Chapter 9
World in a mirror

Symmetries are beautiful

Our hymn to symmetry is a symphony in two movements:

1. symmetry induced interrelations amongst individual orbits
2. - a symmetry equates multiplets of equivalent orbits.
   - group operations that relate distinct tiles do double duty as letters of an alphabet which assigns symbolic itineraries to trajectories.
3. how symmetries affect global densities of trajectories, the factorization of spectral determinants.

Fundamental domain

If invariant under a set of discrete symmetries G, the state space $\mathcal{M}$ is tiled by a set of symmetry-related tiles, and the dynamics can be reduced to dynamics within one such tile, the fundamental domain $\mathcal{M}/G$.

If the symmetry is continuous the dynamics is reduced to a lower-dimensional desymmetrized system $\mathcal{M}/G$.

families of symmetry-related full state space cycles are replaced by fewer, shorter "relative" cycles.

The notion of a prime periodic orbit is replaced by the notion of a relative periodic orbit, the shortest segment of the full state space cycle which tiles the cycle under the action of the group.

Discrete symmetries

A finite group consists of a set of elements

$$G = \{e, g_2, \ldots, g_{|G|}\} \qquad (5)$$

and a group multiplication rule $g_j \circ g_i$ satisfying

1. Closure: If $g_i, g_j \in G$, then $g_j \circ g_i \in G$
2. Associativity: $g_k \circ (g_j \circ g_i) = (g_k \circ g_j) \circ g_i$
3. Identity e: $g \circ e = e \circ g = g$ for all $g \in G$
4. Inverse g^{-1}: For every $g \in G$, there exists a unique element $h = g^{-1} \in G$ such that $h \circ g = g \circ h = e$.

$|G|$, the number of elements, is the order of the group.

Coordinate transformations

linear coordinate transformation $x \to \mathbf{T}x$ maps the vector $x \in \mathcal{M}$ into $\mathbf{T}x \in \mathcal{M}$. coordinate transformation $f(x) \to \mathbf{T}^{-1}f(x)$ changes the coordinate system with respect to which the vector $f(x) \in \mathcal{M}$ is measured. Together, they yield the map in the transformed coordinates:

$$\hat{f}(x) = \mathbf{T}^{-1} f(\mathbf{T}x)\,. \qquad (6)$$

$[d \times d]$ matrix $\mathbf{g}$, is the linear representation of element $g \in G$.

If the coordinate transformation $\mathbf{g}$ belongs to a linear non-singular representation of a discrete (finite) group G, for any element $g \in G$, there exists a number $m \leqslant |G|$ such that

$$g^m \equiv \underbrace{g \circ g \circ \cdots \circ g}_{m \text{ times}} = e \quad \to \quad |\det \mathbf{g}| = 1\,. \qquad (7)$$

As the modulus of its determinant is unity, $\det \mathbf{g}$ is an mth root of 1.

Desymmetrization of Lorenz flow

Lorenz equation is invariant under $G = \{e, R\}$, where R is $[x, y]$-plane rotation by π about the z-axis:

$$(x, y, z) \to R(x, y, z) = (-x, -y, z) \,.$$

$R^2 = 1$ condition decomposes the state space into $\mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^-$, the z-axis $\mathcal{M}^+$ and the $[x, y]$ plane $\mathcal{M}^-$.

The 1-d $\mathcal{M}^+$ subspace is the fixed-point subspace of D_1, with the z-axis points left point-wise invariant under the group action

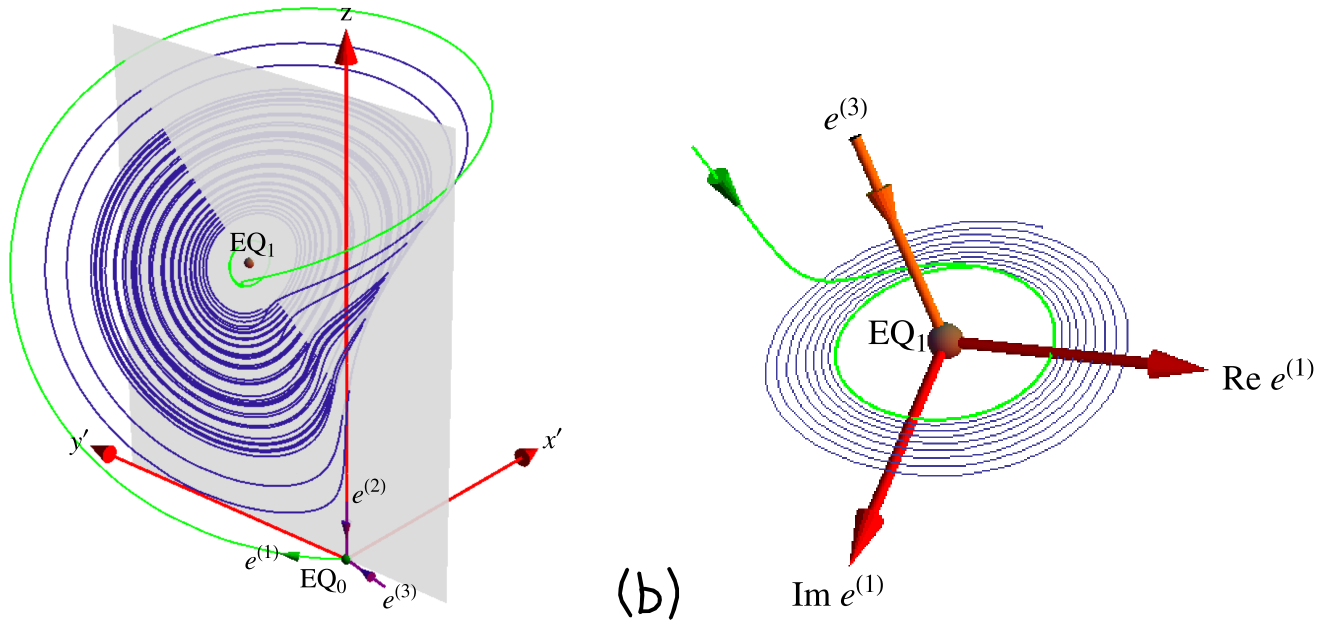
$$\mathrm{Fix}(D_1) = \{x \in \mathcal{M}^+ : g\,x = x \text{ for } g \in \{e, R\}\} \,.$$

A point $x(t)$ in $\mathrm{Fix}(G)$ remains within $x(t) \subseteq \mathrm{Fix}(G)$ for all times; the subspace $\mathcal{M}^+ = \mathrm{Fix}(G)$ is flow invariant. The Lorenz equation is reduced to the exponential contraction to the EQ_0 equilibrium,

$$\dot{z} = -b\,z \,.$$

In higher-dimensional state spaces the flow-invariant $\mathcal{M}^+$ subspace can itself be high-dimensional, with interesting dynamics of its own. This subspace is a topological obstruction: the number of winds of a trajectory around $\to$ a natural symbolic dynamics.

The state space is divided into a half-space fundamental domain $\tilde{\mathcal{M}} = \mathcal{M}/D_1$ and its 180^o rotation $R\tilde{\mathcal{M}}$. Take the fundamental domain $\tilde{\mathcal{M}}$ to be the half-space between the viewer and $\mathcal{P}$. Then the full Lorenz flow is captured by re-injecting back into $\tilde{\mathcal{M}}$ any trajectory that exits it, by a rotation of π around the z axis.



(a) Lorenz attractor plotted in $[x', y', z]$, the doubled-polar angle coordinates, with points related by π-rotation in the $[x, y]$ plane identified. Stable eigenvectors of EQ_0: $\mathbf{e}^{(3)}$ and $\mathbf{e}^{(2)}$, along the z axis. Unstable manifold orbit $W^u(EQ_0)$ (green) is a continuation of the unstable $\mathbf{e}^{(1)}$ of EQ_0.

(b) Blow-up of the region near EQ_1: The unstable eigenplane of EQ_1 is defined by $\mathrm{Re}\,\mathbf{e}^{(2)}$ and $\mathrm{Im}\,\mathbf{e}^{(2)}$, the stable eigenvector $\mathbf{e}^{(3)}$. The descent of the EQ_0 unstable manifold (green) defines the innermost edge of the strange attractor. As it is clear from (a), it also defines its outermost edge.

A state space redefinition that maps a pair of points related by symmetry into a single point accomplished by expressing (x, y) in polar coordinates $(x, y) = (r\cos\vartheta, r\sin\vartheta)$, and then plotting the flow in the "doubled-polar angle representation:"

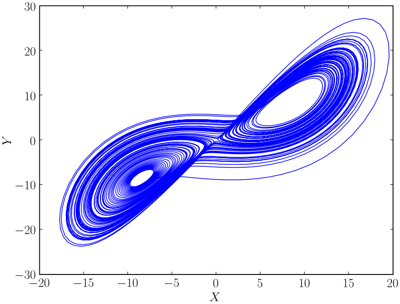
$$\begin{aligned}(x', y') &= (r\cos 2\vartheta, r\sin 2\vartheta)\\ &= ((x^2 - y^2)/r, 2xy/r),\end{aligned}$$

(E. Siminos and J. Halcrow)

Lorenz flow

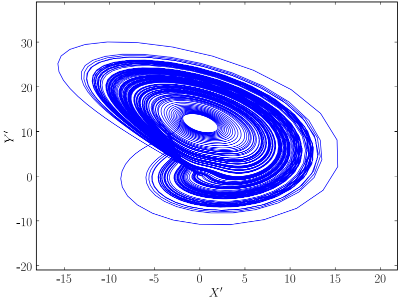
$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = \rho x - y - xz$$
$$\dot{z} = xy - bz$$

EQ_1 comes in 2 copies



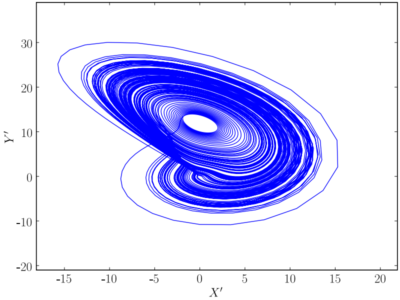
(Lorenz)/Z_2 quotient ('orbit,' 'image') space

- equivariant under $x, y \to -x, -y$



(Lorenz)/Z_2 quotient ('orbit,' 'image') space

- equivariant under
  $x, y \rightarrow -x, -y$
- identify equivalent points, so



(Lorenz)/Z_2 quotient ('orbit,' 'image') space

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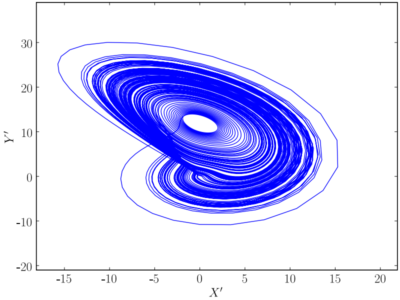
- $EQ1$ comes in 1 copy



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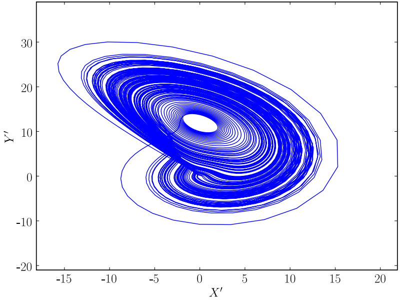
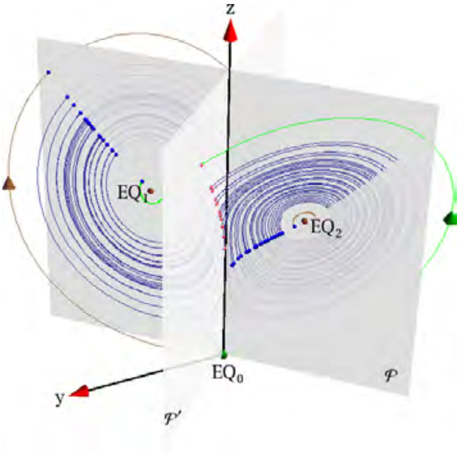
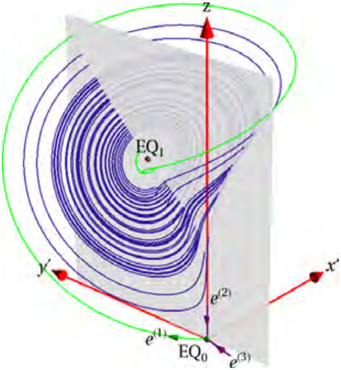


- identify equivalent points by plotting

$$[r, \theta, z] \rightarrow [r, 2\theta, z] = [(x^2 - y^2)/r, 2xy/r, z]$$

Poincaré section



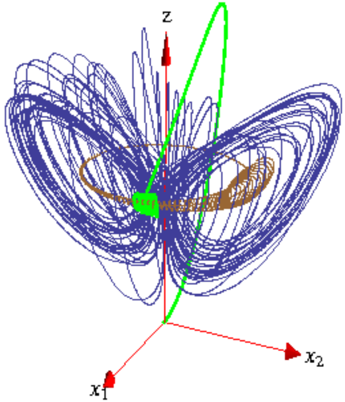


Equivariant flow with continuous symmetry

Gibbon and McGuinness[2] 5-dimensional "Complex Lorenz" model of baroclinic instability in the atmosphere

$$
\begin{aligned}
\dot{x}_1 &= -\sigma x_1 + \sigma y_1 \\
\dot{x}_2 &= -\sigma x_2 + \sigma y_2 \\
\dot{y}_1 &= (r_1 - z)x_1 - r_2 x_2 - y_1 - e y_2 \\
\dot{y}_2 &= r_2 x_1 + (r_1 - z)x_2 + e y_1 - y_2 \\
\dot{z} &= -bz + x_1 y_1 + x_2 y_2
\end{aligned}
$$

[dynamics for $r_1 = 28$, $b = 8/3$, $\sigma = 10$, $a = 1$, $e = 0.01$, $r_2 = 0$]



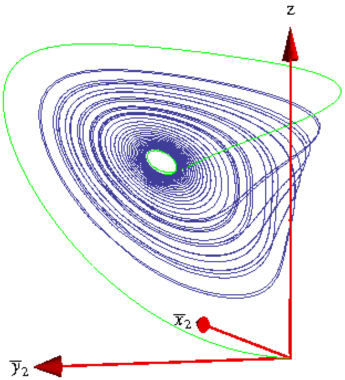
drifts in circles - rotation around z axis $\mathrm{SO}(2)$-equivariant.

[2] Gibbon and McGuinness, *Physica D* **4** (1982)

Complex Lorenz/$\mathrm{SO}(2)$: invariant flow

Quotient $\mathrm{SO}(2)$ by plotting it in an invariant polynomial basis

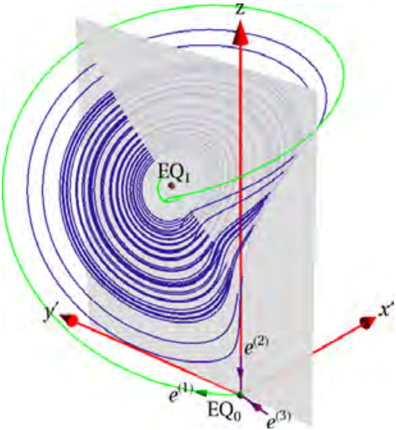
$$\begin{aligned}
\overline{x}_1 &= 0 \quad \text{(group section)}\\
\overline{x}_2 &= (y_1^2 + y_2^2)/r\\
\overline{y}_1 &= -(x_2 y_1 - x_1 y_2)/r\\
\overline{y}_2 &= (x_1 y_1 + x_2 y_2)/r\\
\overline{z} &= z\,, \quad r^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2
\end{aligned}$$

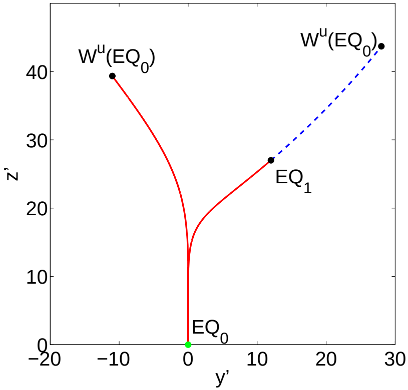


$4d$ flow, no $\mathrm{SO}(2)$-equivariance drift!

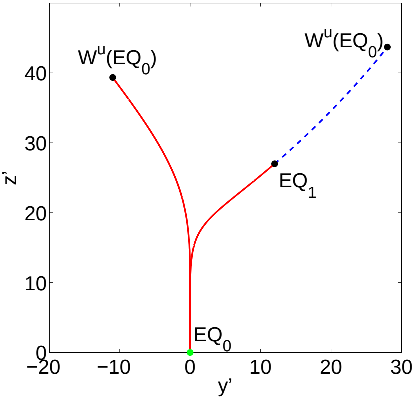
$\approx 1d$ return map; symbolic dynamics, periodic orbits

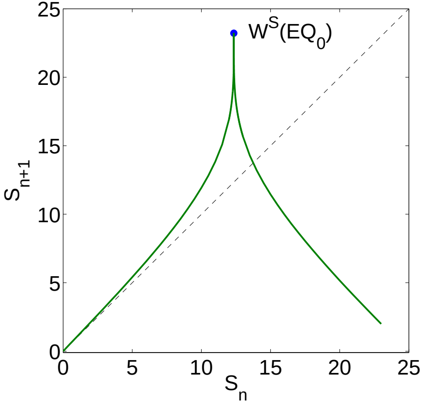
Poincaré section





Return map





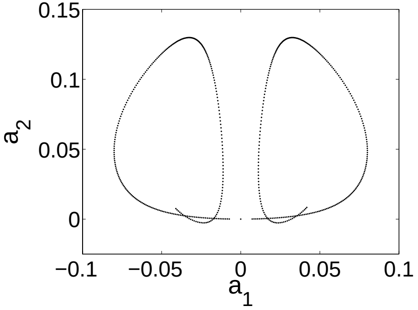
$s =$ unstable manifold arclength along $EQ_1 \to W^u(EQ_0)$

Local unstable manifold return maps

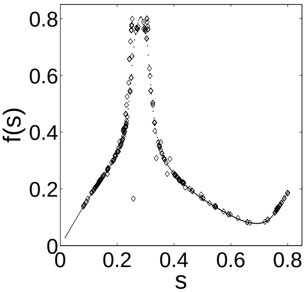
Kuramoto-Sivashinsky: $\infty - d$ state space

but: each repelling Smale horseshoe has its approximate

local $1d$ return map $s \to f(s)$ onto the local unstable manifold:[3]



Poincaré section



return map + cycles

symbolic dynamics, periodic orbits

[3]Lan and Cvitanović, *Phys. Rev. E* (2008)