# Chapter 2

# Go with the flow

> Dynamical systems theory includes an extensive body of knowledge about qualitative properties of generic smooth families of vector fields and discrete maps. The theory characterizes structurally stable invariant sets [...] The logic of dynamical systems theory is subtle. The theory abandons the goal of describing the qualitative dynamics of all systems as hopeless and instead restricts its attention to phenomena that are found in selected systems. The subtlety comes in specifying the systems of interest and which dynamical phenomena are to be analyzed.
>
> — John Guckenheimer

(R. Mainieri, P. Cvitanović and E.A. Spiegel)

We define a *dynamical system* $(\mathcal{M}, f)$ and classify its solutions as equilibria, periodic, and aperiodic. An 'aperiodic' solution is either 'wandering' or belongs to a non–wandering set, which in turn can be decomposed into into chain-recurrent sets. Various cases are illustrated with concrete examples, such as the Rössler and Lorenz systems.

fast track:
chapter 19, p. 345

## 2.1 Dynamical systems

In a dynamical system we observe the world as it evolves with time. We express our observations as numbers and record how they change; given sufficiently detailed information and understanding of the underlying natural laws, we see the future in the present as in a mirror. The motion of the planets against the celestial firmament provides an example. Against the daily motion of the stars from East

section 1.3

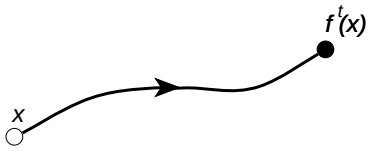


**Figure 2.1:** A trajectory traced out by the evolution rule $f^t$. Starting from the state space point $x$, after a time $t$, the point is at $f^t(x)$.

to West, the planets distinguish themselves by moving among the fixed stars. Ancients discovered that by knowing a sequence of planet's positions–latitudes and longitudes–its future position could be predicted.

For the solar system, tracking the latitude and longitude in the celestial sphere suffices to completely specify the planet's apparent motion. All possible values for positions and velocities of the planets form the *phase space* of the system. More generally, a state of a physical system, at a given instant in time, can be represented by a single point in an abstract space called *state space* $\mathcal{M}$ (mnemonic: curly '$\mathcal{M}$' for a 'manifold'). As the system changes, so does the *representative point* in state space. We refer to the evolution of the totality of such points as a *flow* or *dynamics*, and the function $f^t$ which specifies where the representative point is at time $t$ as the *evolution rule*.

remark 2.1

If there is a definite rule $f$ that tells us how this representative point moves in $\mathcal{M}$, the system is said to be deterministic. For a deterministic dynamical system, the evolution rule takes one point of the state space and maps it into exactly one point. However, this is not always possible. For example, knowing the temperature today is not enough to predict the temperature tomorrow; knowing the value of a stock today will not determine its value tomorrow. The state space can be enlarged, in the hope that in a sufficiently large state space it is possible to determine an evolution rule, so we imagine that knowing the state of the atmosphere, measured over many points over the entire planet should be sufficient to determine the temperature tomorrow. Even that is not quite true, and we are less hopeful when it comes to stocks.

For a deterministic system almost every point has a unique future, so trajectories cannot intersect. We say 'almost' because there might exist a set of measure zero (tips of wedges, cusps, etc.) for which a trajectory is not defined. We may think such sets a nuisance, but it is quite the contrary–they will enable us to partition state space, so that the dynamics can be better understood.

chapter 15

Locally, the state space $\mathcal{M}$ looks like $\mathbb{R}^d$, meaning that a dynamical evolution is an initial value problem, with $d$ numbers sufficient to determine what will happen time $t$ later. The local linear vector space (tangent space) at any given state space point $x \in \mathcal{M}$ can be thought of as a 'chart' (however, we shall use this term in a more restricted sense, only after the continuous time and continuous symmetries have been 'quotiented out', see sects. 3.1 and 13.1). Globally, the state space may be a more complicated *manifold* such as a torus, a cylinder, or some other smooth geometric object. By *manifold* we mean a smooth differentiable $d$-dimensional space which looks like $\mathbb{R}^d$ only locally. For example, the state space of an autonomous Hamiltonian system the flow is confined to a curved constant energy hyper-surface. When we need to stress that the dimension $d$ of $\mathcal{M}$ is greater

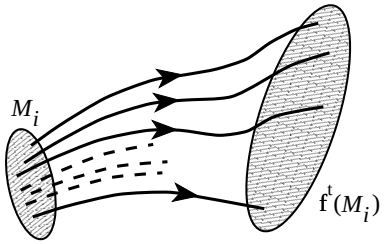


**Figure 2.2:** A flow: The evolution rule $f^t$ can be used to map a region $\mathcal{M}_i$ of the state space into the region $f^t(\mathcal{M}_i)$.

than one, we may refer to the point $x \in \mathcal{M}$ as $x_i$ where $i = 1, 2, 3, \ldots, d$. If the dynamics is described by a set of PDEs (partial differential equations), the state space is the infinite dimensional function space. The evolution rule $f^t : \mathcal{M} \to \mathcal{M}$ tells us where a point $x$ is in $\mathcal{M}$ after a time interval $t$.

The pair $(\mathcal{M}, f)$ constitute a *dynamical system*.

The dynamical systems we will be studying are smooth. This is expressed mathematically by saying that the evolution rule $f^t$ can be differentiated as many times as needed. Its action on a point $x$ is sometimes indicated by $f(x, t)$ to remind us that $f$ is really a function of two variables: the time and a point in state space. Note that time is relative rather than absolute, so only the time interval is necessary. This follows from the fact that a point in state space completely determines all future evolution, and it is not necessary to know anything besides the time interval. The time parameter can be a real variable ($t \in \mathbb{R}$), in which case the evolution is called a *flow*, or an integer ($t \in \mathbb{Z}$), in which case the evolution advances in discrete steps in time, given by *iteration* of a *map*. The evolution parameter need not be the physical time; for example, a time-stationary solution of a partial differential equation is parameterized by spatial variables. In such situations one talks of a 'spatial profile' rather than a 'flow'.

Nature provides us with innumerable dynamical systems. They manifest themselves through their orbits: given a state $x_0$ at initial time $t_0$, the *flow map*

$$f^t : \quad x_0 \to x(x_0, t)$$

yields the state $x(t)$ time $t$ later. This evolution rule traces out a sequence of points $x(t) = f^t(x_0)$, the *orbit* through the point $x_0 = x(0)$. We shall usually omit the $x_0$ label from $x(x_0, t)$. By extension, we can also talk of the evolution of a region $\mathcal{M}_i$ of the state space. The language of continuum mechanics is quite helpful in visualizing such deformations, not only in 3-dimensional space, but also in state spaces of arbitrary dimension. Consider a motion $f$ from the *undeformed* (*reference* or initial) region (a 'body') $\mathcal{M}_i$ to the *deformed* (*current* or final) region $\mathcal{M}_f = f^t(\mathcal{M}_i)$. We may write the motion as a map

$$f^t : \quad \mathcal{M}_i \to \mathcal{M}_f \,, \tag{2.1}$$

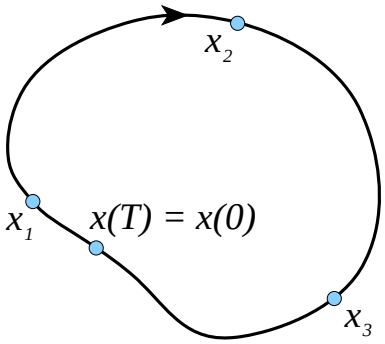


**Figure 2.3:** A periodic point returns to the initial point after a finite time, $x = f^{T_p}(x)$. Periodic orbit $p$ is the set of periodic points $p = \mathcal{M}_p = \{x_1, x_2, \cdots\}$ swept out by the trajectory of any one of them in the finite time $T_p$.

such that every $x_0$ in $\mathcal{M}_i$ is mapped to an $x = f^t(x_0)$ in $\mathcal{M}_f$, as in figure 2.2, where $x$ denotes the state in the deformed region, and $x_0$ represents the state in the initial, undeformed region.

exercise 2.1

The subset of points $\mathcal{M}_{x_0} \subset \mathcal{M}$ that belong to the infinite-time trajectory of a given point $x_0$ is called the *orbit* of $x_0$; we shall talk about forward orbits, backward orbits, periodic orbits, etc.. For a flow, an orbit is a smooth continuous curve; for a map, it is a sequence of points. In this book 'trajectory' refers to a set of points or a curve segment traced out by $x(t)$ over a finite time interval $t$. 'Orbit' refers to the totality of states that can be reached from $x_0$, with state space $\mathcal{M}$ foliated into a union of such orbits (each $\mathcal{M}_{x_0}$ labeled by a single point belonging to the set, $x_0 = x(0)$ for example). Under time evolution a trajectory segment is mapped into another trajectory segment, but points within an orbit are only shifted; the orbit considered as a set is unchanged. Hence an orbit is a *dynamically invariant* notion.

The central idea of ChaosBook is to replace the complicated, ergodic, asymptotic $t \to \infty$ dynamics by a systematic hierarchy of compact *time-invariant* sets or compact orbits (equilibria, periodic orbits, invariant tori, $\cdots$).

### 2.1.1 A classification of possible motions?

> Ah, yes, Judgie, everything will go away someday. It's the waiting that's so exquisitely wearing.
> — Duke Ellington, to Robert Traver

What kinds of orbits are there? This is a grand question, and there are many answers. The following chapters offer some. Here is a first attempt to classify all possible orbits:

| | | |
|---|---|---|
| stationary: | $f^t(x) = x$ | for all $t$ |
| periodic: | $f^t(x) = f^{t+T_p}(x)$ | for a given minimum period $T_p$ |
| aperiodic: | $f^t(x) \neq f^{t'}(x)$ | for all $t \neq t'$ . |

A *periodic orbit* (or a *cycle*) $p$ is the set of points $\mathcal{M}_p \subset \mathcal{M}$ swept out by a trajectory that returns to the initial point in a finite time. We refer to a point on a

periodic orbit as a *periodic point*, see figure 2.3. Periodic orbits form a very small subset of the state space, in the same sense that rational numbers are a set of zero measure on the unit interval.

chapter 5

Periodic orbits and equilibrium points are the simplest examples of 'non-wandering' invariant sets preserved by dynamics. Dynamics can also preserve higher-dimensional smooth compact invariant manifolds; most commonly encountered are the $M$-dimensional tori of Hamiltonian dynamics, with the notion of periodic motion generalized to quasiperiodic (the superposition of $M$ incommensurate frequencies) motion on a smooth torus, and families of solutions related by a continuous symmetry. Further examples are afforded by stable / unstable manifolds (swept by semi-infinite curves originating at an equilibrium along each stability eigenvector) and the most mysterious of all invariant orbits, the infinite time ergodic orbits.

section 15.1

The ancients tried to make sense of all dynamics in terms of periodic motions, epicycles, what we today call 'integrable systems'. The embarrassing truth is that for a generic dynamical system almost all motions are aperiodic. So we refine the classification by dividing aperiodic motions into two subtypes: those that wander off, and those that keep coming back.

A point $x \in \mathcal{M}$ is called a *wandering point*, if there exists an open neighborhood $\mathcal{M}_0$ of $x$ to which the orbit never returns

$$f^t(x) \notin \mathcal{M}_0 \qquad \text{for all} \quad t > t_{\min}\,. \tag{2.2}$$

In physics literature, the dynamics of such a state is often referred to as *transient*.

Wandering points do not take part in the long-time dynamics, so your first task is to prune them from $\mathcal{M}$ as well as you can. What remains envelops the set of the long-time orbits, or the *non-wandering set*.

For times much longer than a typical 'turnover' time, it makes sense to relax the notion of exact periodicity and replace it by the notion of *recurrence*. A point is *recurrent* or *non-wandering*, if for any open neighborhood $\mathcal{M}_0$ of $x$ and any time $t_{min}$ there exists a later time $t$, such that

$$f^t(x) \in \mathcal{M}_0\,. \tag{2.3}$$

In other words, the orbit of a non-wandering point reenters the neighborhood $\mathcal{M}_0$ infinitely often. We shall denote the *non–wandering set* of $f$ by $\Omega$, i.e., the union of all the non-wandering points of $\mathcal{M}$. This non–wandering set of $f$ is key to understanding the long-time behavior of a dynamical system; all calculations undertaken here will be carried out on non–wandering sets.

So much about individual trajectories. What about clouds of initial points? If there exists a connected state space volume that maps into itself under forward

evolution (and you can prove that by the method of Lyapunov functionals, or several other methods available in the literature), the flow is globally contracting onto a subset of $\mathcal{M}$ which we shall refer to as the *attractor*. The attractor may be unique, or there can coexist any number of distinct attracting sets, each with its own *basin of attraction*, the set of all points that fall into the attractor under forward evolution. The attractor can be a fixed point (a *sink*), a periodic orbit (a *limit cycle*), aperiodic, or any combination of the above. The most interesting case is that of an aperiodic recurrent attractor, to which we shall refer loosely as a *strange attractor*. We say 'loosely', as will soon become apparent that diagnosing and proving existence of a genuine, card-carrying strange attractor is a highly nontrivial undertaking; it requires explaining notions like 'transitive' and 'chain-recurrent' that we will be ready to discuss only in sect. 17.1.

example 2.3

Conversely, if we can enclose the non–wandering set $\Omega$ by a connected state space volume $\mathcal{M}_0$ and then show that almost all points within $\mathcal{M}_0$, but not in $\Omega$, eventually exit $\mathcal{M}_0$, we refer to the non–wandering set $\Omega$ as a *repeller*. An example of a repeller is not hard to come by–the pinball game of sect. 1.3 is a simple chaotic repeller. $\Omega$, the *non–wandering set* of $f$, is the union of all of the above, separately invariant sets: attracting/repelling fixed points, strange attractors, repellers, etc..

It would seem, having said that the periodic points are so exceptional that almost all non-wandering points are aperiodic, that we have given up the ancients' fixation on periodic motions. Nothing could be further from truth. As longer and longer cycles approximate more and more accurately finite segments of aperiodic trajectories, we shall establish control over non–wandering sets by defining them as the closure of the union of all periodic points.

Before we can work out an example of a non–wandering set and get a better grip on what chaotic motion might look like, we need to ponder flows in a little more depth.

## 2.2 Flows

> Knowing the equations and knowing the solution are two different things. Far, far away.
>
> — T.D. Lee

A *flow* is a continuous-time dynamical system. The evolution rule $f^t$ is a family of mappings of $\mathcal{M} \to \mathcal{M}$ parameterized by $t \in \mathbb{R}$. Because $t$ represents a time interval, any family of mappings that forms an evolution rule must satisfy:

exercise 2.2

(a) $f^0(x) = x$ (in 0 time there is no motion)

(b) $f^t(f^{t'}(x)) = f^{t+t'}(x)$ (the evolution law is the same at all times)

(c) the mapping $(x, t) \mapsto f^t(x)$ from $\mathcal{M} \times \mathbb{R}$ into $\mathcal{M}$ is continuous.

We shall often find it convenient to represent functional composition by '$\circ$ :' appendix D.1

$$f^{t+s} = f^t \circ f^s = f^t(f^s)\,. \tag{2.4}$$

The family of mappings $f^t(x)$ thus forms a continuous (1-parameter forward Lie semi-) group. Why 'semi-'group? It may fail to form a group if the dynamics is not reversible, and the rule $f^t(x)$ cannot be used to rerun the dynamics backwards in time, with negative $t$; with no reversibility, we cannot define the inverse $f^{-t}(f^t(x)) = f^0(x) = x$, in which case the family of mappings $f^t(x)$ does not form a group. In exceedingly many situations of interest–for times beyond the Lyapunov time, for asymptotic attractors, for dissipative partial differential equations, for systems with noise, for non-invertible maps–the dynamics cannot be run backwards in time, hence, the circumspect emphasis on *semi*groups. On the other hand, there are many settings of physical interest, where dynamics is reversible (such as finite-dimensional Hamiltonian flows), and where the family of evolution maps $f^t$ does form a group.

For infinitesimal times, flows can be defined by differential equations. We write a trajectory, a smooth curve embedded in the state space as

$$x(t+\tau) = f^{t+\tau}(x_0) = f(f(x_0,t),\tau) \tag{2.5}$$

and express the tangent to the curve at point $x(t)$ as exercise 2.3

$$\left.\frac{dx}{d\tau}\right|_{\tau=0} = \partial_\tau f(f(x_0,t),\tau)|_{\tau=0} = \dot{x}(t)\,, \tag{2.6}$$

the time derivative of the evolution rule, a vector evaluated at the point $x(t)$. By considering all possible orbits, we obtain the vector $\dot{x}(t)$ at any point $x \in \mathcal{M}$. This *vector field* is a (generalized) *velocity* field: remark 13.2

$$\dot{x}(t) = v(x)\,. \tag{2.7}$$

Newton's laws, Lagrange's method, or Hamilton's method are all familiar procedures for obtaining a set of differential equations for the vector field $v(x)$ that describes the evolution of a mechanical system. Equations of mechanics may appear different in form from (2.7), as they are often involve higher time derivatives, but an equation that is second or higher order in time can always be rewritten as a set of first order equations.

We are concerned here with a much larger world of general flows, mechanical or not, all defined by a time-independent vector field (2.7). At each point of the state space a vector indicates the local direction in which the orbit evolves. The length of the vector $|v(x)|$ is the *speed* at the point $x$, and the direction and length of $v(x)$ changes from point to point. When the state space is a complicated manifold

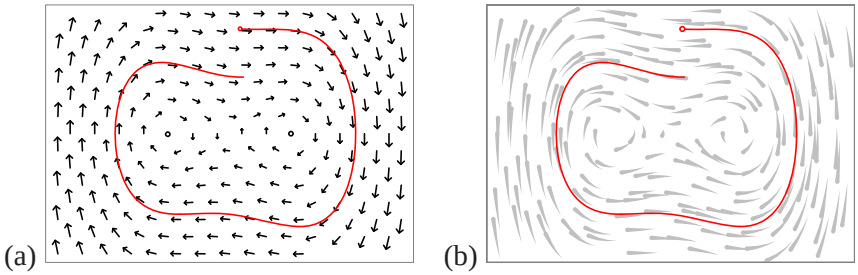

**Figure 2.4:** (a) The 2-dimensional vector field for the Duffing system (2.21), together with a short trajectory segment. (b) The flow lines. Each 'comet' represents the same time interval of a trajectory, starting at the tail and ending at the head. The longer the comet, the faster the flow in that region.

embedded in $\mathbb{R}^d$, one can no longer think of the vector field as being embedded in the state space. Instead, we have to imagine that each point $x$ of state space has a different tangent plane $T\mathcal{M}_x$ attached to it. The vector field lives in the union of all these tangent planes, a space called the *tangent bundle*

$$\mathbf{T}\mathcal{M} = \bigcup_{x\in\mathcal{M}} T\mathcal{M}_x \,.$$

$T\mathcal{M}_x$ is called a *fiber* at $x$, hence the whole thing is called the *fiber bundle*. Locally a fiber bundle looks like the product of two $\mathbb{R}^d$ spaces.

A simple example of a flow defined by a 2-dimensional vector field $v(x)$ is afforded by the unforced Duffing system, figure 2.4. Lorenz flow of figure 2.5, and Rössler flow of figure 2.6 , are representative 3-dimensional flows.

example 2.1 p. 55 example 2.2 p. 55 example 2.3 p. 55

The instantaneous velocity vector $v$ is tangent to the orbit, except at the *equilibrium points*, where it vanishes.

$$\text{If} \qquad v(x_q) = 0 \,, \tag{2.8}$$

$x_q$ is also referred to as a *stationary*, *fixed*, *critical*, *invariant*, *rest*, *stagnation* point, *zero* of the vector field $v$, *standing wave*, *stationary solution*, or *steady state*. Our usage will be 'equilibrium' for a flow, 'fixed point' for a map. The orbit remains forever stuck at $x_q$. Otherwise the orbit passing through $x_0$ at time $t = 0$ can be obtained by integrating the equations (2.7):

$$x(t) = f^t(x_0) = x_0 + \int_0^t d\tau\, v(x(\tau)) \,, \qquad x(0) = x_0 \,. \tag{2.9}$$

We shall consider here only *autonomous* flows, i.e., flows for which the vector field $v_i$ is *stationary*, not explicitly dependent on time. A non-autonomous system

$$\frac{dy}{d\tau} = w(y,\tau) \,, \tag{2.10}$$

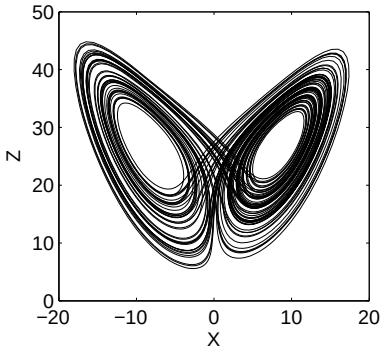


**Figure 2.5:** Lorenz "butterfly" strange attractor. (J. Halcrow)

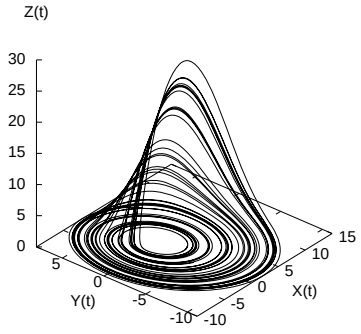


**Figure 2.6:** A trajectory of the Rössler flow at time $t = 250$. (G. Simon)

can always be converted into a system where time does not appear explicitly. To do so, extend ('suspend') state space to be $(d+1)$-dimensional by defining $x = \{y, \tau\}$, with a stationary vector field

exercise 2.4
exercise 2.5

$$v(x) = \begin{bmatrix} w(y,\tau) \\ 1 \end{bmatrix} . \tag{2.11}$$

The new flow $\dot{x} = v(x)$ is autonomous, and the orbit $y(\tau)$ can be read off $x(t)$ by ignoring the last component of $x$.

exercise 6.3

### 2.2.1 Lagrangian and Eulerian viewpoints

Continuum mechanics offers two profoundly different but mathematically equivalent ways to represent a given state space flow, the 'Lagrangian' and the 'Eulerian' viewpoints. From the Eulerian perspective one only cares about what is the state of system here and now; think of a field of grass, each grass blade the local velocity vector. From the Lagrangian viewpoint one cares about where a state space point come fromand where is it going to; think of the state space foliated into a bowl of linguini, each noodle an orbit, marked with a label $x_0$ somewhere along it. In the Eulerian formulation the flow is defined by specifying (2.7), the velocity field $v(x)$. In the Lagrangian formulation it is given by the finite time flow (2.9), i.e., the totality of the trajectories $x(t)$ comprising the deformed region, labeled by their origin $x_0$ in the initial undeformed region. If we mark the orbit

$x(t)$ by its initial point $x_0$, we are describing the flow in the *Lagrangian coordinates*. The *Eulerian* velocity $v(x)$ at a fixed state space position $x$ is equal to the *Lagrangian* velocity $v(x(t))$ at the orbit passing through $x$ at the instant $t$. Because $f^t$ is a single-valued function, any point on the orbit can be used to label the orbit. The transport of the 'material point' $x_0$ at $t = 0$ to its value at the current point $x(t) = f^t(x_0)$ is a coordinate transformation from the Lagrangian coordinates to the Eulerian coordinates.

In numerical work we are given the equations of motion (the *local* Eulerian velocity field $v(x)$), but we care about the solutions of these equations (the *global* Lagrangian flow). Conversely, in experimental work we observe ensembles of Lagrangian trajectories from which we then extract the velocity field (in fluid dynamics this is achieved by particle image velocimetry (PIV)). Once an Eulerian velocity field has been specified or extracted from the observational data, it is straightforward to compute the Lagrangian trajectories, objects of great practical interest in studies of long time dynamics, mixing, and transport.

fast track:
chapter 3, p. 64

## 2.3 Changing coordinates

Problems are handed down to us in many shapes and forms, and they are not always expressed in the most convenient way. In order to simplify a given problem, one may stretch, rotate, bend and mix the coordinates, but in doing so, the vector field will also change. The vector field lives in a (hyper)plane tangent to state space and changing the coordinates of state space affects the coordinates of the tangent space as well, in a way that we will now describe.

Denote by $h$ the *conjugation function* which maps the coordinates of the initial state space $\mathcal{M}$ into the reparameterized state space $\mathcal{M}' = h(\mathcal{M})$, with a point $x \in \mathcal{M}$ related to a point $y \in \mathcal{M}'$ by

$$y = h(x) = (y_1(x), y_2(x), \ldots, y_d(x)) .$$

The change of coordinates must be one-to-one and span both $\mathcal{M}$ and $\mathcal{M}'$, so given any point $y$ we can go back to $x = h^{-1}(y)$. For smooth flows the reparameterized dynamics should support the same number of derivatives as the initial one. If $h$ is a (piecewise) analytic function, we refer to $h$ as a *smooth conjugacy*.

The evolution rule $g^t(y_0)$ on $\mathcal{M}'$ can be computed from the evolution rule $f^t(x_0)$ on $\mathcal{M}$ by taking the initial point $y_0 \in \mathcal{M}'$, going back to $\mathcal{M}$, evolving, and then mapping the final point $x(t)$ back to $\mathcal{M}'$:

$$y(t) = g^t(y_0) = h \circ f^t \circ h^{-1}(y_0) . \tag{2.12}$$

Here '$\circ$' stands for functional composition $h \circ f(x) = h(f(x))$, so (2.12) is a shorthand for $y(t) = h(f^t(h^{-1}(y_0)))$.

The vector field $\dot{x} = v(x)$ is locally tangent to the flow $f^t$; it is related to the flow by differentiation (2.6) along the orbit. The vector field $\dot{y} = w(y)$, $y \in \mathcal{M}'$ locally tangent to $g^t$, follows by the chain rule:

exercise B.2

$$\begin{aligned} w(y) &= \frac{dg^t}{dt}(y)\Big|_{t=0} = \frac{d}{dt}\left(h \circ f^t \circ h^{-1}(y)\right)\Big|_{t=0} \\ &= h'(h^{-1}(y))\, v(h^{-1}(y)) = h'(x)\, v(x)\,. \end{aligned} \tag{2.13}$$

In order to rewrite the right-hand side as a function of $y$, note that the $\partial_y$ differentiation of $h(h^{-1}(y)) = y$ implies

$$\frac{\partial h}{\partial x}\Big|_x \cdot \frac{\partial h^{-1}}{\partial y}\Big|_y = 1 \quad \to \quad \frac{\partial h}{\partial x}(x) = \left[\frac{\partial h^{-1}}{\partial y}(y)\right]^{-1}\,, \tag{2.14}$$

so the equations of motion in the transformed coordinates, with the indices reinstated, are

$$\dot{y}_i = w_i(y) = \left[\frac{\partial h^{-1}}{\partial y}(y)\right]^{-1}_{ij} v_j(h^{-1}(y))\,. \tag{2.15}$$

Imagine the state space as a rubber sheet with the flow lines drawn on it. A coordinate change $h$ corresponds to pulling and tugging on the rubber sheet smoothly, without cutting, gluing, or self-intersections of the distorted rubber sheet. Trajectories that are closed loops in $\mathcal{M}$ will remain closed loops in the new manifold $\mathcal{M}'$, but their shapes will change. Globally, $h$ deforms the rubber sheet in a highly nonlinear manner, but locally it simply rescales and shears the tangent field by the coordinate transformation Jacobian matrix $\partial_j h_i$, yielding the simple transformation law (2.13) for the velocity fields.

Time itself is a parametrization of points along flow lines, and it can also be reparameterized, $s = s(t)$, with the concomitant modification of (2.15). An example is the 2-body collision regularization of the helium Hamiltonian (8.8), to be undertaken in appendix B.2.

## 2.4 Life in extreme dimensions

> Sometimes I've believed as many as six impossible things before breakfast.
>
> — Lewis Carroll

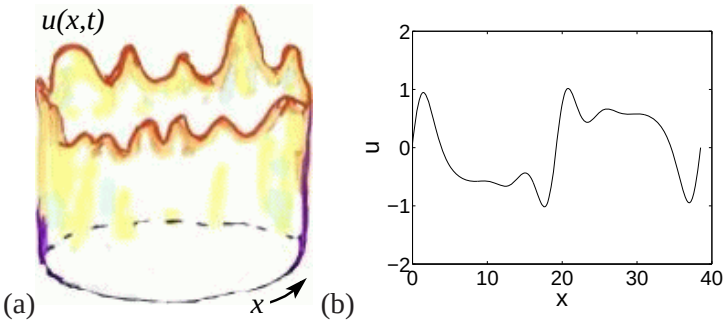


**Figure 2.7:** (a) The Ring of Fire, visualized as a Bunsen burner flame flutter, with $u = u(x,t)$ the velocity of the flame front at position $x$ and time $t$. (b) A profile of the velocity $u$ of the flame front at fixed time instant $t$ folded out on a plane, with spatial periodicity $u(x,t) = u(x+40,t)$ (from ref. [1]).

Systems described by partial differential equations [PDEs] are said to be 'infinite dimensional' dynamical systems, because in order to uniquely specify the state of a spatially extended 'field', one needs infinitely many numbers, one for the value of the field at each configuration space point. Even though the state space is infinite-dimensional, the long-time dynamics of many such systems of physical interest is finite-dimensional, contained within a 'strange attractor' or an 'inertial manifold'. Most of us find it hard to peer into four dimensions. How are we to visualize -and why we would have any hope of visualizing- dynamics in such extreme dimensions? A representative point is a point, and its trajectory is a curve in any 2- or 3-dimensional projection, so that is not so hard. What is hard is to get an understanding of relative disposition of different states. The coordinates have to be chosen thoughtfully, as in a randomly picked coordinate frame most orbits of interest will appear minuscule.

A dynamical system is specified by the pair $(\mathcal{M}, f)$, where $d$ numbers uniquely determine a state of the system, or the representative point $x$ in the state space manifold $\mathcal{M}$. Here we focus on how one constructs such state space, and how one visualizes a representative point $x$ and its trajectory $f^t(x)$ time $t$ later. We shall return to dynamics, i.e., the evolution rule $f^t$ that maps a state space region $\mathcal{M}_i$ of the state space into the region $f^t(\mathcal{M}_i)$ (see figure 2.2) for such systems in chapter 29, where we describe in some detail time-evolution equations for spatially-extended systems, and discuss 'turbulence' that such systems may exhibit.

### 2.4.1 Fluttering flame front

Consider the flame front flutter of gas burning on your kitchen stove. Such 'Bunsen burner', invented by Göttingen chemistry prodigy Robert Bunsen in 1855, entered popular culture in 1963 as Johnny Cash *et al.* [2] "Ring of Fire". Its flame front instabilities are perhaps the most familiar example of a nonlinear system that exhibits 'turbulence' (or, more modestly, 'spatiotemporally chaotic behavior'): a typical 'configuration' or 'physical' space visualization is sketched in figure 2.7. Its state can be described by the 'flame front velocity' $u = u(x,t)$ on a periodic domain $u(x,t) = u(x+L,t)$.

Spatial, 'configuration' or 'physical' space visualization of a state of such system, figure 2.7, or a fixed time snapshot of velocity and vorticity fields in $3D$ Navier-Stokes, or a visualization of the flame front flutter in time, figure 2.8, or

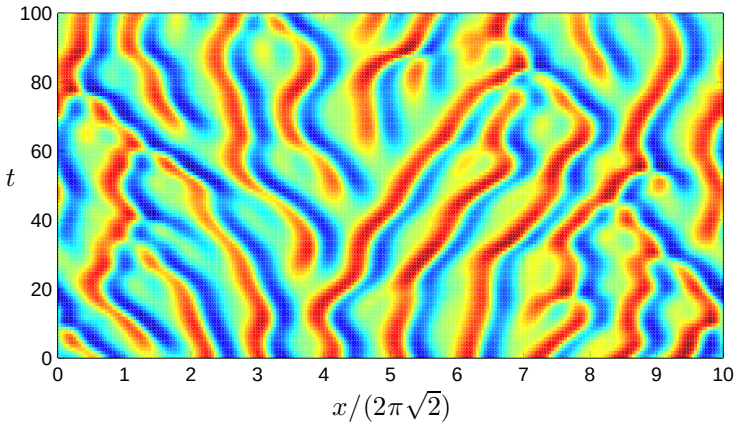


**Figure 2.8:** A spatiotemporal plot of a typical Ring of Fire "turbulent" solution, periodic domain $u(x,t) = u(x + 20\pi\sqrt{2}, t)$ is obtained by plotting the profile of figure 2.7 (b) for successive time instants (vertical axes). The color indicates the value of $u$ at a given position and instant in time (from ref. [3]).

a time-evolving video of a fluid, offer little insight into detailed *dynamics* of such systems. To understand the dynamics, one must turn to the complementary, and often much more illuminating state space representations. In this context 'flow' refers to a $d$-dimensional flow in the dynamical state space, not the flow of a fluid, and 'velocity' to the state space tangent field $\dot{x} = v(x)$, not to the $3D$ configuration space fluid velocity field $u(x,t) \in \mathbb{R}^3$. A 'representative point' is a full specification of the state $x \in \mathcal{M}$ of the system, In today's experiments or numerical simulations, this is a set of anything from 16 to $10^6$ numbers, a complete snapshot of the flame front figure 2.7 or the state of volume of turbulent fluid in a pipe at an instant in time.

### 2.4.2 Constructing a state space

Think globally, act locally.
— Patrick Geddes

At this juncture, our everyday, plumber's visual intuition actually interferes with dynamical visualization of state space of a spatially-extended systems: while the spatial dimension of the Ring of Fire is 1, its dimension as a dynamical system is $\infty$. Absorbing this simple fact of life is the same rite of passage as going from the 1 degree of freedom quantum mechanical oscillator to the 'second quantization' of quantum field theory, with its infinitely many quantum oscillator degrees of freedom.

To develop some intuition about such dynamics we turn to experiments, or numerical simulations, such as the Ring of Fire time evolution, figure 2.8. The first thing we note is that while the dynamics might be 'turbulent', for many such systems the long-time solutions tend to be smooth. That suggests that a discretization, perhaps aided by interpolations such as $n$-point spatial derivatives might give us a representation of the dynamics of reasonable accuracy.

**Discrete mesh:** You can subdivide the configuration domain into a sufficiently fine discrete grid of $N$ boxes, replace space derivatives in the governing equations by approximate discrete derivatives, and integrate a finite set of first order differ-

ential equations for the discretized spatial components $u_j(t) = u(jL/N, t)$, by any integration routine you trust. Most often that's the best you can do.

The next thing we note is that the solutions for many physical systems of physical interest tend to be not only smooth, but also that the laws that govern them are invariant in form under operations such as translations. For example, the fluttering flame front governing equations should be configuration-space rotationally equivariant, time-translationally invariant, and invariant under reflection $x \to -x$, $u \to -u$.

**Spectral methods:** The spatial periodicity $u(x,t) = u(x+L,t)$ then suggests that it might be convenient to work in the Fourier space,

$$u(x,t) = \sum_{k=-\infty}^{+\infty} \tilde{u}_k(t)\, e^{iq_k x}\,, \tag{2.16}$$

where $\tilde{u}_k = x_k + i\,y_k = |\tilde{u}_k| e^{i\phi_k}$, $q_k = 2\pi k/L$, $L$ is the domain size, $x$ is the spatial coordinate and $\tau$ is time. The velocity field $u(x,t)$ is real, so $\tilde{u}_k = \tilde{u}^*_{-k}$, and we can replace the sum by an $k \geq 0$ sum. Thus a state of a spatially 1-dimensional extended system can described by an infinite set of complex Fourier coefficients $\tilde{u}_k(t)$. This is an example of an infinite-dimensional state space alluded to on page 45, in this section's introduction.

Intuitively the flame front is smooth, so Fourier coefficients $\tilde{u}_k$ drop off fast with $k$, and truncations of (29.7) to finite numbers of terms can yield highly accurate states. In numerical computations this state space is truncated to a finite number of real dimensions. For example, a state might be specified by $2N$ real Fourier coefficients, or 'computational degrees of freedom'

$$x = (x_1, y_1, x_2, y_2, \ldots, x_N, y_N)^{\mathsf{T}}\,. \tag{2.17}$$

More sophisticated variants of such truncations are called in the literature *Gälerkin truncations*, or *Gälerkin projections*.

Once a trajectory is computed in Fourier space, we can recover and plot the corresponding spatiotemporal pattern $u(x,t)$ over the configuration space, as in figure 2.7 and figure 2.8, by inverting (29.7). Spatiotemporal patterns give us a qualitative picture of the flow and a physical intuition about the energetics of the flow, but no detailed dynamical information; for that, tracking the evolution in a high-dimensional state space is much more informative.

### 2.4.3 State space, as visualized by dummies

This is dedicated to Student X
— Professore Dottore Gatto Nero

The question is: how is one to look at such state space flow? The laziest thing to do is to examine the trajectory's projections onto any three computational degrees of freedom, let's say the first three Fourier modes $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$. Why would you do that? Well, that's what computer spews out. This won't do. Let's accept that you do not know much about high dimensions, but you have been born someplace where they force you to watch grown men kick a ball, for hours on end. Your choice of $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ coordinates means that you (or the TV camera) are standing at a corner of the field. Far, far away, at the opposite end of the field, there is action - but you only see a few little moving silhouettes, and can hardly see the ball.

Or, if you scholarly kind, and would rather while hours away evaluating $\Gamma$-functions, here is a precise way of saying the same: chose a direction in a high-dimensional state space, call it your basis vector $\mathbf{e}^{(1)}$. Now pick a state $\mathbf{u}$ in state space at random. That gives you a second vector. What is the angle between these two vectors? The cosine of that angle you compute by evaluating the 'dot' product (or L2 norm)

$$\langle \mathbf{u}|\mathbf{e}^{(1)}\rangle = \frac{1}{V}\int_\Omega dx\, \mathbf{u}\cdot\mathbf{e}^{(1)}\,, \quad \|\mathbf{u}\|^2 = \langle \mathbf{u}|\mathbf{u}\rangle\,. \tag{2.18}$$

exercise 2.11

Once you finish the exercise 2.11 you will know what every computer scientist knows: the expectation value of the angle between any two high-dimensional vectors picked at random is $90^o$, with a very small variance. In other words, in high dimension and with a random coordinate system, every distant silhouette is vanishingly small. And your lazy $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ coordinates are a random choice, the turbulent state might require $10^5$ such coordinates to be accurately resolved.

So, if you were a referee, or a cameraman, would your really just stand there, in the far corner of the field?

### 2.4.4 Exact state-space portraiture: go where the action is

(J.F. Gibson and P. Cvitanović)

You are interested into dynamics and especially the recurrent dynamics, so cross the field, and identify, by long-time numerical simulations or other means, prominent states that characterize the observed recurrent coherent structures of interest to you. If you form a basis set from them, and project the evolving state $x(t)$ onto this basis, coordinates so formed will capture close recurrences to these states. That is, form orthonormal basis functions $\{\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(n)}\}$ from a set of linearly independent fluid states and produce a state-space trajectory

$$x(t) = (x_1(t), x_2(t), \cdots, x_n(t), \cdots)\,, \qquad x_n(t) = \langle \mathbf{u}(t)|\mathbf{e}^{(n)}\rangle \tag{2.19}$$

in the $\{\mathbf{e}^{(n)}\}$ coordinate frame. The projection of the trajectory can be viewed in any of the $2d$ planes $\{\mathbf{e}^{(m)}, \mathbf{e}^{(n)}\}$ or in $3d$ perspective views $\{\mathbf{e}^{(\ell)}, \mathbf{e}^{(m)}, \mathbf{e}^{(n)}\}$. The

dimensionality is lower than the full state space, so in such projections trajectories can appear to cross. It is important to understand that this is low-dimensional *visualization*, not low-dimensional *modeling*, a truncation to fewer computational degrees of freedom. The dynamics are computed with fully-resolved direct numerical simulations and projected onto basis sets to produce low-dimensional state-space portraits, tailored to specific purposes and specific regions of state space. The resulting portraits depend on the *physical states* involved and not on the choice of a numerical representation. The portraits reveal dynamical information visually, providing insight into dynamics that can guide further analysis.

There is an infinity of possible basis sets, but two types of bases appear particularly natural: (a) a global basis, determined by a set of dynamically important states, or (b) a local basis, defined, for example, in terms of a given equilibrium and its linear stability eigenvectors.

With this road map in hand, we can take a stroll through the state space of a spatiotemporally turbulent flow. Like many dynamical narratives, this might turn into a long trek through unfamiliar landscape with many landmarks of local interest. It is amazing that such a promenade is possible even in $10^5$ dimensions. But a detailed road map is a necessary prerequisite for solving at least three of your outstanding problems: (a) uncovering the interrelations between (in principle infinite number of) unstable invariant solutions of a turbulent flow, (b) a partition of state space is a needed for a systematic exploration of turbulent dynamics, and (c) linear stability eigenvectors and their unstable-manifold continuations will be needed to control and chaperon a given spatiotemporal state to a desired target state.

In summary, when dealing with spatiotemporally extended systems, you'll need dual vision - you will have to think both in the configuration space, and in the state space ( click on the video link here).

## 2.5 Computing trajectories

> On two occasions I have been asked [by members of Parliament], 'Pray, Mr. Babbage, if you put into the machine wrong figures, will the right answers come out?' I am not able rightly to apprehend the kind of confusion of ideas that could provoke such a question.
>
> — Charles Babbage

You have not learned dynamics unless you know how to integrate numerically whatever dynamical equations you face. Sooner or later, you need to implement some finite time-step prescription for integration of the equations of motion (2.7). The simplest is the Euler integrator which advances the trajectory by $\delta\tau \times$ velocity at each time step:

$$x_i \to x_i + v_i(x)\,\delta\tau\,. \tag{2.20}$$

This might suffice to get you started, but as soon as you need higher numerical accuracy, you will need something better. There are many excellent reference texts and computer programs that can help you learn how to solve differential equations numerically using sophisticated numerical tools, such as pseudo-spectral methods or implicit methods. If a 'sophisticated' integration routine takes days and gobbles up terabits of memory, you are using brain-damaged high level software. Try writing a few lines of your own Runge-Kutta code in some mundane everyday language. While you absolutely need to master the requisite numerical methods, this is neither the time nor the place to expound upon them; how you learn them is your business. And if you have developed some nice routines for solving problems in this text or can point another student to some, let us know.

exercise 2.6

exercise 2.7

exercise 2.9

exercise 2.10

## Résumé

Start from a state space point and evolve it for a finite time, you trace out its *trajectory*. Evolve it forward and backward for infinite time, you get the *orbit*, the set of all states reachable by evolution from a given state space point. An orbit is a time-invariant notion: time evolution marches points along it, but the set itself does not change. The *flow* describes the time evolution of all state space points, i.e., the totality of all orbits: the evolution law $f$ turns the state space into a bowl of spaghetti, with each individual spaghetto an orbit.

Chaotic dynamics with a low-dimensional attractor can be visualized as a succession of nearly periodic but unstable motions. In the same spirit, turbulence in spatially extended systems can be described in terms of recurrent spatiotemporal patterns. Pictorially, dynamics drives a given spatially extended system through a repertoire of unstable patterns; as we watch a turbulent system evolve, every so often we catch a glimpse of a familiar pattern. For any finite spatial resolution and finite time, the system follows approximately a pattern belonging to a finite repertoire of possible patterns. The long-term dynamics can be thought of as a walk through the space of such patterns. Recasting this image into mathematics is the subject of this book.

The state-space portraits are *dynamically intrinsic*, since the projections are defined in terms of solutions of the equations of motion, and *representation independent*, since the L2 product (2.18) is independent of the numerical representation. The method can be applied to any high-dimensional dissipative flow. Production of state-space portraits requires numerical data of configuration space fields evolving in time (obtained obtained from simulation or experiment), estimates of important physical states (such as equilibria and their linear stability eigenfunctions), and a method of computing the inner product between velocity fields over the physical domain.

## Commentary

**Remark 2.1** 'State space' or 'phase space?' In ChaosBook, *state space* is the set of admissible states in a general $d$- or $\infty$-dimensional dynamical system. The term *phase space* is reserved for Hamiltonian state spaces of $2D$-dimensions, where $D$ is the number of Hamiltonian degrees of freedom. If the state space is a continuous smooth manifold much of the literature [4, 5] refers to it as 'phase space,' but we find the control engineering usage sharper: in the state space (or 'time-domain') description of an autonomous physical system, the state of the system is represented as a vector within the 'state space,' space whose axes are the state variables, and the evolution of a state is given by differential equations which are first-order in time. Hopf [6] would refer to such a state as an 'instantaneous phase' of the system obeying a 'differential law of the phase motion'. The distinction made here is needed in a text where one treats deterministic dynamical systems, stochastic systems and quantum-mechanical systems. The term 'phase' has a precise meaning in wave mechanics, quantum mechanics and dynamics of integrable systems at the heart of Hamilton's formulation of Newtonian mechanics, while 'state space' is more descriptive of the way the notion is used in the general theory of dynamical systems. Further confusion arises when prefix spatio- as in 'spatiotemporal' is used in reference to states extended in the (1, 2, or 3-dimensional) physical configuration space. They may exhibit spatial wave-like behaviors, but their *state space* is $\infty$-dimensional.

Much of the literature denotes the vector field in a first order differential equation (2.7) by $f(x)$ or $F(x)$ or even $X(x)$, and its integral for time $t$ by the 'time-$t$ forward map' or 'flow map' $x(x_0, t) = \Phi(x_0, t)$, or $\phi_t(x_0)$, or something else. Here we treat maps and flows on an equal footing, and we save Greek letters for matters quantum-mechanical. We reserve the notation $f^t(x)$ for *maps* such as (2.9) and refer to a state space velocity *vector field* as $v(x)$. We come to regret this choice very far into the text, only by the time we delve into Navier-Stokes equations.

**Remark 2.2** Rössler and Duffing flows. The Duffing system (2.21) arises in the study of electronic circuits [7]. The Rössler flow (2.27) is the simplest flow which exhibits many of the key aspects of chaotic dynamics. It was introduced in ref. [8] as a set of equations describing no particular physical system, but capturing the essence of Lorenz chaos in the most simple of smooth flows. Otto Rössler, a man of classical education, was inspired in this quest by that rarely cited grandfather of chaos, Anaxagoras (456 B.C.). This and references to earlier work can be found in refs. [9, 10, 11]. We recommend in particular the inimitable Abraham and Shaw illustrated classic [12] for its beautiful sketches of many flows, including the Rössler flow. Timothy Jones [13] has a number of interesting simulations on a Drexel website.

**Remark 2.3** Lorenz equation. The Lorenz equation (2.22) is the most celebrated early illustration of "deterministic chaos" [5] (but not the first - that honor goes to Dame Cartwright [15]). Lorenz's paper, which can be found in reprint collections refs. [16, 17], is a pleasure to read, and it is still one of the best introductions to the physics motivating such models (read more about Lorenz here). The equations, a set of ODEs in $\mathbb{R}^3$, exhibit strange attractors. W. Tucker [18, 19, 20] has proven rigorously via interval arithmetic that the Lorenz attractor is strange for the original parameters (no stable orbits) and that it has a long stable periodic orbit for slightly different parameters. In contrast to the hyperbolic strange attractors such as the weakly perturbed cat map, the Lorenz attractor is

structurally unstable. Frøyland [21] has a nice brief discussion of Lorenz flow. Frøyland and Alfsen [22] plot many periodic and heteroclinic orbits of the Lorenz flow; some of the symmetric ones are included in ref. [21]. Guckenheimer-Williams [23] and Afraimovich-Bykov-Shilnikov [24] offer an in-depth discussion of the Lorenz equation. The most detailed study of the Lorenz equation was undertaken by Sparrow [25]. For a geophysics derivation, see Rothman course notes [26]. For a physical interpretation of $\rho$ as "Rayleigh number," see Jackson [27] and Seydel [28]. The Lorenz truncation to 3 modes is so drastic that the model bears no relation to the geophysical hydrodynamics problem that motivated it. For a detailed pictures of Lorenz invariant manifolds consult Vol II of Jackson [27]. The Lorenz attractor is a very thin fractal – as we saw, stable manifold thickness is of the order $10^{-4}$ – whose fractal structure has been accurately resolved by D. Viswanath [29, 30]. If you wonder what analytic function theory has to say about Lorenz, check ref. [31]. Refs. [32, 33] might also be of interest. (continued in remark 11.1)

**Remark 2.4** Diagnosing chaos. In sect. 1.3.1 we have stated that a deterministic system exhibits 'chaos' if its orbits are locally unstable (positive Lyapunov exponent) and globally mixing (positive entropy). In sect. 6.2 we shall define Lyapunov exponents and discuss their evaluation, but already at this point it would be handy to have a few quick numerical methods to diagnose chaotic dynamics. Laskar's *frequency analysis* method [34] is useful for extracting quasi-periodic and weakly chaotic regions of state space in Hamiltonian dynamics with many degrees of freedom. For pointers to other numerical methods, see ref. [35].

**Remark 2.5** High-dimensional flows and their visualizations. Dynamicist's vision of turbulence was formulated by Eberhard Hopf in his seminal 1948 paper [4], see appendix A.5 Much about high-dimensional state spaces is counterintuitive. The literature on why the expectation value of the angle between any two high-dimensional vectors picked at random is $90^o$ is mostly about spikey spheres: see the draft of the Hopcroft and Kannan [36] book and Ravi Kannan's course; lecture notes by Hermann Flaschka on *Some geometry in high-dimensional spaces*; Wegman and Solka [37] visualizations of high-dimensional data; Spruill paper [38]; a lively mathoverflow.org thread on "Intuitive crutches for higher dimensional thinking."

appendix A.5

The 'good' coordinates, introduced in ref. [39] and described here are akin in spirit to the low-dimensional projections of the POD modeling [40], in that both methods aim to capture key features and dynamics of the system in just a few dimensions. But the method described here is very different from POD in a key way: we construct basis sets from *exact solutions of the fully-resolved dynamics* rather than from the empirical eigenfunctions of the POD. Exact solutions and their linear stability modes (a) characterize the spatially-extended states precisely, as opposed to the truncated expansions of the POD, (b) allow for different basis sets and projections for different purposes and different regions of state space, (c) our low-dimensional projections are not meant to suggest low-dimensional ODE models; they are only visualizations, every point in these projections is still a point the full state space, and (d) the method is not limited to Fourier mode bases.

(J.F. Gibson and P. Cvitanović)

**Remark 2.6** Dynamical systems software: J.D. Meiss [41] has maintained for many years *Sci.nonlinear FAQ* which is now in part superseded by the SIAM Dynamical Systems website `www.dynamicalsystems.org`. The website glossary contains most

of Meiss's FAQ plus new ones, as well as an up-to-date software list [42] with links to DSTool, xpp, AUTO, etc.. Springer on-line *Encyclopaedia of Mathematics* maintains links to dynamical systems software packages on eom.springer.de/D/d130210.htm. Kuznetsov [43] Appendix D.9 gives an exhaustive overview of software available in 2004. (see also remark 15.1)

## 2.6 Examples

10. Try to leave out the part that readers tend to skip.
— Elmore Leonard's Ten Rules of Writing.

The reader is urged to study the examples collected at the ends of chapters. If you want to return back to the main text, click on [click to return] pointer on the margin.

**Example 2.1** ***A 2-dimensional vector field*** $v(x)$***:*** *A simple example of a flow is afforded by the unforced Duffing system*

$$\begin{aligned} \dot{x}(t) &= y(t) \\ \dot{y}(t) &= -0.15\,y(t) + x(t) - x(t)^3 \end{aligned} \tag{2.21}$$

*plotted in figure 2.4. The velocity vectors are drawn superimposed over the configuration coordinates* $(x(t), y(t))$ *of state space* $\mathcal{M}$*, but they belong to a different space, the* tangent bundle $\mathbf{T}\mathcal{M}$*.*

**Example 2.2** ***Lorenz strange attractor:*** *Edward Lorenz arrived at the equation*

$$\dot{x} = v(x) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma(y-x) \\ \rho x - y - xz \\ xy - bz \end{bmatrix} \tag{2.22}$$

*by a drastic simplification of the Rayleigh-Benard flow. Lorenz fixed* $\sigma = 10$, $b = 8/3$, *and varied the "Rayleigh number"* $\rho$*. For* $0 < \rho < 1$ *the equilibrium* $EQ_0 = (0,0,0)$ *at the origin is attractive. At* $\rho = 1$ *it undergoes a pitchfork bifurcation into a pair of equilibria at*

remark 2.3

$$x_{EQ_{1,2}} = (\pm\sqrt{b(\rho-1)}, \pm\sqrt{b(\rho-1)}, \rho-1)\,, \tag{2.23}$$

*We shall not explore the Lorenz flow dependence on the* $\rho$ *parameter in what follows, but here is a brief synopsis: the* $EQ_0$ 1*-dimensional unstable manifold closes into a homoclinic orbit at* $\rho = 13.56\ldots$*. Beyond that, an infinity of associated periodic orbits are generated, until* $\rho = 24.74\ldots$*, where* $EQ_{1,2}$ *undergo a Hopf bifurcation.*

*All computations that follow will be performed for the Lorenz parameter choice* $\sigma = 10, b = 8/3, \rho = 28$*. For these parameter values the long-time dynamics is confined to the strange attractor depicted in figure 2.5, and the positions of its equilibria are marked in figure 11.2. (continued in example 3.4)*

click to return: p. 42

**Example 2.3** ***Rössler strange attractor:*** *The Duffing flow of figure 2.4 is bit of a bore–every orbit ends up in one of the two attractive equilibrium points. Let's construct a flow that does not die out, but exhibits a recurrent dynamics. Start with a harmonic oscillator*

$$\dot{x} = -y\,, \qquad \dot{y} = x\,. \tag{2.24}$$

*The solutions are $re^{it}$, $re^{-it}$, and the whole $x$-$y$ plane rotates with constant angular velocity $\dot\theta = 1$, period $T = 2\pi$. Now make the system unstable by adding*

$$\dot x = -y\,, \qquad \dot y = x + ay\,, \qquad a > 0\,, \tag{2.25}$$

*or, in radial coordinates, $\dot r = ar\sin^2\theta$, $\dot\theta = 1 + (a/2)\sin 2\theta$. The plane is still rotating with the same average angular velocity, but trajectories are now spiraling out. Any flow in the plane either escapes, falls into an attracting equilibrium point, or converges to a limit cycle. Richer dynamics requires at least one more dimension. In order to prevent the trajectory from escaping to $\infty$, kick it into 3rd dimension when $x$ reaches some value $c$ by adding*

$$\dot z = b + z(x - c)\,, \qquad c > 0\,. \tag{2.26}$$

*As $x$ crosses $c$, $z$ shoots upwards exponentially, $z \simeq e^{(x-c)t}$. In order to bring it back, start decreasing $x$ by modifying its equation to*

$$\dot x = -y - z\,.$$

*Large $z$ drives the trajectory toward $x = 0$; there the exponential contraction by $e^{-ct}$ kicks in, and the trajectory drops back toward the $x$-$y$ plane. This frequently studied example of an autonomous flow is called the* Rössler flow

$$\begin{aligned} \dot x &= -y - z \\ \dot y &= x + ay \\ \dot z &= b + z(x - c)\,, \qquad a = b = 0.2\,, \quad c = 5.7 \end{aligned} \tag{2.27}$$

*(for definitiveness, we fix the parameters $a$, $b$, $c$ in what follows). The system is as simple as they get–it would be linear, were it not for the sole bilinear term $zx$. Even for so 'simple' a system the nature of long-time solutions is far from obvious.* exercise 2.8

*There are two repelling equilibrium points (2.8):*

$$\begin{aligned} x_\pm &= (\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4ab/c^2})(c, -c/a, c/a) \\ x_- &\approx (ab/c, -b/c, b/c)\,, \qquad x_+ \approx (c, -c/a, c/a) \\ (x_-, y_-, z_-) &= (\,0.0070,\ -0.0351,\ 0.0351\,) \\ (x_+, y_+, z_+) &= (\,5.6929,\ -28.464,\ 28.464\,) \end{aligned} \tag{2.28}$$

*One is close to the origin by construction. The other, some distance away, exists because the equilibrium condition has a 2nd-order nonlinearity.*

*To see what solutions look like in general, we need to resort to numerical integration. A typical numerically integrated long-time trajectory is sketched in figure 2.6 (see also figure 14.10). Trajectories that start out sufficiently close to the origin seem to converge to a* strange attractor*. We say 'seem' as there exists no proof that such an attractor is asymptotically aperiodic–it might well be that what we see is but a long transient on a way to an attractive periodic orbit. For now, accept that figure 2.6 and similar figures in what follows are examples of 'strange attractors.'*

*The Rössler flow is the simplest flow which exhibits many of the key aspects of chaotic dynamics; we shall use it and the 3-pinball (see chapter 9) systems throughout ChaosBook to motivate introduction of Poincaré sections, return maps, symbolic dynamics, cycle expansions, and much else. Rössler flow is integrated in exercise 2.7,*

*its equilibria are determined in exercise 2.8, its Poincaré sections constructed in exercise 3.1, and the corresponding return Poincaré map computed in exercise 3.2. Its volume contraction rate is computed in exercise 4.3, its topology investigated in exercise 4.4, the shortest Rössler flow cycles are computed and tabulated in exercise 7.1, and its Lyapunov exponents evaluated in exercise 6.4. (continued in exercise 2.8 and example 3.3)* *(R. Paškauskas)*

click to return: p. 42

The exercises that you should do have **underlined titles**. The rest (**smaller type**) are optional. Difficult problems are marked by any number of *** stars.

## Exercises

2.1. **Orbits do not intersect.** An orbit in the state space $\mathcal{M}$ is the set of points one gets by evolving $x \in \mathcal{M}$ forwards and backwards in time:

$$\mathcal{M}_x = \{y \in \mathcal{M} : \quad f^t(x) = y \quad \text{for } t \in \mathbb{R}\}\,.$$

Show that if two trajectories intersect, then they are the same curve.

2.2. **Evolution as a group.** The trajectory evolution $f^t$ is a one-parameter semigroup, where (2.4)

$$f^{t+s} = f^t \circ f^s\,.$$

Show that it is a commutative semigroup.

In this case, the commutative character of the semigroup of evolution functions comes from the commutative character of the time parameter under addition. Can you think of any other semigroup replacing time?

2.3. **Almost ODE's.**

(a) Consider the point $x$ on $\mathbb{R}$ evolving according $\dot{x} = e^{\dot{x}}$. Is this an ordinary differential equation?

(b) Is $\dot{x} = x(x(t))$ an ordinary differential equation?

(c) What about $\dot{x} = x(t+1)$?

2.4. **All equilibrium points are fixed points.** Show that a point of a vector field $v$ where the velocity is zero is a fixed point of the dynamics $f^t$.

2.5. **Gradient systems.** Gradient systems (or 'potential problems') are a simple class of dynamical systems for which the velocity field is given by the gradient of an auxiliary function, the 'potential' $\phi$

$$\dot{x} = -\nabla\phi(x)$$

where $x \in \mathbb{R}^d$, and $\phi$ is a function from that space to the reals $\mathbb{R}$.

(a) Show that the velocity of the particle is in the direction of most rapid decrease of the function $\phi$.

(b) Show that all extrema of $\phi$ are fixed points of the flow.

(c) Show that it takes an infinite amount of time for the system to reach an equilibrium point.

(d) Show that there are no periodic orbits in gradient systems.

2.6. **Runge-Kutta integration.** Implement the fourth-order Runge-Kutta integration formula (see, for example, ref. [**?**]) for $\dot{x} = v(x)$:

$$\begin{aligned} x_{n+1} &= x_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(\delta\tau^5) \\ k_1 &= \delta\tau\, v(x_n)\,, \quad k_2 = \delta\tau\, v(x_n + k_1/2) \\ k_3 &= \delta\tau\, v(x_n + k_2/2) \\ k_4 &= \delta\tau\, v(x_n + k_3)\,. \end{aligned}$$

If you already know your Runge-Kutta, program what you believe to be a better numerical integration routine, and explain what is better about it.

2.7. **Rössler flow.** Use the result of exercise 2.6 or some other integration routine to integrate numerically the Rössler flow (2.27). Does the result look like a 'strange attractor'?

2.8. **Equilibria of the Rössler flow.**

(a) Find all equilibrium points $(x_q, y_q, z_q)$ of the Rössler system (2.27). How many are there?

(b) Assume that $b = a$. As we shall see, some surprisingly large, and surprisingly small numbers arise in this system. In order to understand their size, introduce parameters

$$\epsilon = a/c\,,\ D = 1 - 4\epsilon^2\,,\ p^{\pm} = (1 \pm \sqrt{D})/2\,.$$

Express all the equilibria in terms of $(c, \epsilon, D, p^{\pm})$, expand to the first order in $\epsilon$, and evaluate for $a = b = 0.2$, $c = 5.7$ in (2.27). In the case studied $\epsilon \approx 0.03$, so these estimates are quite accurate. (continued in exercise 3.1)

(Rytis Paškauskas)

2.9. **Can you integrate me?** Integrating equations numerically is not for the faint of heart. It is not always possible to establish that a set of nonlinear ordinary

differential equations has a solution for all times and there are many cases were the solution only exists for a limited time interval, as, for example, for the equation $\dot{x} = x^2\,,\ x(0) = 1\,.$

(a) For what times do solutions of

$$\dot{x} = x(x(t))$$

exist? Do you need a numerical routine to answer this question?

(b) Let's test the integrator you wrote in exercise 2.6. The equation

$$\ddot{x} = x \tag{2.29}$$

with initial conditions $x(0) = 2$ and $\dot{x} = 0$ has the solution $x(t) = e^{-t}(1 + e^{2\,t})\,.$ Can your integrator reproduce this solution for the interval $t \in [0, 10]$? Check your solution by plotting the error as compared to the exact result.

(c) Test your integrator for

$$\ddot{x} = -x \tag{2.30}$$

with the same initial conditions and integration interval.

(d) Now we will try something a little harder. The equation is going to be third order

$$\dddot{x} + 0.6\ddot{x} + \dot{x} - |x| + 1 = 0\,,$$

which can be checked–numerically–to be chaotic. For initial conditions, we will always use $\ddot{x}(0) = \dot{x}(0) = x(0) = 0\,.$ Can you reproduce the result $x(12) = 0.8462071873$ (all digits are significant)? Even though the equation being integrated is chaotic, the time intervals are not long enough for the exponential separation of trajectories to be noticeable (the exponential growth factor is $\approx 2.4$).

(e) Determine the time interval for which the solution of $\dot{x} = x^2, x(0) = 1$ exists.

2.10. **Classical collinear helium dynamics.** In order to apply periodic orbit theory to quantization of helium we shall need to compute classical periodic orbits of the helium system. In this exercise we commence their evaluation for the collinear helium atom (8.8)

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - \frac{Z}{r_1} - \frac{Z}{r_2} + \frac{1}{r_1 + r_2}\,.$$

The nuclear charge for helium is $Z = 2$. Colinear helium has only 3 degrees of freedom and the dynamics can be visualized as a motion in the $(r_1, r_2)$, $r_i \geq 0$ quadrant. In $(r_1, r_2)$-coordinates the potential is singular for $r_i \to 0$ nucleus-electron collisions. These 2-body collisions can be regularized by rescaling the coordinates, with details given in sect. B.2. In the transformed coordinates $(x_1, x_2, p_1, p_2)$ the Hamiltonian equations of motion take the form

$$\begin{aligned}
\dot{P}_1 &= 2Q_1\left[2 - \frac{P_2^2}{8} - Q_2^2(1 + \frac{Q_2^2}{R^4})\right] \\
\dot{P}_2 &= 2Q_2\left[2 - \frac{P_1^2}{8} - Q_1^2(1 + \frac{Q_1^2}{R^4})\right] \\
\dot{Q}_1 &= \frac{1}{4}P_1Q_2^2\,, \quad \dot{Q}_2 = \frac{1}{4}P_2Q_1^2\,.
\end{aligned} \tag{2.31}$$

where $R = (Q_1^2 + Q_2^2)^{1/2}$.

(a) Integrate the equations of motion by the fourth order Runge-Kutta computer routine of exercise 2.6 (or whatever integration routine you like). A convenient way to visualize the 3-dimensional state space orbit is by projecting it onto the 2-dimensional $(r_1(t), r_2(t))$ plane. (continued in exercise 3.4)

(Gregor Tanner, Per Rosenqvist)

2.11. **In high dimensions any two vectors are (nearly) orthogonal.** Among humble plumbers laboring with extremely high-dimensional ODE discretizations of fluid and other PDEs, there is an inclination to visualize the $\infty$-dimensional state space flow by projecting it onto a basis constructed from a few random coordinates, let's say the 2nd Fourier mode along the spatial $x$ direction against the 4th Chebyshev mode along the $y$ direction. It's easy, as these are typically the computational degrees of freedom. As we will now show, it's easy but not smart, with vectors representing the dynamical states of interest being almost orthogonal to any such random basis.

Suppose your state space $\mathcal{M}$ is a real 10 247-dimensional vector space, and you pick from it two vectors $x_1, x_2 \in \mathcal{M}$ at random. What is the angle between them likely to be?

By asking for 'angle between two vectors' we have implicitly assumed that there exist is a dot product

$$x_1{}^{\top} \cdot x_2 = \| x_1 \| \| x_2 \| \cos(\theta_{12})\,,$$

so let's make these vectors unit vectors, $\left\| x_j \right\| = 1\,.$ When you think about it, you would be hard put to say what 'uniform probability' would mean for a vector $x \in \mathcal{M} = \mathbb{R}^{10\,247}$, but for a unit vector it is obvious: probability that $x$ direction lies within a solid angle $d\Omega$ is $d\Omega/$(unit hyper-sphere surface).

So what is the surface of the unit sphere (or, the total solid angle) in $d$ dimensions? One way to compute it is to evaluate the Gaussian integral

$$I_d = \int_{-\infty}^{\infty} dx_1 \cdots dx_d\, e^{-\frac{1}{2}(x_1^2 + \cdots + x_d^2)} \tag{2.32}$$

in cartesian and polar coordinates. Show that

(a) In cartesian coordinates $I_d = (2\pi)^{d/2}$.

(b) Recast the integrals in polar coordinate form. You know how to compute this integral in 2 and 3 dimensions. Show by induction that the surface $S_{d-1}$ of unit $d$-ball, or the total solid angle in even and odd dimensions is given by

$$S_{2k} = \frac{2(2\pi)^k}{(2k-1)!!}, \qquad S_{2k+1} = \frac{2\pi^{k+1}}{k!}. \quad (2.33)$$

(c) Show, by examining the form of the integrand in the polar coordinates, that for arbitrary, perhaps even complex dimension $d \in \mathbb{C}$

$$S_{d-1} = 2\pi^{d/2}/\Gamma(d/2).$$

In Quantum Field Theory integrals over 4-momenta are brought to polar form and evaluated as functions of a complex dimension parameter $d$. This procedure is called the 'dimensional regularization'.

(d) Check your formula for $d = 2$ (1-sphere, or the circle) and $d = 3$ (2-sphere, or the sphere).

(e) What limit does $S_d$ does tend to for large $d$? (Hint: it's not what you think. Try Sterling's formula).

So now that we know the volume of a sphere, what is a the most likely angle between two vectors $x_1, x_2$ picked at random? We can rotate coordinates so that $x_1$ is aligned with the '$z$-axis' of the hypersphere. An angle $\theta$ then defines a meridian around the '$z$-axis'.

(f) Show that probability $P(\theta)d\theta$ of finding two vectors at angle $\theta$ is given by the area of the meridional strip of width $d\theta$, and derive the formula for it:

$$P(\theta) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(d/2)}{\Gamma((d-1)/2)}.$$

(One can write analytic expression for this in terms of beta functions, but it is unnecessary for the problem at hand).

(g) Show that for large $d$ the probability $P(\theta)$ tends to a normal distribution with mean $\theta = \pi/2$ and variance $1/d$.

So, in $d$-dimensional vector space the two random vectors are nearly orthogonal, within accuracy of $\theta = \pi/2 \pm 1/d$.

If you are a humble plumber, and the notion of a vector space is some abstract hocus-pocus to you, try thinking this way. Your 2nd Fourier mode basis vector is something that wiggles twice along your computation domain. Your turbulent state is very wiggly. The product of the two functions integrated over the computational domain will average to zero, with a small leftover. We have just estimated that with dumb choices of coordinate bases this leftover will be of order of $1/10\,247$, which is embarrassingly small for displaying a phenomenon of order $\approx 1$.

Several intelligent choices of coordinates for state space projections are described in Gibson *et al.* [48] and the web tutorial ChaosBook.org/tutorials.

Sara A. Solla and P. Cvitanović

## References


[2.1] Y. Lan and P. Cvitanović, Unstable recurrent patterns in Kuramoto-Sivashinsky dynamics, Phys. Rev. E **78**, 026208 (2008), arXiv:0804.2474.

[2.2] J. Cash, J. Carter Cash, and M. Kilgore, The Ring of Fire, 1963.

[2.3] P. Cvitanović, R. L. Davidchack, and E. Siminos, On the state space geometry of the Kuramoto-Sivashinsky flow in a periodic domain, SIAM J. Appl. Dyn. Syst. **9**, 1 (2010), arXiv:0709.2944.

[2.4] E. Hopf, A mathematical example displaying features of turbulence, Comm. Appl. Math. **1**, 303 (1948).

[2.5] E. N. Lorenz, Deterministic nonperiodic flow, J. Atmos. Sci. **20**, 130 (1963).

[2.6] E. Hopf, Statistical hydromechanics and functional calculus, J. Rat. Mech. Anal. **1**, 87 (1952).

[2.7] G. Duffing, *Erzwungene Schwingungen bei veränderlicher Eigenfrequenz und ihre technische Bedeutung*, Samlung Vieweg No. 41-42 (Vieweg und Sohn, Braunschweig, 1918).

[2.8] O. E. Rössler, An equation for continuous chaos, Phys. Lett. A **57**, 397 (1976).

[2.9] J. Peinke, J. Parisi, O. E. Rössler, and R. Stoop, *Encounter with chaos - Self organised hierarchical complexity in semiconductor experiments* (Springer, New York, 1992).

[2.10] R. Gilmore and M. Lefranc, *The Topology of Chaos* (Wiley, New York, 2003).

[2.11] J. M. T. Thompson and H. B. Stewart, *Nonlinear Dynamics and Chaos: Geometrical Methods for Engineers and Scientists* (Wiley, New York, 2002).

[2.12] R. H. Abraham and C. D. Shaw, *Dynamics - The Geometry of Behavior* (Wesley, Reading, MA, 1992).

[2.13] T. Jones, Symmetry of chaos animations, 1994, lagrange.physics.drexel.edu/flash.

[2.14] M. Axenides and E. Floratos, Strange attractors in dissipative Nambu mechanics: Classical and quantum aspects, JHEP **1004**, 036 (2010), arXiv:0910.3881.

[2.15] M. L. Cartwright and J. E. Littlewood, On nonlinear differential equations of the second order, J. London Math. Soc. **20**, 180 (1945).

[2.16] P. Cvitanović, *Universality in Chaos* (Adam Hilger, Bristol, 1989).

[2.17] B.-L. Hao, *Chaos II* (World Scientific, Singapore, 1990).

[2.18] W. Tucker, The Lorentz attractor exists, C. R. Acad. Sci. Paris Sér. I Math. **328**, 1197 (1999).

[2.19] W. Tucker, A rigorous ODE solver and smale's 14th problem, Found. Comp. Math **2**, 53 (2002).

[2.20] M. Viana, Whats new on Lorenz strange attractors?, Math. Intelligencer **22**, 6 (2000).

[2.21] J. Frø yland, *Chaos and coherence* (Inst. of Phys. Publ., Bristol, 1992).

[2.22] J. Frø yland and K. H. Alfsen, Lyapunov-exponent spectra for the Lorenz model, Phys. Rev. A **29**, 2928 (1984).

[2.23] J. Guckenheimer and R. Williams, Structural stability of the Lorenz attractor, Publ. Math. IHES **50**, 55 (1979).

[2.24] V. S. Afraimovich, B. B. Bykov, and L. P. Shilnikov, On the appearence and the structure of the Lorenz attractor, Dokl. Akad. Nauk SSSR **234**, 336 (1987), In Russian.

[2.25] C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos and Strange Attractors* (Springer, New York, 1982).

[2.26] D. Rothman, Nonlinear Dynamics I: Chaos, 2006.

[2.27] E. A. Jackson, *Perspectives of nonlinear dynamics: Vol. 1 and 2* (Cambridge Univ. Press, Cambridge, 1989).

[2.28] R. Seydel, *From equilibrium to chaos: Practical bifurcation and stability analysis* (Elsevier, New York, 1988).

[2.29] D. Viswanath, Symbolic dynamics and periodic orbits of the Lorenz attractor, Nonlinearity **16**, 1035 (2003).

[2.30] D. Viswanath, The fractal property of the Lorenz attractor, Physica D **190**, 115 (2004).

[2.31] D. Viswanath and S. Şahutoğlu, Complex singularities and the Lorenz attractor, SIAM review **52**, 294 (2010).

[2.32] P. Manneville and Y. Pomeau, Different ways to turbulence in dissipative dynamical systems, Physica D **1**, 219 (1980).

[2.33] J. B. Laughlin and P. C. Martin, Transition to turbulence of a statically stressed fluid, Phys. Rev. Lett. **33**, 1189 (1974).

[2.34] J. Laskar, The chaotic behavior of the solar system: A numerical estimate of the chaotic zones, Icarus **88**, 266 (1990).

[2.35] C. Skokos, Alignment indices: a new, simple method for determining the ordered or chaotic nature of orbits, J. Phys. A **34**, 10029 (2001).

[2.36] J. Hopcroft and R. Kannan, Foundations of data science, In preparation, 2014.

[2.37] E. J. Wegman and J. L. Solka, On some mathematics for visualizing high dimensional data, Sankhyā: Indian J. Statistics, Ser. A , 429 (2002).

[2.38] M. C. Spruill, Asymptotic distribution of coordinates on high dimensional spheres, Elect. Comm. in Probab. **12**, 234 (2007).

[2.39] J. F. Gibson, J. Halcrow, and P. Cvitanović, Visualizing the geometry of state-space in plane Couette flow, J Fluid Mech. **611**, 107 (2008), arXiv:0705.3957.

[2.40] N. Aubry, P. Holmes, J. L. Lumley, and E. Stone, The dynamics of coherent structures in the wall region of turbulent boundary layer, J. Fluid Mech. **192**, 115 (1988).

[2.41] J. D. Meiss, Sci.nonlinear FAQ, Computational Resources, 2003, amath.colorado.edu/faculty/jdm/faq.html.

[2.42] S. D. Systems, DSWeb Dynamical Systems Software, 2015, www.dynamicalsystems.org.

[2.43] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory* (Springer, New York, 2004).

[2.44] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes in Fortran* (Cambridge Univ. Press, Cambridge, 1996).