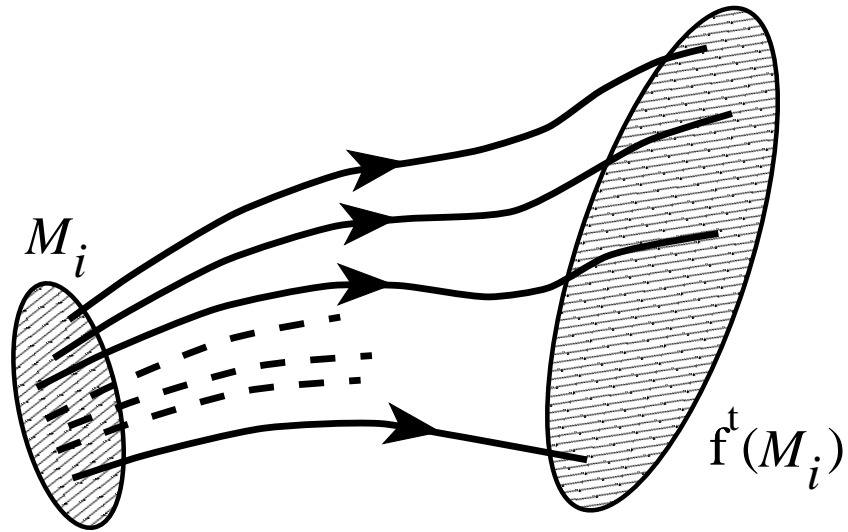


# Chapter 4 Local stability

# Flows transport neighborhoods



next: transport the neighborhood of  $x(t)$

# Equations of variations

Flow transports displacement  $x(t) + \delta x(t)$   
along trajectory  $x(t) = f^t(x_0)$ .

equations of variations for infinitesimal neighborhood:

$$\dot{x}_i + \delta \dot{x}_i = v_i(x + \delta x) \approx v_i(x) + \sum_j \frac{\partial v_i}{\partial x_j} \delta x_j.$$

Together

$$\dot{x}_i = v_i(x), \quad \delta \dot{x}_i = \sum_j A_{ij}(x) \delta x_j$$

where stability matrix

$$A_{ij}(x) = \frac{\partial v_i(x)}{\partial x_j}$$

is the instantaneous rate of shearing of  $x(t)$  neighborhood

# fundamental matrix

infinitesimal neighborhood after a finite time:

$$f_i^t(x_0 + \delta x) = f_i^t(x_0) + \sum_j \frac{\partial f_i^t(x_0)}{\partial x_{0j}} \delta x_j + \dots,$$

Linearized neighborhood transported by the Jacobian

$$\delta x(t) = J^t(x_0) \delta x(0), \quad J_{ij}^t(x_0) = \left. \frac{\partial x_i(t)}{\partial x_j} \right|_{x=x_0}.$$

# Stability of trajectories

exponential of a constant matrix

$$e^{tA} = \lim_{m \rightarrow \infty} \left( \mathbf{1} + \frac{t}{m} \mathbf{A} \right)^m .$$

tax-accountant's discrete step definition of an exponential

local rate of neighborhood distortion  $\mathbf{A}(x)$  depends on  $x(t)$

$$\begin{aligned} \mathbf{J}^t &= \lim_{m \rightarrow \infty} \prod_{n=m}^1 \left( \mathbf{1} + \delta t \mathbf{A}(x_n) \right) \\ &= \lim_{m \rightarrow \infty} e^{\delta t \mathbf{A}(x_m)} e^{\delta t \mathbf{A}(x_{m-1})} \dots e^{\delta t \mathbf{A}(x_2)} e^{\delta t \mathbf{A}(x_1)} , \\ &\quad \delta t = (t - t_0)/m, x_n = x(t_0 + n\delta t). \end{aligned}$$

limit of this procedure:

$$J_{ij}^t(x_0) = \left[ \mathbf{T} e^{\int_0^t d\tau \mathbf{A}(x(\tau))} \right]_{ij},$$

where  $\mathbf{T}$  stands for time-ordered integration.

the fundamental matrices are multiplicative along the flow,

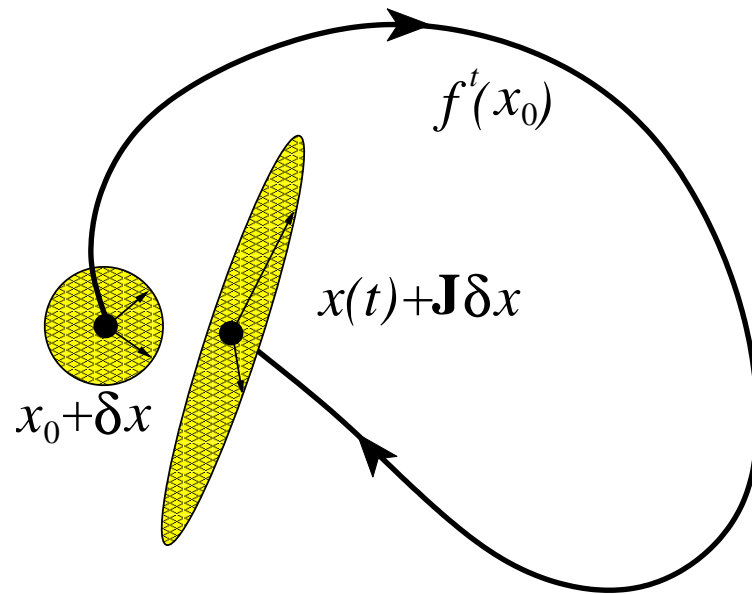
$$J^{t+t'}(x) = J^{t'}(x') J^t(x), \quad \text{where } x' = f^t(x).$$

$\Lambda_k$  = kth **stability eigenvalue** of the finite time fundamental matrix  $M^t$ .

$\lambda_k$  = kth **stability exponent**

$$|\Lambda_k| = e^{t\lambda_k}, \quad \Lambda_k = \Lambda_k(x_0, t), \quad \lambda_k = \lambda_k(x_0, t).$$

# fundamental matrix



Jacobian maps a spherical neighborhood of  $x_0$  into an ellipsoidal neighborhood time  $t$  later

Neighbors separate along **unstable directions**,  
 approach each other along **stable directions**,  
 creep along the **marginal directions**

## Stability of equilibria

Stability matrix  $\mathbf{A} = \mathbf{A}(x_q)$  evaluated at an equilibrium point  $x_q$  is constant

$$f^t(x) = x_q + e^{\mathbf{A}t}(x - x_q) + \dots,$$
$$J^t(x_q) = e^{\mathbf{A}t} \quad \mathbf{A} = \mathbf{A}(x_q).$$

For a constant  $\mathbf{A}$  the fundamental matrix

$$x(t) = e^{t\mathbf{A}}x(0).$$

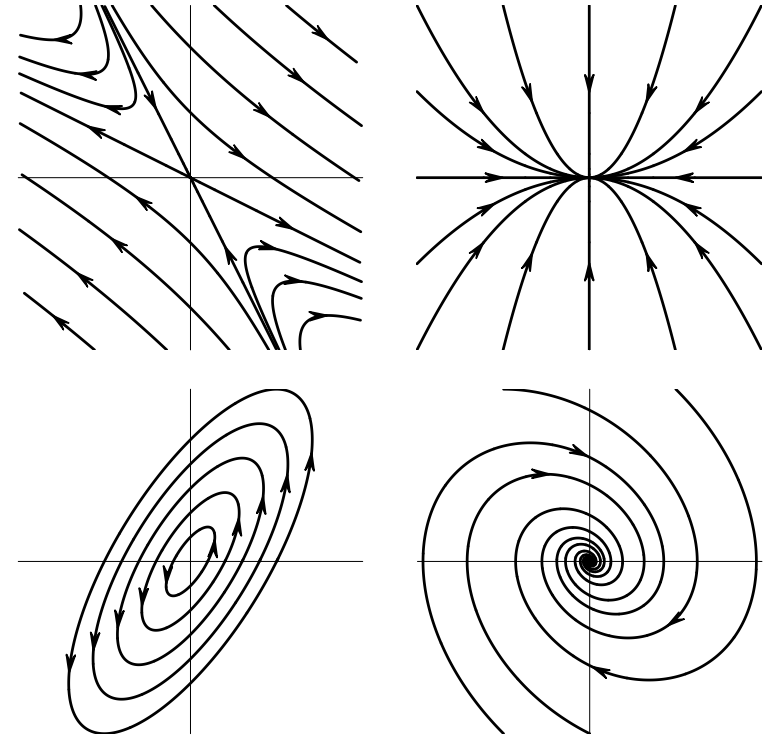
is the solution of the linear equation

$$\dot{x} = \mathbf{A}x.$$

so study **linear** flows first:

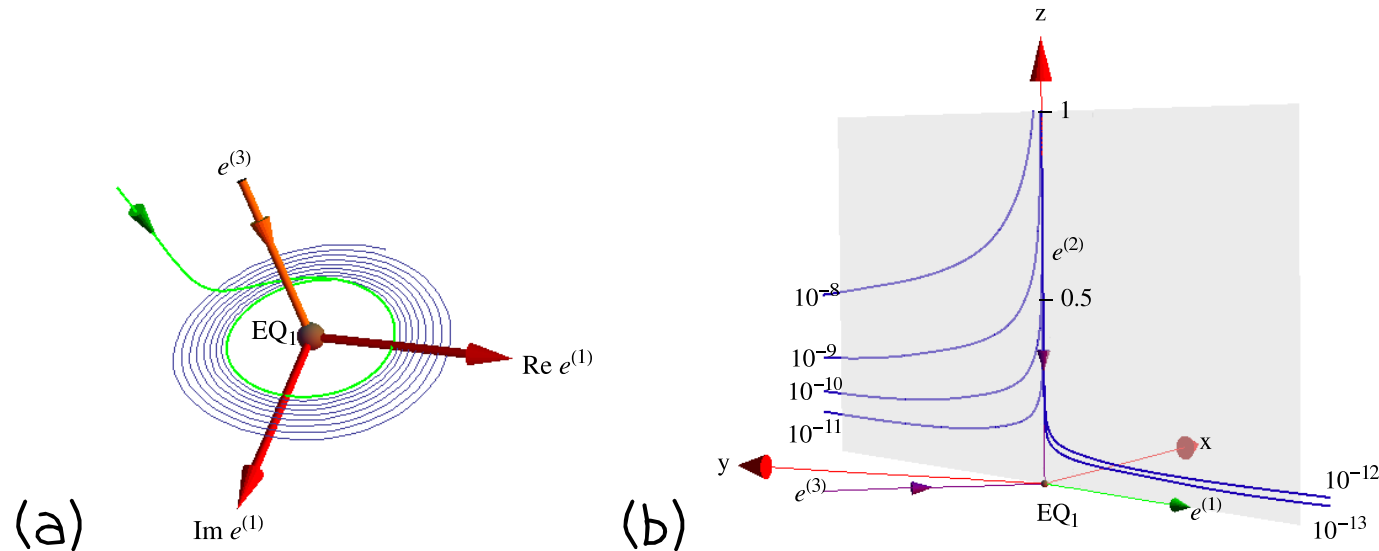
## 2-dimensional flows

Streamlines for several typical 2-dimensional flows:  
saddle (hyperbolic),  
in node (attracting),  
center (elliptic),  
in spiral.



# Stability of Lorenz flow equilibria

the flow is organized by its 3 equilibria



(a) The unstable eigenplane spanned by  $Re e^{(1)}$  and  $Im e^{(1)}$ , the stable eigenvector  $e^{(3)}$ ,  $EQ_1$  equilibrium.

(b) flow near the  $EQ_0$  equilibrium: unstable eigenvector  $e^{(1)}$ , stable eigenvectors  $e^{(2)}$ ,  $e^{(3)}$ . Note the strong  $\lambda^{(1)}$  expansion: the

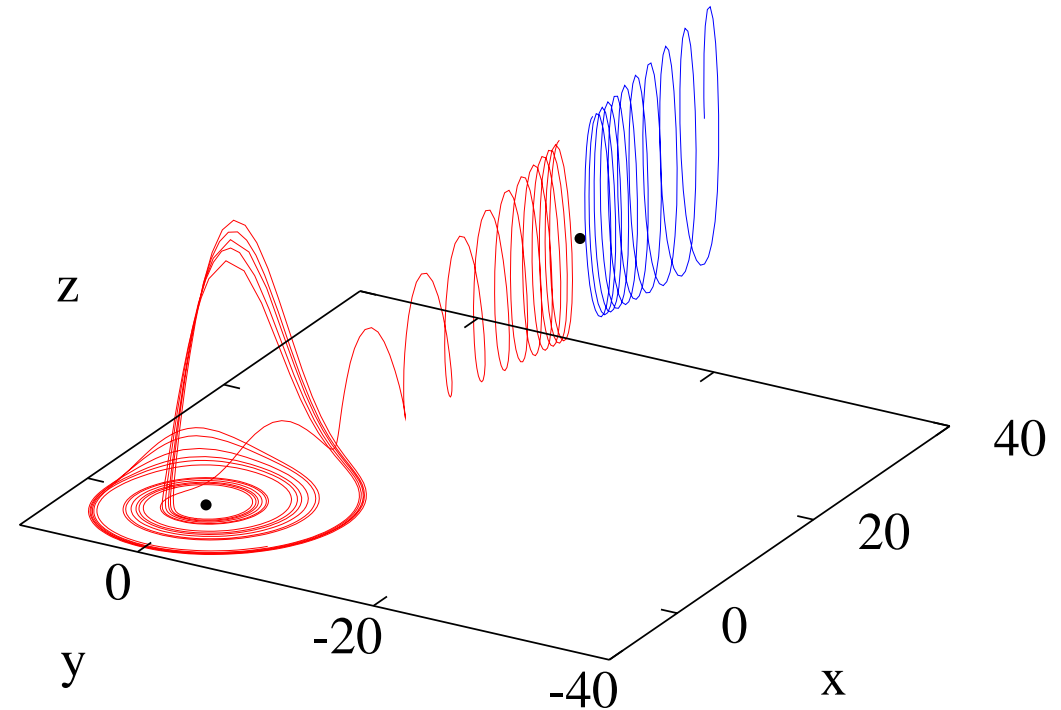
$EQ_0$  equilibrium is unreachable, and the  $EQ_1 \rightarrow EQ_0$  heteroclinic connection never observed in simulations.

(E. Siminos)

# Stability of Rössler flow equilibria

two equilibrium points  
 $(x^-, y^-, z^-)$   $(x^+, y^+, z^+)$

stable manifold of  
 "+" equilibrium point  
 = attraction basin  
 boundary:



right of the "+" equilibrium trajectories escape,  
 left of the "+" spiral toward the "-" equilibrium point  
 → seem to wander chaotically for all times.

linearized stability exponents

$$(\lambda_1^-, \lambda_2^- \pm i\vartheta_2^-) = (-5.686, \quad 0.0970 \pm i0.9951)$$

$$(\lambda_1^+, \lambda_2^+ \pm i\vartheta_2^+) = (0.1929, \quad -4.596 \times 10^{-6} \pm i5.428)$$

The  $\lambda_2^- \pm i\vartheta_2^-$  eigenvectors span a plane

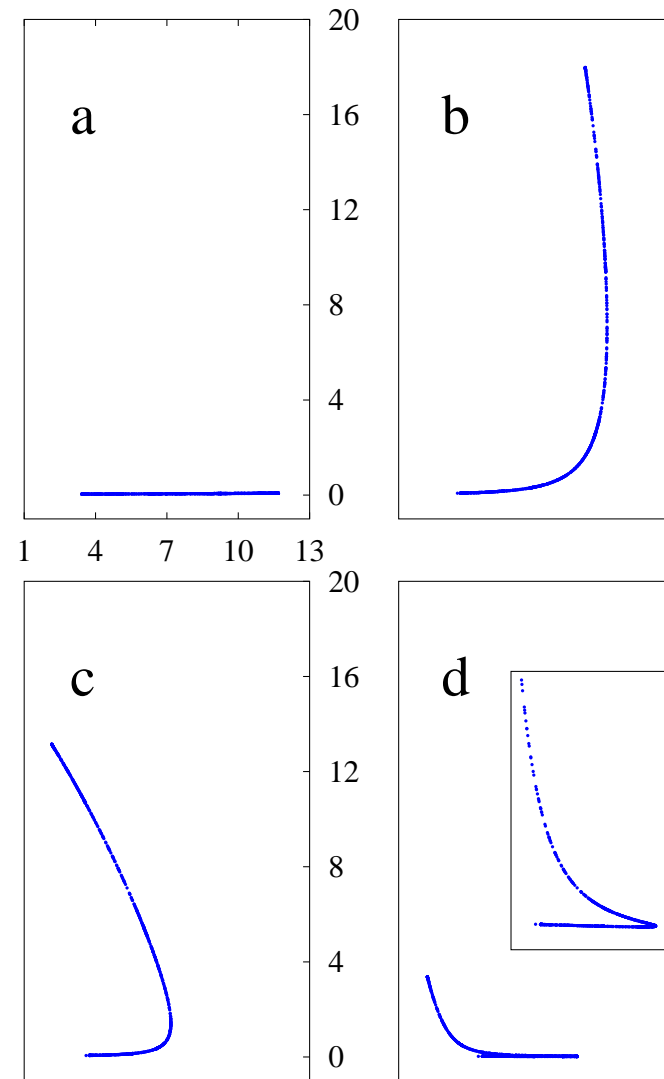
this plane rotates with angular period  $T_- \approx |2\pi/\vartheta_2^-| = 6.313$

a trajectory that starts near the “-” equilibrium point spirals away per one rotation with multiplier  $\Lambda_{\text{radial}} \approx \exp(\lambda_2^- T_-) = 1.84$

each Poincaré section return, contracted into the stable manifold by amazing factor of  $\Lambda_1 \approx \exp(\lambda_1^- T_-) = 10^{-15.6}$  (!)

start with a 1 mm interval pointing in the contracting  $\Lambda_1$  eigendirection.

After one Poincaré return the interval is of order of  $10^{-4}$  fermi



Rössler Poincaré return map is in practice 1 - dimensional