Chapter 2
Flows

Dynamical systems

state space $\mathcal{M}$

representative point $x(t) \in \mathcal{M}$: a physical system at instant in time

dynamics: $f^t(x_0)$ = representative point time t later

deterministic dynamics: evolution rule f maps a point into exactly one point at time t.

dynamical system: the pair $(\mathcal{M}, f)$

$\mathcal{M} \approx \mathbb{R}^d$, d numbers determine next state.

Dynamical systems

flow: evolution in continuous time $t \in \mathbb{R}$:

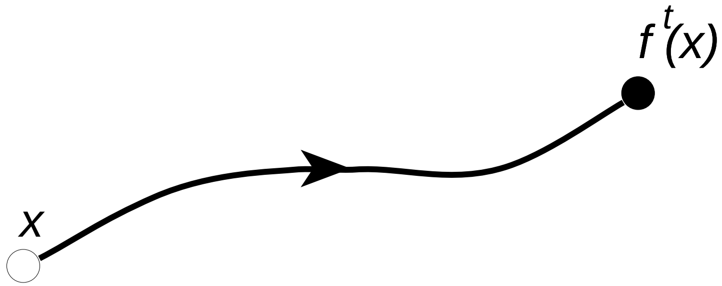
iteration of a map: evolution advances in discrete time steps, integer time $t \in \mathbb{Z}$:

trajectory: evolution rule traces out curve $x(t) = f^t(x_0)$, through the point $x_0 = x(0)$.

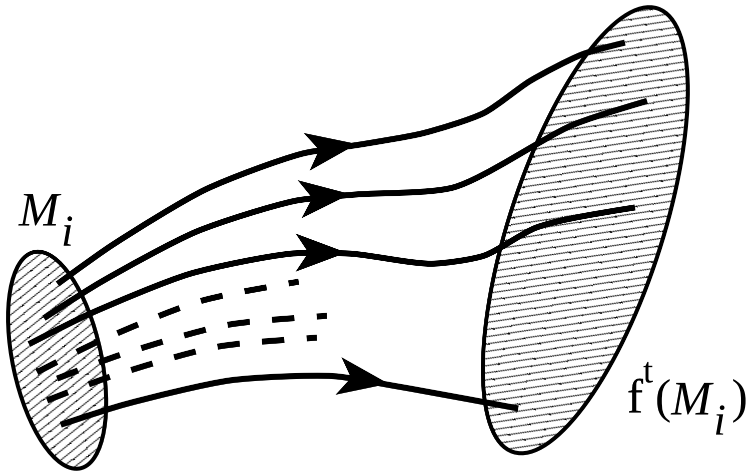
orbit of x_0: subset in $\mathcal{M}$ of points reached by the (possibly infinite) trajectory of x_0

For a flow, an orbit is a continuous curve; for a map, it is a sequence of points.

smooth dynamical system: f^t can be differentiated as many times as needed.



trajectory traced out by the evolution rule f^t. Starting from the phase space point x, after a time t, the point is at $f^t(x)$.



evolution rule f^t maps a region $\mathcal{M}_i$ of the phase space into the region $f^t(\mathcal{M}_i)$.

What are the possible trajectories?

$$\begin{aligned} \text{stationary:}\quad & f^t(x) = x && \text{for all } t \\ \text{periodic:}\quad & f^t(x) = f^{t+T_p}(x) && \text{for a given minimum period } T_p \\ \text{aperiodic:}\quad & f^t(x) \neq f^{t'}(x) && \text{for all } t \neq t' \ . \end{aligned}$$

A periodic orbit corresponds to a trajectory that returns exactly to the initial point in a finite time.

Periodic orbits - a very small subset of the phase space, in the same sense that rational numbers are a set of zero measure on the unit interval.

Flows

For infinitesimal times, flows can be defined by differential equations – a generalized vector field

$$v(x) = \dot{x}(t) .$$

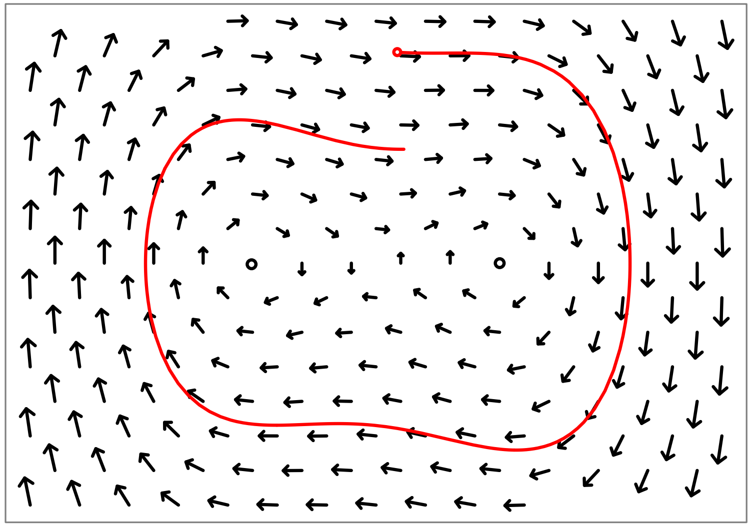
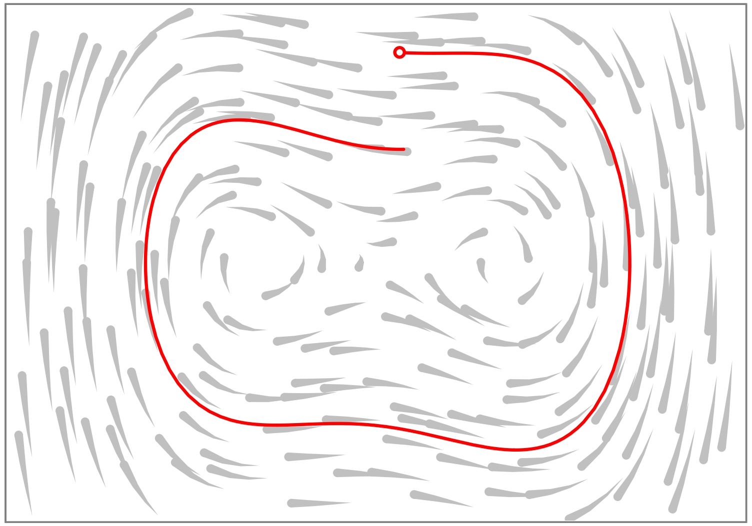
Examples:

Newton's laws for a mechanical system

general flows, mechanical or not, defined by a time-independent vector field $v(x)$

Unforced Duffing system

$$\dot{x}(t) = y(t)$$
$$\dot{y}(t) = -0.15\, y(t) + x(t) - x(t)^3$$

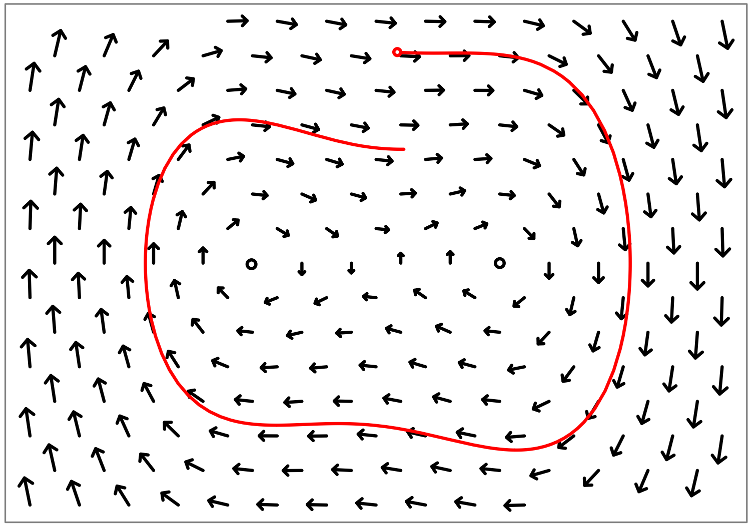
(a)  (b) 

velocity vectors superimposed over the configuration coordinates $(x(t), y(t)) \in \mathcal{M}$, but they belong to a different space, the tangent bundle $\mathbf{T}\mathcal{M}$.

Equilibria

x_q is an equilibrium point

$$\text{if} \qquad v(x_q) = 0 \,,$$



The Duffing flow – bit of a bore – every trajectory ends up in one of the two attractive equilibrium points.

Lorenz "butterfly" strange attractor. (J. Halcrow)

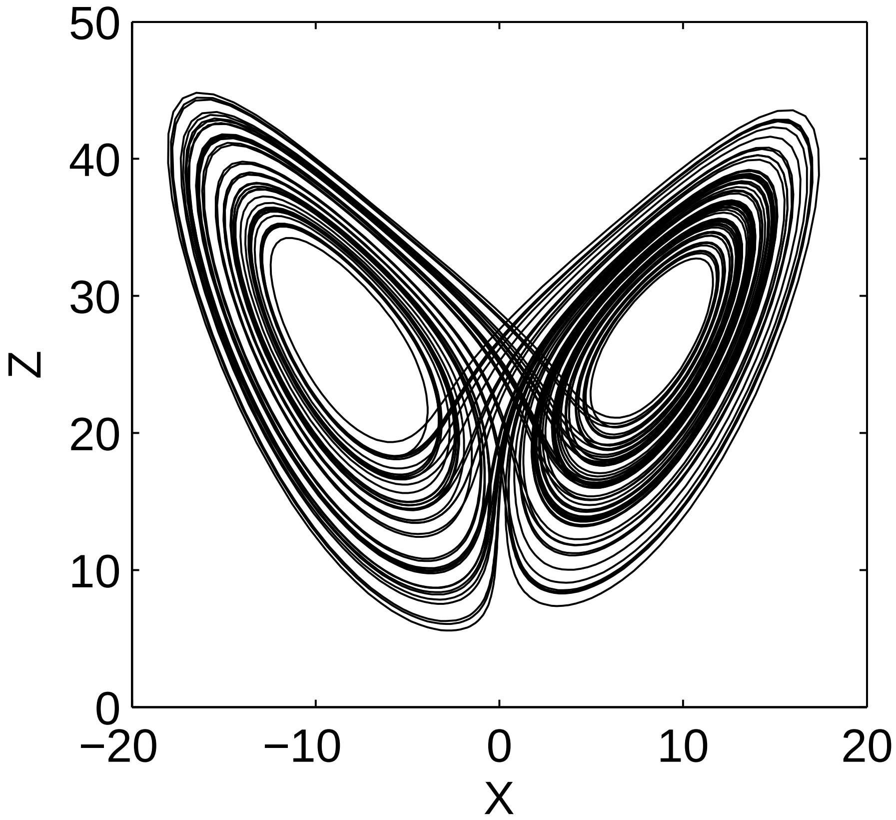
$$\dot{x} = v(x) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma(y-x) \\ \rho x - y - xz \\ xy - bz \end{bmatrix} \qquad (1)$$

Lorenz fixed $\sigma = 10$, $b = 8/3$, and varied the "Rayleigh number" ρ.

$0 < \rho < 1$ the equilibrium $EQ_0 = (0,0,0)$ attractive. At $\rho = 1$ a pitchfork bifurcation into a pair of equilibria at

$$x_{EQ1,2} = (\pm\sqrt{b(\rho-1)}, \pm\sqrt{b(\rho-1)}, \rho-1),$$

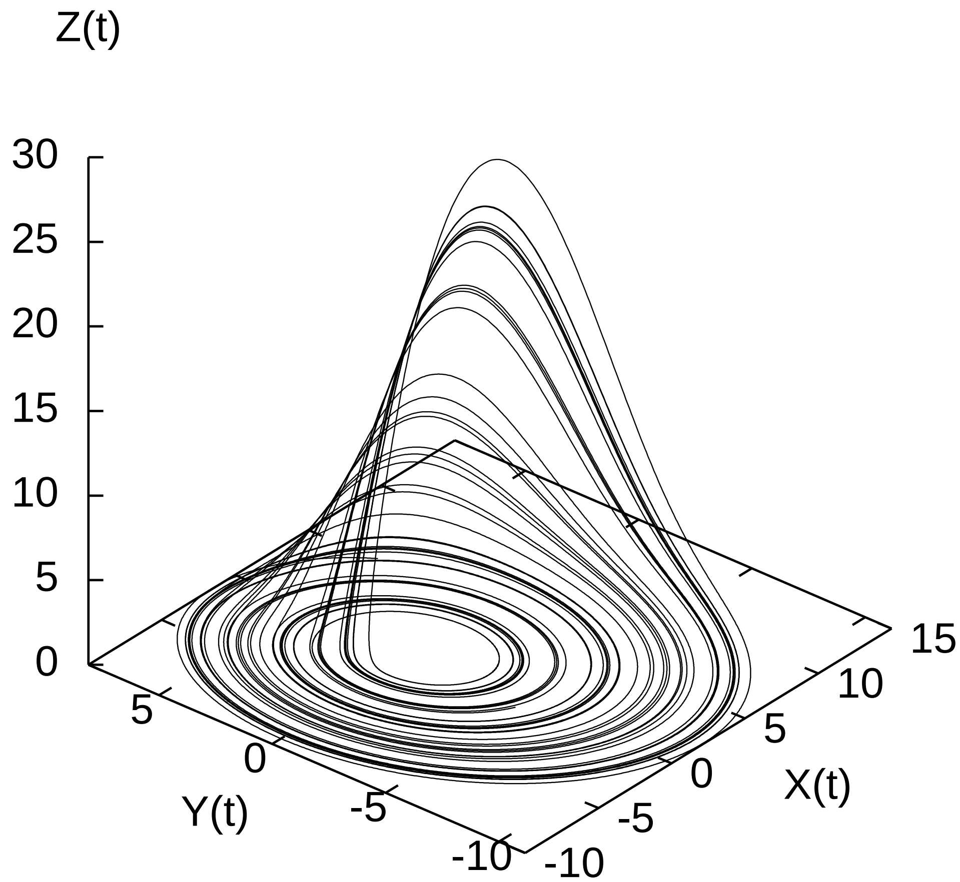
Lorenz strange attractor



Rössler flow

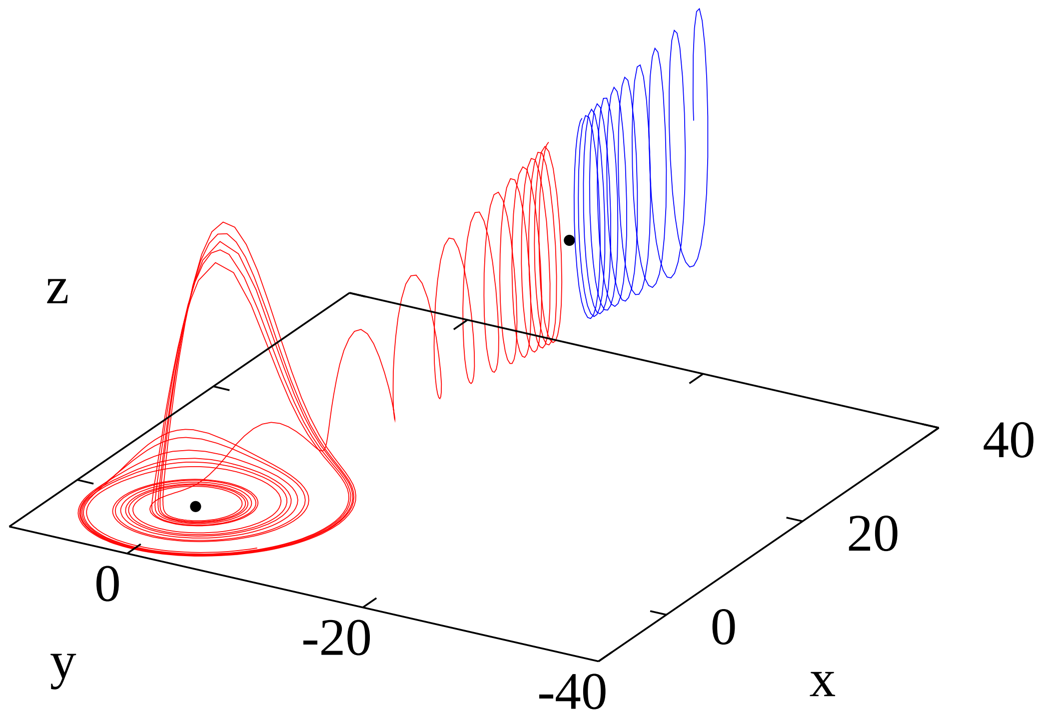
$$\dot{x} = -y - z$$
$$\dot{y} = x + ay$$
$$\dot{z} = b + z(x - c),$$
$$a = b = 0.2, \quad c = 5.7.$$

A typical numerically integrated long-time trajectory



Equilibria of Rössler flow

Two trajectories of the Rössler flow initiated in the neighborhood of the "+" equilibrium point



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2 repelling equilibrium points (no dynamics there!)

$$(x^-, y^-, z^-) = (0.0070, -0.0351, 0.0351)$$
$$(x^+, y^+, z^+) = (5.6929, -28.464, 28.464)$$

Résumé

A dynamical system -- a flow, or an iterated map -- is defined by specifying a pair $(\mathcal{M}, f)$, where $\mathcal{M}$ is the phase space and $f : \mathcal{M} \to \mathcal{M}$. The key concepts in exploration of the long time dynamics are the notions of recurrence and of the non--wandering set of f, the union of all the non-wandering points of $\mathcal{M}$.