, $L = 96$. (a) Spatiotemporal plot of a typical vector ($j = 46$) in the isolated region. (b) Spatial power spectra of vectors of indices $j = 1$, 16, 32, 38, 44, 52, 60, 68, 76, 84. (c) Top panel: peak wavenumber in the power spectra (red circles) and $k = [j/2] \cdot 2\pi/L$ (black line). Bottom panel: DOS violation fraction $\nu_\tau^{(j,j+1)}$, neighboring vectors. (d) (e) Angle distributions between pairs of vectors.



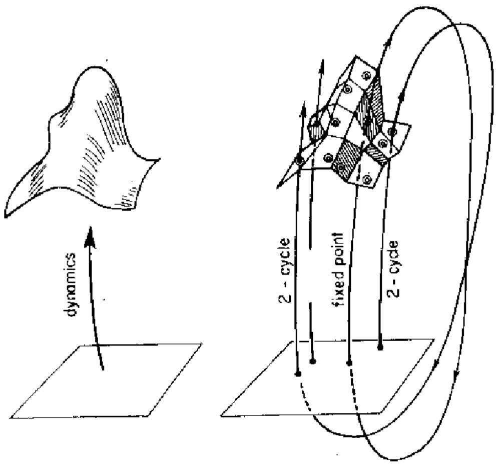


(right panel) Complex Ginzburg-Landau, amplitude turbulence regime, $L = 64$. (a,b) Spatiotemporal plots of the phase component of a typical vector $j = 91$ in the isolated region (c) Lyapunov spectrum; inset: close-up around threshold. (d) Time fraction $\nu_\tau^{(j,j+1)}$ of DOS violation, as a function of τ ($j = 78, 82, 86, 90, 94$, from top to bottom).

charting the state space

for turbulent/chaotic systems a set of Poincaré sections is needed to capture the dynamics. The choice of sections should reflect the dynamically dominant patterns seen in the solutions of nonlinear PDEs

we propose to construct a global atlas of the dimensionally reduced state space $\bar{\mathcal{M}}$ by deploying linear Poincaré sections $\mathcal{P}^{(j)}$ across neighborhoods of the qualitatively most important patterns $y'^{(j)}$



this is the periodic-orbit implementation of the idea of state space tessellation we shall refer to these states as *templates*, each represented in the state space $\mathcal{M}$ of the system by a *template point* y'

together with the velocity field at this point, a template defines a linear Poincaré section, an affine hyperplane $y \in \mathcal{P}$,

$$v(y') \cdot (y - y') = 0 \,,$$

locally normal to the $v(y')$ at the template point y'

each Poincaré section $\mathcal{P}^{(j)}$, provides a local chart at $y'^{(j)}$ for a neighborhood of an important, qualitatively distinct class of solutions (2-rolls states, 3-rolls states, etc.); together they ‘Voronoi’ tessellate the curved manifold in which the reduced strange attractor is embedded by a finite set of hyperplane tiles a Poincaré section is a $(d-1)$-dimensional hyperplane. If we pick another template point $y'^{(2)}$, it comes along with its own Poincaré section

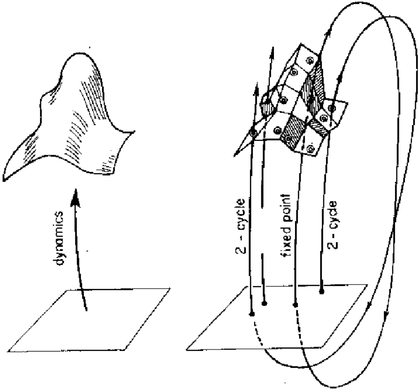
any neighboring pair of $(d-1)$-dimensional Poincaré sections intersects in a 'ridge' ('boundary,' 'edge'), a $(d-2)$-dimensional hyperplane, easy to compute a global atlas so constructed should be sufficiently fine-grained: each ‘chart’ or ‘tile,’ bounded by ridges to neighboring Poincaré sections, should be sufficiently small follow an ant as it traces out a trajectory $y^{(1)}(\tau)$, confined to the Poincaré section $\mathcal{P}^{(1)}$ The moment $\langle (y^{(1)}(\tau) - y'^{(2)}) | v^{(2)} \rangle$ changes sign, the ant has crossed the ridge and continues its merry stroll within the $\mathcal{P}^{(2)}$ Poincaré section there is a rub, though - you need to know how to pick the neighboring templates. This is a reflection of the flaw inherent in use of a Poincaré section hyperplane globally: a Poincaré section is derived from the Euclidean notion of distance, but for nonlinear flows the distance has to be measured curvilinearly, along unstable manifolds we propose to construct a global atlas by deploying a set of linear Poincaré sections in neighborhoods of the most important equilibria and/or periodic orbits as local charts

physical task is to, for a given dynamical flow, pick a set of qualitatively distinct templates whose Poincaré sections are locally tangent to the strange attractor

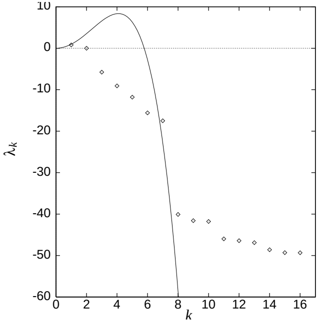
state space tessellation by periodic orbits

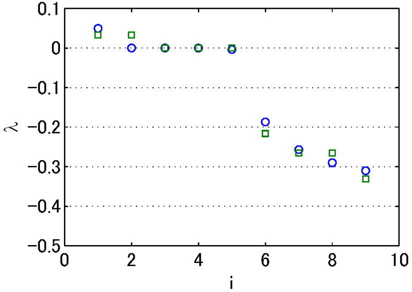
two 1-cycles

a 2-cycle that alternates between the neighborhoods of the two 1-cycles, shadowing first one of the two 1-cycles, and then the other



smooth dynamics (left frame) tesselated by the skeleton of periodic points, together with their linearized neighborhoods, (right frame)





(left) Lyapunov exponents λ_k versus k for the periodic orbit $\overline{1}$ compared with the stability eigenvalues of the $u(x, t) = 0$ stationary solution $k^2 - \nu k^4$. λ_k for $k \geq 8$ fall below the numerical accuracy of integration and are not meaningful

(right) Lyapunov exponents λ_j for the full state space, periodic b.c. KS for $L = 22$, from a 124 real Fourier modes (blue circles) long-time simulation overlayed on the $T_p = 10.2534$ periodic orbit (green squares)

periodic orbits

the idea is to coarsely cover the *continuous-symmetry reduced* nonlinear strange attractor with a set of tangent hyperplanes

any adjacent pair intersects in a ‘ridge’ hyperplane of one less dimension

the task:

for a given strange attractor, pick a set of Poincaré section-fixing points, such that each local section is approximately tangent to the strange attractor

résumé

if a physical flow is confined to a lower-dimensional manifold, one should use this fact to implement a dimensionality reduction

we have described dimensionality reduction by the method linear Poincaré sections, a linear procedure particularly simple and practical to implement

while a Poincaré section intersects each trajectory in a neighborhood of a template only once, extended globally any Poincaré section intersects a longer trajectory segment multiple times. So it makes no sense physically to use one Poincaré section globally

We propose instead to construct a global atlas by deploying sets of linear Poincaré sections as charts of neighborhoods of the most important (relative) equilibria and/or (relative) periodic orbits

résumé

such global atlas should be sufficiently fine-grained

the atlas so constructed retains the dimensionality of the original problem. The full dynamics is faithfully retained, we are *not* constructing a lower-dimensional model of the dynamics

neighborhoods of unstable equilibria and periodic orbits are dominated by their unstable and least contracting stable eigenvalues and are, for all practical purposes, low-dimensional. Traversals of the ridges are, however, higher dimensional. For example, crossing from the neighborhood of a two-rolls state into the neighborhood of a three-rolls state entails going through a pattern ‘defect,’ a rapid transient whose precise description requires many Fourier modes

résumé

nevertheless, the recent progress on separation of ‘physical’ and ‘hyperbolically isolated’ covariant Lyapunov vectors described here gives us hope that the proposed atlas could provide a systematic and controllable framework for construction of lower-dimensional models of ‘turbulent’ dynamics of dissipative PDEs

sources

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bonus points:

do it for Navier-Stokes!