Chapter 5

Cycle stability

> We owe it to a book to withhold judgment until we reach page 100.
>
> —Henrietta McNutt, George Johnson's seventh-grade English teacher

TOPOLOGICAL FEATURES of a dynamical system –singularities, periodic orbits, and the ways in which the orbits intertwine– are invariant under a general continuous change of coordinates. Equilibria and periodic orbits are *flow-invariant* sets, in the sense that the flow only shifts points along a periodic orbit, but the periodic orbit as the set of periodic points remains unchanged in time. Surprisingly, there also exist quantities that depend on the notion of metric distance between points, but nevertheless do not change value under a smooth change of coordinates. Local quantities such as the eigenvalues of equilibria and periodic orbits, and global quantities such as Lyapunov exponents, metric entropy, and fractal dimensions are examples of properties of dynamical systems independent of coordinate choice.

We now turn to the first, local class of such invariants, linear stability of equilibria and periodic orbits of flows and maps. This will give us metric information about local dynamics, as well as the key concept, the concept of a *neighborhood* of a point x: its size is primarily determined by the number of expanding directions, and the rates of expansion along them: contracting directions play only a secondary role (see sect. 5.6).

If you already know that the eigenvalues of periodic orbits are invariants of a flow, skip this chapter.

fast track:
chapter 7, p. 134

As noted on page 41, a trajectory can be stationary, periodic or aperiodic. For chaotic systems almost all trajectories are aperiodic–nevertheless, equilibria and periodic orbits turn out to be the key to unraveling chaotic dynamics. Here we note a few of the properties that make them so precious to a theorist.

5.1 Equilibria

> At the still point, there the dance is.
>
> —T. S. Eliot, *Four Quartets - Burnt Norton* [00:15:30]

For a start, consider the case where x_q is an equilibrium point (2.8). Expanding around the equilibrium point x_q, using the fact that the stability matrix $A = A(x_q)$ in (4.2) is constant, and integrating, $f^t(x) = x_q + e^{At}(x - x_q) + \cdots$, we verify that the simple formula (4.15) applies also to the Jacobian matrix of an equilibrium point,

$$J^t_q = e^{A_q t}\,, \qquad J^t_q = J^t(x_q)\,,\ A_q = A(x_q)\,. \tag{5.1}$$

As an equilibrium point is stationary, time plays no role, and the eigenvalues and the eigenvectors of stability matrix A_q evaluated at the equilibrium point x_q,

$$A_q\,\mathbf{e}^{(j)} = \lambda^{(j)}_q\,\mathbf{e}^{(j)}\,, \tag{5.2}$$

describe the linearized neighborhood of the equilibrium point, with stability exponents $\lambda^{(j)}_q = \mu^{(j)}_q \pm i\omega^{(j)}_q$ independent of any particular coordinate choice.

- If all $\mu^{(j)} < 0$, then the equilibrium is stable, or a *sink*.
- If some $\mu^{(j)} < 0$, and other $\mu^{(j)} > 0$, the equilibrium is hyperbolic, or a *saddle*.
- If all $\mu^{(j)} > 0$, then the equilibrium is repelling, or a *source*.
- If some $\mu^{(j)} = 0$, think again (you have a symmetry or a bifurcation).

The eigenvectors (5.2) are also the eigenvectors of the Jacobian matrix, $J^t_q\,\mathbf{e}^{(j)} = \exp(t\lambda^{(j)}_q)\,\mathbf{e}^{(j)}$.

5.2 Periodic orbits

An obvious virtue of periodic orbits is that they are *topological* invariants: a fixed point remains a fixed point for any choice of coordinates, and similarly a periodic orbit remains periodic in any representation of the dynamics. Any reparametrization of a dynamical system that preserves its topology has to preserve topological relations between periodic orbits, such as their relative inter-windings and knots. So the mere existence of periodic orbits suffices to partially organize the spatial layout of a non–wandering set. No less important, as we shall now show, is the fact that cycle eigenvalues are *metric* invariants: they determine the relative sizes of neighborhoods in a non–wandering set.

We start by noting that due to the multiplicative structure (4.20) of Jacobian matrices, the Jacobian matrix for the rth repeat of a prime cycle p of period T is

$$J^{rT}(x) = J^T(f^{(r-1)T}(x))\cdots J^T(f^T(x))J^T(x) = J_p(x)^r\,, \tag{5.3}$$

where $J_p(x) = J^T(x)$ is the Jacobian matrix for a single traversal of the prime cycle p, $x \in \mathcal{M}_p$ is any point on the cycle, and $f^{rT}(x) = x$ as $f^t(x)$ returns to x every multiple of the period T. Hence, it suffices to restrict our considerations to the stability of prime cycles.

fast track:
sect. 5.3, p. 100

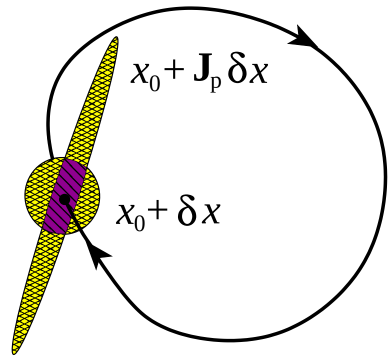

Figure 5.1: For a prime cycle p, Floquet matrix J_p returns an infinitesimal spherical neighborhood of $x_0 \in \mathcal{M}_p$ stretched into an ellipsoid, with overlap ratio along the eigendirection $\mathbf{e}^{(j)}$ of $J_p(x)$ given by the Floquet multiplier $|\Lambda_j|$. These ratios are invariant under smooth nonlinear reparametrizations of state space coordinates, and are intrinsic property of cycle p.

5.2.1 Cycle stability

The time-dependent T-periodic vector fields, such as the flow linearized around a periodic orbit, are described by Floquet theory. Hence from now on we shall refer to a Jacobian matrix evaluated on a periodic orbit p either as a $[d\times d]$ *Floquet matrix* J_p or a $[(d-1)\times(d-1)]$ *monodromy matrix* M_p, to its eigenvalues Λ_j as *Floquet multipliers* (4.7), and to $\lambda_p^{(j)} = \mu_p^{(j)} + i\omega_p^{(j)}$ as *Floquet exponents*. The stretching/contraction rates per unit time are given by the real parts of Floquet exponents

appendix C.2.1

$$\mu_p^{(j)} = \frac{1}{T_p} \ln \left|\Lambda_{p,j}\right| . \tag{5.4}$$

The factor $1/T_p$ in the definition of the Floquet exponents is motivated by its form for the linear dynamical systems, for example (4.31). (Parenthetically, a Floquet exponent is not a Lyapunov exponent (6.11) evaluated on one period the prime cycle p; read chapter 6). When Λ_j is real, we do care about $\sigma^{(j)} = \Lambda_j/|\Lambda_j| \in \{+1, -1\}$, the sign of the jth Floquet multiplier. If $\sigma^{(j)} = -1$ and $|\Lambda_j| \neq 1$, the corresponding eigen-direction is said to be *inverse hyperbolic*. Keeping track of this by case-by-case enumeration is an unnecessary nuisance, so most of our formulas will be stated in terms of the Floquet multipliers Λ_j rather than in the terms of the multiplier signs $\sigma^{(j)}$, exponents $\mu^{(j)}$ and phases $\omega^{(j)}$.

section 7.3

In dynamics the expanding directions, $|\Lambda_e| > 1$, have to be taken care of first, while the contracting directions $|\Lambda_c| < 1$ tend to take care of themselves, hence we always order multipliers Λ_k in order of decreasing magnitude $|\Lambda_1| \geq |\Lambda_2| \geq \ldots \geq |\Lambda_d|$. Since $|\Lambda_j| = e^{t\mu^{(j)}}$, this is the same as ordering by $\mu^{(1)} \geq \mu^{(2)} \geq \ldots \geq \mu^{(d)}$. We sort the Floquet multipliers $\{\Lambda_{p,1}, \Lambda_{p,2}, \ldots, \Lambda_{p,d}\}$ of the Floquet matrix evaluated on the p-cycle into three sets $\{e, m, c\}$

$$\begin{aligned}
\text{expanding:} \quad & \{\Lambda\}_e &= \{\Lambda_{p,j} : \left|\Lambda_{p,j}\right| > 1\} \\
& \{\lambda\}_e &= \{\lambda_p^{(j)} :\ \mu_p^{(j)} > 0\} \\
\text{marginal:} \quad & \{\Lambda\}_m &= \{\Lambda_{p,j} : \left|\Lambda_{p,j}\right| = 1\} \\
& \{\lambda\}_m &= \{\lambda_p^{(j)} :\ \mu_p^{(j)} = 0\} \\
\text{contracting:} \quad & \{\Lambda\}_c &= \{\Lambda_{p,j} : \left|\Lambda_{p,j}\right| < 1\} \\
& \{\lambda\}_c &= \{\lambda_p^{(j)} :\ \mu_p^{(j)} < 0\} .
\end{aligned} \tag{5.5}$$

In what follows, the volume of expanding manifold will play an important role. We denote by Λ_p (no jth eigenvalue index) the product of *expanding* Floquet

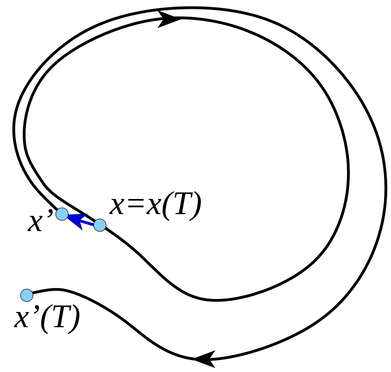

Figure 5.2: An unstable periodic orbit repels every neighboring trajectory $x'(t)$, except those on its center and stable manifolds.

multipliers

$$\Lambda_p = \prod_e \Lambda_{p,e} \,. \qquad (5.6)$$

As J_p is a real matrix, complex eigenvalues always come in complex conjugate pairs, $\Lambda_{p,i+1} = \Lambda^*_{p,i}$, so the product (5.6) is always real.

A periodic orbit of a continuous-time flow, or of a map, or a fixed point of a map is

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- *stable*, a *sink* or a *limit cycle* if all $|\Lambda_j| < 1$ (real parts of all of its Floquet exponents, other than the vanishing longitudinal exponent for perturbations tangent to the cycle, see sect. 5.3.1, are strictly negative, $0 > \mu^{(1)} \geq \mu^{(j)}$).

- *hyperbolic* or a saddle, unstable to perturbations outside its stable manifold if some $|\Lambda_j| > 1$, and other $|\Lambda_j| < 1$ (a set of $\mu^{(j)} \geq \mu_{min} > 0$ is strictly positive, the rest is strictly negative).

- *elliptic*, *neutral* or *marginal* if all $|\Lambda_j| = 1$ ($\mu^{(j)} = 0$).

- *partially hyperbolic*, if $\mu^{(j)} = 0$ for a subset of exponents (other than the longitudinal one).

- *repelling*, or a *source*, unstable to any perturbation if *all* $|\Lambda_j| > 1$ (all Floquet exponents, other than the vanishing longitudinal exponent, are strictly positive, $\mu^{(j)} \geq \mu^{(d)} > 0$).

The region of system parameter values for which a periodic orbit p is stable is called the *stability window* of p. The set of initial points that are asymptotically attracted to $\mathcal{M}_p$ as $t \to +\infty$ (for a fixed set of system parameter values) is called the *basin of attraction* of limit cycle p. Repelling and hyperbolic cycles are unstable to generic perturbations, and thus said to be *unstable*, see figure 5.2.

section 7.4

If *all* Floquet exponents (other than the vanishing longitudinal exponent) of *all* periodic orbits of a flow are strictly bounded away from zero, the flow is said to be *hyperbolic*. Otherwise the flow is said to be *nonhyperbolic*. A confined smooth flow or map is generically nonhyperbolic, with partial ellipticity or marginality expected only in presence of continuous symmetries, or for bifurcation parameter values. As we shall see in chapter 10, in presence of continuous symmetries equilibria and periodic orbits are not likely solutions, and their role is played by higher-dimensional tori, relative equilibria and relative periodic orbits. For

Hamiltonian flows the symplectic Sp(d) symmetry (Liouville phase-space volume conservation, Poincaré invariants) leads to a proliferation of elliptic and partially hyperbolic tori.

section 7.5

Henriette Roux: In my 61,506-dimensional computation of a Navier-Stokes equilibrium I generated about 30 eigenvectors before I wanted to move on. How many of these eigenvectors are worth generating for a particular solution and why?

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A: A rule of the thumb is that you need all equilibrium eigenvalues / periodic orbit Floquet exponents with positive real parts, and at least those negative exponents whose magnitude is less or comparable to the largest expanding eigenvalue. More precisely; keep adding the next least contracting eigenvalue to the sum of the preceding ones as long as the sum is positive (Kaplan-Yorke criterion). Then, just to be conservative, double the number of eigenvalues you keep. You do not need to worry about the remaining (60 thousand!) eigen-directions for which the negative eigenvalues are of larger magnitude, as they always contract: nonlinear terms cannot mix them up in such a way that expansion in some directions overwhelms the strongly contracting ones.

example 5.1
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5.3 Floquet multipliers are invariant

The 1-dimensional map Floquet multiplier (5.21) is a product of derivatives over all points around the cycle, and is therefore independent of which periodic point is chosen as the initial one. In higher dimensions the form of the Floquet matrix $J_p(x_0)$ in (5.3) does depend on the choice of coordinates and the initial point $x_0 \in \mathcal{M}_p$. Nevertheless, as we shall now show, the cycle *Floquet multipliers* are intrinsic property of a cycle in any dimension. Consider the ith eigenvalue, eigenvector pair $(\Lambda_j, \mathbf{e}^{(j)})$ computed from J_p evaluated at a periodic point x,

$$J_p(x)\,\mathbf{e}^{(j)}(x) = \Lambda_j\,\mathbf{e}^{(j)}(x)\,, \qquad x \in \mathcal{M}_p\,. \tag{5.7}$$

Consider another point on the cycle at time t later, $x' = f^t(x)$ whose Floquet matrix is $J_p(x')$. By the semigroup property (4.20), $J^{T+t} = J^{t+T}$, and the Jacobian matrix at x' can be written either as

$$J^{T+t}(x) = J^T(x')\,J^t(x) = J_p(x')\,J^t(x)\,,$$

or $J^t(x)\,J_p(x)$. Multiplying (5.7) by $J^t(x)$, we find that the Floquet matrix evaluated at x' has the same Floquet multiplier,

$$J_p(x')\,\mathbf{e}^{(j)}(x') = \Lambda_j\,\mathbf{e}^{(j)}(x')\,, \quad \mathbf{e}^{(j)}(x') = J^t(x)\,\mathbf{e}^{(j)}(x)\,, \tag{5.8}$$

but with the eigenvector $\mathbf{e}^{(j)}$ transported along the flow $x \to x'$ to $\mathbf{e}^{(j)}(x') = J^t(x)\,\mathbf{e}^{(j)}(x)$. Hence, in the spirit of the Floquet theory (appendix C.2.1) one can define time-periodic eigenvectors (in a co-moving 'Lagrangian frame')

$$\mathbf{e}^{(j)}(t) = e^{-\lambda^{(j)}t}\,J^t(x)\,\mathbf{e}^{(j)}(0)\,, \qquad \mathbf{e}^{(j)}(t) = \mathbf{e}^{(j)}(x(t))\,, \quad x(t) \in \mathcal{M}_p\,. \tag{5.9}$$

J_p evaluated anywhere along the cycle has the same set of Floquet multipliers $\{\Lambda_1, \Lambda_2, \cdots, 1, \cdots, \Lambda_{d-1}\}$. As quantities such as tr $J_p(x)$, det $J_p(x)$ depend only

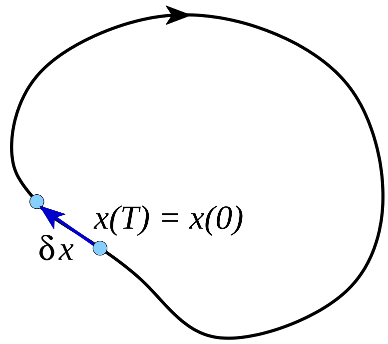

Figure 5.3: Any two points along a periodic orbit p are mapped into themselves after one cycle period T, hence a longitudinal displacement $\delta x = v(x_0)\delta t$ is mapped into itself by the cycle Jacobian matrix J_p.

on the eigenvalues of $J_p(x)$ and not on the starting point x, in expressions such as $\det\left(\mathbf{1} - J_p^r(x)\right)$ we may omit reference to x,

$$\det\left(\mathbf{1} - J_p^r\right) = \det\left(\mathbf{1} - J_p^r(x)\right) \quad \text{for any } x \in \mathcal{M}_p \,. \tag{5.10}$$

We postpone the proof that the cycle Floquet multipliers are smooth conjugacy invariants of the flow to sect. 5.4; time-forward map (5.8) is the special case of this general property of smooth manifolds and their tangent spaces.

5.3.1 Marginal eigenvalues

The presence of marginal eigenvalues signals either a continuous symmetry of the flow (which one should immediately exploit to simplify the problem), or a non-hyperbolicity of a flow (a source of much pain, hard to avoid). In that case (typical of parameter values for which bifurcations occur) one has to go beyond linear stability, deal with Jordan type subspaces (see example 4.3), and sub-exponential growth rates, such as t^α. For flow-invariant solutions such as periodic orbits, the time evolution is itself a continuous symmetry, hence a periodic orbit of a flow always has a *marginal Floquet multiplier*, as we now show.

The Jacobian matrix $J^t(x)$ transports the velocity field $v(x)$ by (4.9), $v(x(t)) = J^t(x_0)\, v(x_0)$. In general the velocity at point $x(t)$ does not point in the same direction as the velocity at point x_0, so this is not an eigenvalue condition for J^t; the Jacobian matrix computed for an arbitrary segment of an arbitrary trajectory has no invariant meaning. However, if the orbit is periodic, $x(T_p) = x(0)$, after a complete period

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exercise 5.1

$$J_p(x)\, v(x) = v(x) \,, \qquad x \in \mathcal{M}_p \,. \tag{5.11}$$

Two successive points on the cycle initially distance $\delta x = x'(0) - x(0)$ apart, are separated by the exactly same distance after a completed period $\delta x(T) = \delta x$, see figure 5.3, hence for a periodic orbit of a *flow* the velocity field v at any point along cycle is an eigenvector $\mathbf{e}^{(\|)}(x) = v(x)$ of the Jacobian matrix J_p with the unit Floquet multiplier, zero Floquet exponent

$$\Lambda_\| = 1 \,, \qquad \lambda^{(\|)} = 0 \,. \tag{5.12}$$

exercise B.3

The continuous invariance that gives rise to this marginal Floquet multiplier is the invariance of a cycle (the set $\mathcal{M}_p$) under a time translation of its points along the cycle. As we shall see in sect. 5.5, this marginal stability direction can be eliminated by cutting the cycle by a Poincaré section and replacing the continuous flow Floquet matrix by the Floquet matrix of the Poincaré return map.

If the flow is governed by a time-independent Hamiltonian, the energy is conserved, and that leads to an additional marginal Floquet multiplier (we shall show in sect. 7.4 that due to the symplectic invariance (7.27) real eigenvalues come in pairs). Further marginal eigenvalues arise in presence of continuous symmetries, as discussed in chapter 10.

5.4 Floquet multipliers are metric invariants

In sect. 5.3 we established that for a given flow, the Floquet multipliers are intrinsic to a given cycle, independent of the starting point along the cycle. Now we prove a much stronger statement: cycle Floquet multipliers are *smooth conjugacy* or *metric invariants* of the flow, the same in *any* representation of the dynamical system. That follows by elementary differential geometry considerations:

If the same dynamics is given by a map f in x coordinates, and a map g in the $y = h(x)$ coordinates, then f and g (or any other good representation) are related by a *smooth conjugacy*, a reparameterization and a coordinate transformation $g = h \circ f \circ h^{-1}$ which maps nearby points of f into nearby points of g. As both f and g are arbitrary representations of the dynamical system, the explicit form of the conjugacy h is of no interest, only the properties invariant under any transformation h are of general import. Furthermore, a good representation should not mutilate the data; the mapping h must be a *smooth conjugacy* which maps nearby points of f into nearby points of g.

This smoothness guarantees that the cycles are not only topological invariants, but that their linearized neighborhoods are also metric invariants. For a fixed point $f(x) = x$ of a 1-dimensional map this follows from the chain rule for derivatives,

$$\begin{aligned} g'(y) &= h'(f \circ h^{-1}(y)) f'(h^{-1}(y)) \frac{1}{h'(x)} \\ &= h'(x) f'(x) \frac{1}{h'(x)} = f'(x) \,. \end{aligned} \tag{5.13}$$

In d dimensions the relationship between the maps in different coordinate representations is again $g \circ h = h \circ f$. The chain rule now relates J', the Jacobian matrix of the map g, to the Jacobian matrix of map f:

$$J'(y)_{ij} = \Gamma(f(x))_{ik} J(x)_{kl} \Gamma(x)^{-1}_{lj} \,, \tag{5.14}$$

where

$$\Gamma(x)_{ik} = \frac{\partial y_i}{\partial x_k} \qquad \text{and} \qquad \Gamma(x)^{-1}_{ik} = \frac{\partial x_i}{\partial y_k} \,.$$

If x is an equilibrium point, $x = f(x)$, Γ is the matrix inverse of Γ^{-1}, and (5.14) is a *similarity* transformation and thus preserves eigenvalues. It is easy to verify that in the case of period n_p cycle $J'_p(y)$ and $J_p(x)$ are again related by a similarity transformation. (Note, though, that this is not true for $J^r(x)$ with $r \neq n_p$). As stability of a flow can always be reduced to stability of a Poincaré return map, a Floquet multiplier of any cycle, for a flow or a map in arbitrary dimension, is a metric invariant of the dynamical system.

exercise B.3

The ith Floquet (multiplier, eigenvector) pair $(\Lambda_i, \mathbf{e}^{(i)})$ are computed from J evaluated at a periodic point x, $J(x)\,\mathbf{e}^{(i)}(x) = \Lambda_i\,\mathbf{e}^{(i)}(x)\,, \quad x \in \mathcal{M}_p$. Multiplying by $\Gamma(x)$ from the left, and inserting $\mathbf{1} = \Gamma(x)^{-1}\Gamma(x)$, we find that the J evaluated at $y = h(x)$ has the same Floquet multiplier,

$$J'_p(y)\,\mathbf{e}^{(i)}(y)' = \Lambda_i\,\mathbf{e}^{(i)}(y)'\,, \tag{5.15}$$

but with the eigenvector $\mathbf{e}^{(i)}(x)$ mapped to $\mathbf{e}^{(i)}(y)' = \Gamma(x)\,\mathbf{e}^{(i)}(x)$.

5.5 Stability of Poincaré map cycles

(R. Paškauskas and P. Cvitanović)

If a continuous flow periodic orbit p pierces the Poincaré section $\mathcal{P}$ once, the section point is a fixed point of the Poincaré return map P with stability (4.25)

$$\hat{J}_{ij} = \left(\delta_{ik} - \frac{v_i\,U_k}{(v \cdot U)}\right) J_{kj}\,, \tag{5.16}$$

with all primes dropped, as the initial and the final points coincide, $x' = f^T(x) = x$. If the periodic orbit p pierces the Poincaré section n times, the same observation applies to the nth iterate of P.

We have already established in (4.26) that the velocity $v(x)$ is a zero eigenvector of the Poincaré section Floquet matrix, $\hat{J} v = 0$. Consider next $(\Lambda_\alpha, \mathbf{e}^{(\alpha)})$, the full state space αth (eigenvalue, eigenvector) pair (5.7), evaluated at a periodic point on a Poincaré section,

$$J(x)\,\mathbf{e}^{(\alpha)}(x) = \Lambda_\alpha\,\mathbf{e}^{(\alpha)}(x)\,, \qquad x \in \mathcal{P}\,. \tag{5.17}$$

Multiplying (5.16) by $\mathbf{e}^{(\alpha)}$ and inserting (5.17), we find that the full state space Floquet matrix and the Poincaré section Floquet matrix $\hat{J}$ have the same Floquet multiplier

$$\hat{J}(x)\,\hat{\mathbf{e}}^{(\alpha)}(x) = \Lambda_\alpha\,\hat{\mathbf{e}}^{(\alpha)}(x)\,, \qquad x \in \mathcal{P}\,, \tag{5.18}$$

where $\hat{\mathbf{e}}^{(\alpha)}$ is a projection of the full state space eigenvector onto the Poincaré section:

$$(\hat{\mathbf{e}}^{(\alpha)})_i = \left(\delta_{ik} - \frac{v_i\ U_k}{(v \cdot U)}\right)(\mathbf{e}^{(\alpha)})_k\,. \tag{5.19}$$

Hence, $\hat{J}_p$ evaluated on any Poincaré section point along the cycle p has the same set of Floquet multipliers $\{\Lambda_1, \Lambda_2, \cdots \Lambda_d\}$ as the full state space Floquet matrix J_p, except for the marginal unit Floquet multiplier (5.12).

As established in (4.26), due to the continuous symmetry (time invariance) $\hat{J}_p$ is a rank $d-1$ matrix. We shall refer to the rank $[(d-1-N)\times(d-1-N)]$ submatrix with $N-1$ continuous symmetries quotiented out as the *monodromy matrix* M_p (from Greek *mono-* = alone, single, and *dromo* = run, racecourse, meaning a single run around the stadium). Quotienting continuous symmetries is discussed in chapter 10 below.

5.6 There goes the neighborhood

In what follows, our task will be to determine the size of a *neighborhood* of $x(t)$, and that is why we care about the Floquet multipliers, and especially the unstable (expanding) ones.

Nearby points aligned along the stable (contracting) directions remain in the neighborhood of the trajectory $x(t) = f^t(x_0)$; the ones to keep an eye on are the points which leave the neighborhood along the unstable directions: all chaos arises from flights along these these directions. The sub-volume $|\mathcal{M}_{x_0}| = \prod_i^e \Delta x_i$ of the set of points which get no further away from $f^t(x_0)$ than L, the typical size of the system, is fixed by the condition that $\Delta x_i \Lambda_i = O(L)$ in each expanding direction i. Hence the neighborhood size scales as $|\mathcal{M}_{x_0}| \propto O(L^{d_e})/|\Lambda_p| \propto 1/|\Lambda_p|$ where Λ_p is the product of expanding Floquet multipliers (5.6) only; contracting ones play a secondary role. Discussion of sect. 1.5.1, figure 1.9, and figure 5.1 illustrate intersection of initial volume with its return, and chapters 12 and 18 illustrate the key role that the unstable directions play in systematically partitioning the state space of a given dynamical system. The contracting directions are so secondary that even infinitely many of them (for example, the infinity of contracting eigen-directions of the spatiotemporally chaotic dynamics described by a PDE will not matter.

So the dynamically important information is carried by the expanding sub-volume, not the total volume computed so easily in (4.29). That is also the reason why the dissipative and the Hamiltonian chaotic flows are much more alike than one would have naively expected for 'compressible' *vs.* 'incompressible' flows. In hyperbolic systems what matters are the expanding directions. Whether the contracting eigenvalues are inverses of the expanding ones or not is of secondary importance. As long as the number of unstable directions is finite, the same theory applies both to the finite-dimensional ODEs and infinite-dimensional PDEs.

Résumé

Periodic orbits play a central role in any invariant characterization of the dynamics, because (a) their existence and inter-relations are a *topological*, coordinate-independent property of the dynamics, and (b) their Floquet multipliers form an infinite set of *metric invariants*: The Floquet multipliers of a periodic orbit remain invariant under any smooth nonlinear change of coordinates $f \to h \circ f \circ h^{-1}$. Let us summarize the linearized flow notation used throughout the ChaosBook.

Differential formulation, flows: Equations

$$\dot{x} = v \,, \qquad \dot{\delta x} = A\,\delta x$$

govern the dynamics in the tangent bundle $(x, \delta x) \in \mathbf{T}\mathcal{M}$ obtained by adjoining the d-dimensional tangent space $\delta x \in T\mathcal{M}_x$ to every point $x \in \mathcal{M}$ in the d-dimensional state space $\mathcal{M} \subset \mathbb{R}^d$. The *stability matrix* $A = \partial v/\partial x$ describes the instantaneous rate of shearing of the infinitesimal neighborhood of $x(t)$ by the flow.

Finite time formulation, maps: A discrete sets of trajectory points $\{x_0, x_1, \cdots, x_n, \cdots\} \in \mathcal{M}$ can be generated by composing finite-time maps, either given as

$x_{n+1} = f(x_n)$, or obtained by integrating the dynamical equations

$$x_{n+1} = f(x_n) = x_n + \int_{t_n}^{t_{n+1}} d\tau\, v(x(\tau))\,, \qquad \Delta t_n = t_{n+1} - t_n\,, \tag{5.20}$$

for a discrete sequence of times $\{t_0, t_1, \cdots, t_n, \cdots\}$, specified by some criterion such as strobing or Poincaré sections. In the discrete time formulation the dynamics in the tangent bundle $(x, \delta x) \in \mathbf{T}\mathcal{M}$ is governed by

$$x_{n+1} = f(x_n)\,, \quad \delta x_{n+1} = J(x_n)\,\delta x_n\,, \qquad J(x_n) = J^{\Delta t_n}(x_n)\,,$$

where $J(x_n) = \partial x_{n+1}/\partial x_n = \int d\tau\ \exp(A\,\tau)$ is the 1-time step Jacobian matrix.
Stability of invariant solutions: The linear stability of an equilibrium $v(x_q) = 0$ is described by the eigenvalues and eigenvectors $\{\lambda^{(j)}, \mathbf{e}^{(j)}\}$ of the stability matrix A evaluated at the equilibrium point, and the linear stability of a periodic orbit $f^T(x) = x$, $x \in \mathcal{M}_p$,

$$J_p(x)\,\mathbf{e}^{(j)}(x) = \Lambda_j\,\mathbf{e}^{(j)}(x)\,, \qquad \Lambda_j = \sigma^{(j)} e^{\lambda^{(j)}T}\,,$$

by its Floquet multipliers, vectors and exponents $\{\Lambda_j, \mathbf{e}^{(j)}\}$, where $\lambda^{(j)} = \mu^{(j)} \pm i\omega^{(j)}$. For every continuous symmetry there is a marginal eigen-direction, with $\Lambda_j = 1$, $\lambda^{(j)} = 0$. With all $1 + N$ continuous symmetries quotiented out (Poincaré sections for time, slices for continuous symmetries of dynamics, see chapter 10.4.3) linear stability of a periodic orbit (and, more generally, of a partially hyperbolic torus) is described by the $[(d\text{-}1\text{-}N)\times(d\text{-}1\text{-}N)]$ monodromy matrix, all of whose Floquet multipliers $|\Lambda_j| \neq 1$ are generically strictly hyperbolic,

$$M_p(x)\,\mathbf{e}^{(j)}(x) = \Lambda_j\,\mathbf{e}^{(j)}(x)\,, \qquad x \in \mathcal{M}_p/G\,.$$

We shall show in chapter 11 that extending the linearized stability hyperbolic eigen-directions into stable and unstable manifolds yields important global information about the topological organization of state space. What matters most are the expanding directions. The physically important information is carried by the unstable manifold, and the expanding sub-volume characterized by the product of expanding Floquet multipliers of J_p. As long as the number of unstable directions is finite, the theory can be applied to flows of arbitrarily high dimension.

in depth:
appendix C, p. 822

fast track:
chapter 9.4, p. 174

Commentary

Remark 5.1 Periodic orbits *vs.* 'cycles'. Throughout this text, the terms 'periodic orbit' and 'cycle' (which has many other uses in mathematics) are used interchangeably; while 'periodic orbit' is more precise, 'pseudo-cycle' is easier on the ear than 'pseudo-periodic-orbit.' In Soviet times obscure abbreviations were a rage, but here we shy away from acronyms such as UPOs (Unstable Periodic Orbits). We refer to unstable periodic orbits simply as 'periodic orbits', and the stable ones 'limit cycles'. Lost in the mists of time is the excitement experienced by the first physicist to discover that there are periodic orbits other than the limit cycles reached by mindless computation forward in time; but once one understands that there are at most several stable limit cycles (SPOs?) as opposed to the Smale horseshoe infinities of unstable cycles (UPOs?), what is gained by prefix 'U'? A bit like calling all bicycles 'unstable bicycles'.

Remark 5.2 Periodic orbits and Floquet theory. Study of time-dependent and T-periodic vector fields is a classical subject in the theory of differential equations [1, 2]. In physics literature Floquet exponents often assume different names according to the context where the theory is applied: they are called Bloch phases in the discussion of Schrödinger equation with a periodic potential [3], or quasi-momenta in the quantum theory of time-periodic Hamiltonians. Here a discussion of Floquet theory is given in appendix C.2.1. For further reading on periodic orbits, consult Moehlis and K. Josić [**?**] Scholarpedia.org article.

5.7 Examples

The reader is urged to study the examples collected here. To return back to the main text, click on [click to return] pointer on the margin.

Example 5.1 ***Stability of cycles of 1-dimensional maps:*** *The stability of a prime cycle p of a 1-dimensional map follows from the chain rule (4.42) for stability of the n_pth iterate of the map*

$$\Lambda_p = \frac{d}{dx_0} f^{n_p}(x_0) = \prod_{m=0}^{n_p-1} f'(x_m) \,, \quad x_m = f^m(x_0) \,. \tag{5.21}$$

Λ_p is a property of the cycle, not the initial periodic point, as taking any periodic point in the p cycle as the initial one yields the same Λ_p.

A critical point *x_c is a value of x for which the mapping $f(x)$ has vanishing derivative, $f'(x_c) = 0$. A periodic orbit of a 1-dimensional map is* stable *if*

$$\left|\Lambda_p\right| = \left|f'(x_{n_p}) f'(x_{n_p-1}) \cdots f'(x_2) f'(x_1)\right| < 1 \,,$$

and superstable *if the orbit includes a critical point, so that the above product vanishes. For a stable periodic orbit of period n the slope Λ_p of the nth iterate $f^n(x)$ evaluated on a periodic point x (fixed point of the nth iterate) lies between -1 and 1. If $\left|\Lambda_p\right| > 1$, p-cycle is* unstable.

Example 5.2 ***Stability of cycles for maps:*** *No matter what method one uses to determine unstable cycles, the theory to be developed here requires that their Floquet multipliers be evaluated as well. For maps a Floquet matrix is easily evaluated by picking any periodic point as a starting point, running once around a prime cycle, and multiplying the individual periodic point Jacobian matrices according to (4.22). For example, the Floquet matrix M_p for a prime cycle p of length n_p of the Hénon map (3.17) is given by (4.43),*

$$M_p(x_0) = \prod_{k=n_p}^{1} \begin{pmatrix} -2ax_k & b \\ 1 & 0 \end{pmatrix} \,, \qquad x_k \in \mathcal{M}_p \,,$$

and the Floquet matrix M_p for a 2-dimensional billiard prime cycle p of length n_p

$$M_p = (-1)^{n_p} \prod_{k=n_p}^{1} \begin{pmatrix} 1 & \tau_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r_k & 1 \end{pmatrix}$$

follows from (8.11) of chapter 8 below. The decreasing order in the indices of the products in above formulas is a reminder that the successive time steps correspond to multiplication from the left, $M_p(x_1) = M(x_{n_p}) \cdots M(x_1)$. We shall compute Floquet multipliers of Hénon map cycles once we learn how to find their periodic orbits, see exercise 13.10.

click to return: p. 100

Exercises

5.1. **A limit cycle with analytic Floquet exponent.** There are only two examples of nonlinear flows for which the Floquet multipliers can be evaluated analytically. Both are cheats. One example is the 2-dimensional flow

$$\begin{aligned}\dot{q} &= p + q(1 - q^2 - p^2) \\ \dot{p} &= -q + p(1 - q^2 - p^2)\,.\end{aligned}$$

Determine all periodic solutions of this flow, and determine analytically their Floquet exponents. Hint: go to polar coordinates $(q, p) = (r\cos\theta, r\sin\theta)$. G. Bard Ermentrout

5.2. **The other example of a limit cycle with analytic Floquet exponent.** What is the other example of a nonlinear flow for which the Floquet multipliers can be evaluated analytically? Hint: email G.B. Ermentrout.

5.3. **Yet another example of a limit cycle with analytic Floquet exponent.** Prove G.B. Ermentrout wrong by solving a third example (or more) of a nonlinear flow for which the Floquet multipliers can be evaluated analytically.

References

[5.1] G. Floquet, “Sur les equations differentielles lineaires à coefficients periodique,” *Ann. Ecole Norm. Ser. 2*, **12**, 47 (1883).

[5.2] E. L. Ince, *Ordinary Differential Equations* (Dover, New York 1953).

[5.3] N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Holt, Rinehart and Winston, New York 1976).