By drawing more diagrams [e.g., 2d scales as $N(1/N^4)N^4$, with the three factors coming from the quartic coupling, the propagators, and the sum over colors, respectively], you can convince yourself that planar diagrams dominate in the large N limit, all scaling as N. For a challenge, try to prove it. Evidently, there is a topological flavor to all this.

The reduction to planar diagrams is a vast simplification but there are still an infinite number of diagrams. At this stage in our mastery of field theory, we still can't solve large N QCD. (As I started writing this book, there were tantalizing clues, based on insight and techniques developed in string theory, that a solution of large N QCD might be within sight. As I now go through the final revision, that hope has faded.)

The double-line formalism has a natural interpretation. Group theoretically, the matrix gauge potential A^i_j transforms just like $\bar{q}^i q_j$ (but assuredly we are not saying that the gluon is a quark-antiquark bound state) and the two lines may be thought of as describing a quark and an antiquark propagating along, with the arrows showing the direction in which color is flowing.

Random matrix theory

There is a much simpler theory, structurally similar to large N QCD, that actually can be solved. I am referring to random matrix theory.

Exaggerating a bit, we can say that quantum mechanics consists of writing down a matrix known as the Hamiltonian and then finding its eigenvalues and eigenvectors. In the early 1950s, when confronted with the problem of studying the properties of complicated atomic nuclei, Eugene Wigner proposed that instead of solving the true Hamiltonian in some dubious approximation we might generate large matrices randomly and study the distribution of the eigenvalues—a sort of statistical quantum mechanics. Random matrix theory has since become a rich and flourishing subject, with an enormous and growing literature and applications to numerous areas of theoretical physics and even to pure mathematics (such as operator algebra and number theory.)[2] It has obvious applications to disordered condensed matter systems and less obvious applications to random surfaces and hence even to string theory. Here I will content myself with showing how 't Hooft's observation about planar diagrams works in the context of random matrix theory.

Let us generate N by N hermitean matrices φ randomly according to the probability

$$P(\varphi) = \frac{1}{Z} e^{-N \operatorname{tr} V(\varphi)} \tag{2}$$

[2] For a glimpse of the mathematical literature, see D. Voiculescu, ed., *Free Probability Theory*.

with $V(\varphi)$ a polynomial in φ. For example, let $V(\varphi) = \frac{1}{2}m^2\varphi^2 + g\varphi^4$. The normalization $\int D\varphi P(\varphi) = 1$ fixes

$$Z = \int d\varphi e^{-N \operatorname{tr} V(\varphi)} \tag{3}$$

The limit $N \to \infty$ is always understood.

As in Chapter VI.7 we are interested in $\rho(E)$, the density of eigenvalues of φ. To make sure that you understand what is actually meant, let me describe what we would do were we to evaluate $\rho(E)$ numerically. For some large integer N, we would ask the computer to generate a hermitean matrix φ with the probability $P(\varphi)$ and then to solve the eigenvalue equation $\varphi v = Ev$. After this procedure had been repeated many times, the computer could plot the distribution of eigenvalues in a histogram that eventually approaches a smooth curve, called the density of eigenvalues $\rho(E)$.

We already developed the formalism to compute $\rho(E)$ in (VI.7.1): Compute the real analytic function $G(z) \equiv \langle (1/N) \operatorname{tr}[1/z - \varphi] \rangle$ and $\rho(E) = -(1/\pi) \lim_{\varepsilon \to 0} \operatorname{Im} G(E + i\varepsilon)$. The average $\langle \cdots \rangle$ is taken with the probability $P(\varphi)$:

$$\langle O(\varphi) \rangle = \frac{1}{Z} \int D\varphi e^{-N \operatorname{tr} V(\varphi)} O(\varphi)$$

You see that my choice of notation, φ for the matrix and $V(\varphi) = \frac{1}{2}m^2\varphi^2 + g\varphi^4$ as an example, is meant to be provocative. The evaluation of Z is just like the evaluation of a path integral, but for an action $S(\varphi) = N \operatorname{tr} V(\varphi)$ that does not involve $\int d^d x$. Random matrix theory can be thought of as a quantum field theory in $(0 + 0)$-dimensional spacetime!

Various field theoretic methods, such as Feynman diagrams, can all be applied to random matrix theory. But life is sweet in $(0 + 0)$-dimensional spacetime: There is no space, no time, no energy, and no momentum and hence no integral to do in evaluating Feynman diagrams.

The Wigner semicircle law

Let us see how this works for the simple case $V(\varphi) = \frac{1}{2}m^2\varphi^2$ (we can always absorb m into φ but we won't). Instead of $G(z)$, it is slightly easier to calculate

$$G^i_j(z) \equiv \left\langle \left(\frac{1}{z - \varphi} \right)^i_j \right\rangle = \delta^i_j G(z)$$

The last equality follows from invariance under unitary transformations:

$$P(\varphi) = P(U^\dagger \varphi U) \tag{4}$$

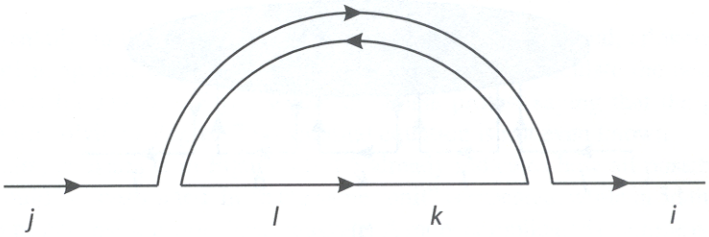


Figure VII.4.3

Expand

$$G^i_j(z) = \sum_{n=0}^{\infty} \frac{1}{z^{2n+1}} \langle (\varphi^{2n})^i_j \rangle \tag{5}$$

Do the Gaussian integral

$$\frac{1}{Z} \int D\varphi e^{-N \operatorname{tr} \frac{1}{2} m^2 \varphi^2} \varphi^i_k \varphi^l_j = \frac{1}{Z} \int D\varphi e^{-N \frac{1}{2} m^2 \sum_{p,q} \varphi^p_q \varphi^q_p} \varphi^i_k \varphi^l_j = \delta^i_j \delta^l_k \frac{1}{Nm^2} \tag{6}$$

Setting $k = l$ and summing, we find the $n = 1$ term in (5) is equal to $(1/z^3)\delta^i_j (1/m^2)$.

Just as in any field theory we can associate a Feynman diagram with each of the terms in (5). For the $n = 1$ term, we have Figure VII.4.3. The matrix character of φ lends itself naturally to 't Hooft's double-line formalism and thus we can speak of quark and gluon propagators with a good deal of ease. The Feynman rules are given in Figure VII.4.4. We recognize φ as the gluon field and (6) as the gluon propagator. Indeed, we can formulate our problem as follows: Given the bare quark propagator $1/z$, compute the true quark propagator $G(z)$ with all interaction effects taken into account.

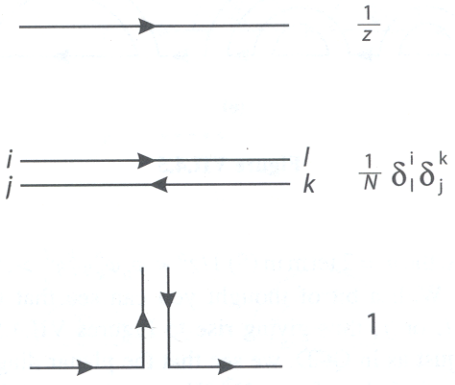


Figure VII.4.4

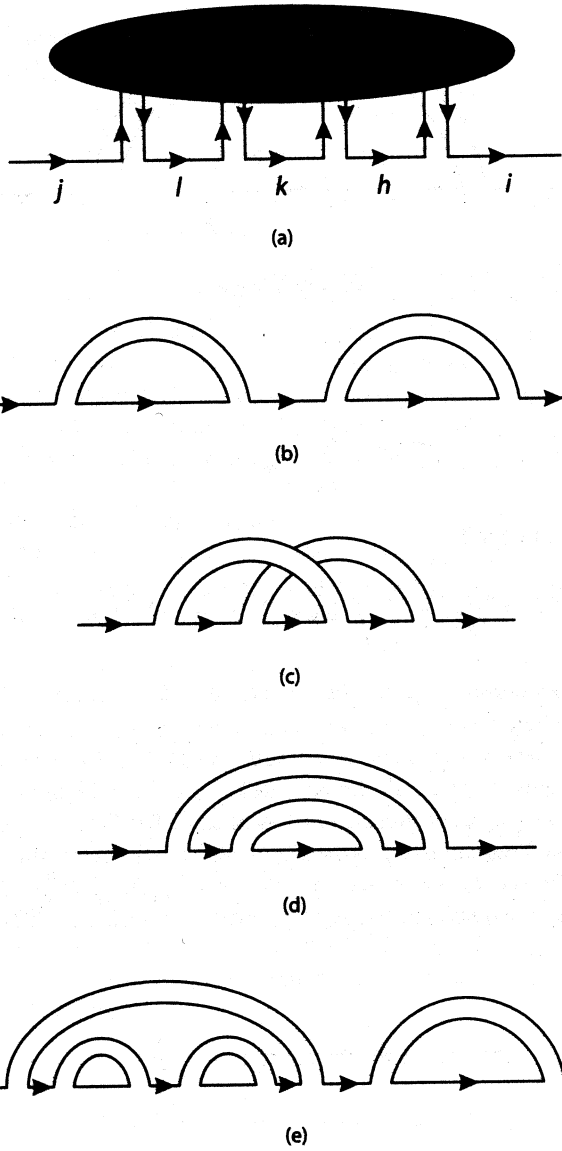


Figure VII.4.5

Let us now look at the $n = 2$ term in (5) $1/z^5 < \varphi_h^i \varphi_k^h \varphi_i^k \varphi_j^l >$, which we represent in Figure VII.4.5a. With a bit of thought you can see that the index i can be contracted with k, l, or j, thus giving rise to Figures VII.4.5b, c, d. Summing over color indices, just as in QCD, we see that the planar diagrams in 5b and 5d dominate the diagram in 5c by a factor N^2. We can take over 't Hooft's observation that planar diagrams dominate.

Incidentally, in this example, you see how large N is essential, allowing us to get rid of nonplanar diagrams. After all, if I ask you to calculate the density of eigenvalues for say $N = 7$ you would of course protest saying that the general formula for solving a degree-7 polynomial equation is not even known.

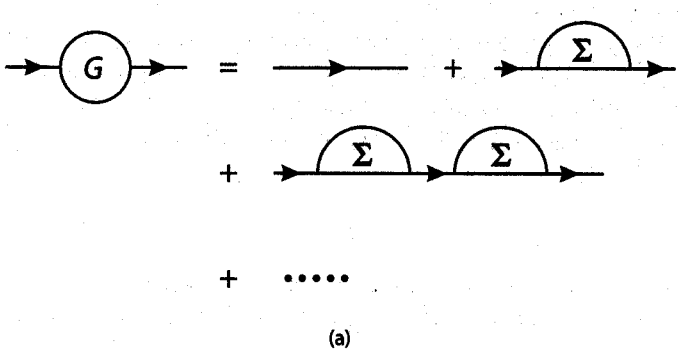
The simple example in Figure VII.4.5 already indicates how all possible diagrams could be constructed. In 5b the same "unit" is repeated, while in 5d the same "unit" is nested inside a more basic diagram. A more complicated example is shown in 5e. You can convince yourself that for $N = \infty$ all diagrams contributing to $G(z)$ can be generated by either "nesting" existing diagrams inside an overarching gluon propagator or "repeating" an existing structure over and over again. Translate the preceding sentence into two equations: "Repeat" (see Figure VII.4.6a),

$$\begin{aligned} G(z) &= \frac{1}{z} + \frac{1}{z}\Sigma(z)\frac{1}{z} + \frac{1}{z}\Sigma(z)\frac{1}{z}\Sigma(z)\frac{1}{z} + \cdots \\ &= \frac{1}{z - \Sigma(z)} \end{aligned} \tag{7}$$

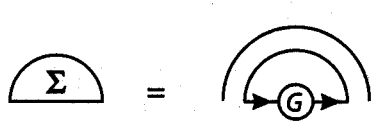
and "nest" (see Figure VII.4.6b),

$$\Sigma(z) = \frac{1}{m^2}G(z) \tag{8}$$

Combining these two equations we obtain a simple quadratic equation for $G(z)$ that we can immediately solve to obtain



(a)



(b)

Figure VII.4.6

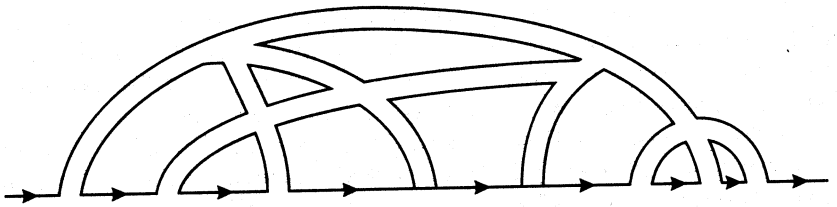


Figure VII.4.7

$$G(z) = \frac{m^2}{2}\left(z - \sqrt{z^2 - \frac{4}{m^2}}\right) \tag{9}$$

(From the definition of $G(z)$ we see that $G(z) \to 1/z$ for large z and thus we choose the negative root.) We immediately deduce that

$$\rho(E) = \frac{2}{\pi a^2}\sqrt{a^2 - E^2} \tag{10}$$

where $a^2 = 4/m^2$. This is a famous result known as Wigner's semicircle law.

The Dyson gas

I hope that you are struck by the elegance of the large N planar diagram approach. But you might have also noticed that the gluons do not interact. It is as if we have solved quantum electrodynamics while we have to solve quantum chromodynamics. What if we have to deal with $V(\varphi) = \frac{1}{2}m^2\varphi^2 + g\varphi^4$? The $g\varphi^4$ term causes the gluons to interact with each other, generating horrible diagrams such as the one in Figure VII.4.7. Clearly, diagrams proliferate and as far as I know nobody has ever been able to calculate $G(z)$ using the Feynman diagram approach.

Happily, $G(z)$ can be evaluated using another method known as the Dyson gas approach. The key is to write

$$\varphi = U^\dagger \Lambda U \tag{11}$$

where Λ denotes the N by N diagonal matrix with diagonal elements equal to λ_i, $i = 1, \ldots, N$. Change the integration variable in (3) from φ to U and Λ:

$$Z = \int dU \int \left(\Pi_i d\lambda_i\right) J e^{-N\sum_k V(\lambda_k)} \tag{12}$$

with J the Jacobian. Since the integrand does not depend on U we can throw away the integral over U. It just gives the volume of the group $SU(N)$. Does this remind you of chapter VII.1? Indeed, in (11) U corresponds to the unphysical gauge degrees of freedom—the relevant degrees of freedom are the eigenvalues $\{\lambda_i\}$. As an exercise you can use the Faddeev-Popov method to calculate J.