Symbolic Dynamics of Coupled Map Lattices

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Abstract

We show that coupled lattices of unimodal maps admit a simple generating partition which can be analyzed using the methods of symbolic dynamics. These methods have been very successful at rigorously defining and classifying chaotic motion in certain one and two dimensional maps. In the case of coupled map lattices (CMLs), which are models of high-dimensional dynamical systems, we find a generating partition, a generalized Gray ordering, and rules for determining the admissibility of symbol sequences. A specific case utilizing coupled tent maps is presented.

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The study of complex motion is greatly simplified by investigating models that employ a coarse space-time discretization. Such models, typified by coupled map lattices (CMLs), have been shown to reproduce the essential features of turbulence in physical, chemical, and biological systems [1]. A further simplification can be achieved by considering iterates of the resulting map through a generating partition which reduces the chaotic motion to a purely symbolic signal with an associated grammar. The study of such signals is called *symbolic dynamics* [2]. An effective state-space-time discretization connects dynamical systems theory to the study of formal languages, and hence, to computer science, information theory, and automata [3].

It is our goal in this letter to describe the grammar of the symbolic dynamics resulting from a generic coupled map lattice. Symbolic dynamical methods have been studied as a possible way to understand space-time chaos [4], and have recently been used [5] to bound entropy and determine ergodic properties of CMLs [6]. Attempts to understand Sinai-Ruelle-Bowen measures of general CMLs have relied on showing that good symbolic representations of local dynamics are preserved in the presence of weak interactions [7]; specifically, for uncoupled map lattices that have Markov partitions, weak coupling will tend to preserve the Markov property [8]. In this work we are focusing on the symbolic dynamics due to generating partitions, which exist in a much broader class of CMLs, including those with strong coupling. On the basis of the resulting sequence space we can build concise models of turbulence in CMLs compatible with those used recently for the targeting and control of low-dimensional chaos [9–11], for probing the limits of synchronization [12], and for efficient chaos communication schemes [13–15]. We introduce generalized Gray ordering and a pruning front for this purpose.

We consider a map lattice with N sites labeled $i = 1 \dots N$. Each site is described by a state x_n^i in the interval $I = [a, b]$ and with local dynamics $f_i : I \to I$. A typical coupled map lattice (CML) as introduced by Kaneko is written as

$$x_{n+1}^i = (1-\epsilon) f_i(x_n^i) + \epsilon/2[f_{i-1}(x_n^{i-1}) + f_{i+1}(x_n^{i+1})], \tag{1}$$

along with rules for the boundary sites [16] . The behavior of the lattice with respect to the coupling parameter ϵ is of primary interest here; the dynamics of the local maps f_i are well-understood. In particular, we choose f with a good symbolic representation.

Such a function can be taken from the family of unimodal (single-humped) maps over the interval. We label the location of the maxima as the critical point x_c and divide the interval into two sets $\mathcal{P} = \{P_0, P_1\}$, where $P_0 = [a, x_c)$ and $P_1 = (x_c, b]$. The point $x_0 \in I$ is mapped to the semi-infinite symbol sequence $\pi(x_0) = \mathbf{s} = s_0 s_1 s_2 s_3 \ldots$ where

$$s_n = \begin{cases} 0 \text{ if } f^n(x_0) \in P_0, \\ 1 \text{ if } f^n(x_0) \in P_1. \end{cases} \tag{2}$$

The critical point can be assigned either symbol. In this formulation we are concerned with finding rules that describe the set of sequences that are allowed by f. The grammar of a one dimensional map has been classically understood in terms of the kneading theory of Milnor and Thurston [17–20]. Once found, the set of allowable sequences can be related to the state space through the inverse mapping

$$r(\mathbf{s}) = \bigcap_{n=0}^{\infty} f^{-n}(P_{s_n}). \tag{3}$$

For a faithful representation of chaotic dynamics, we require that $r(\mathbf{s})$ converge to a single point in the interval, or to the null set if $\mathbf{s}$ is not permitted by f. A partition that satisfies this property is termed *generating*. Note that for non-invertible maps f^{-1} has multiple solutions, one on each monotonic segment of f. For a partition to be generating each of these branches must be uniquely labeled, otherwise points that share a common image would not be distinguishable in sequence space.

Returning to the CML, denote by F the product function of f_i onto each site, and by A an $N \times N$ coupling matrix. Then (1) can be generalized as the composition $H = A \circ F$, where A is chosen consistent with $H : I^N \to I^N$. Assuming A is non-singular, the elements of H^{-1} are

$$[H^{-1}(\mathbf{x_0})]_i = f_i^{-1}\left(\sum_j A_{ij}^{-1} x_0^j\right), \tag{4}$$

keeping in mind that f_i^{-1} is multi-valued. Because f_i^{-1} acts alone on the ith element the associated generating partition $\mathcal{P}^i$ of the solitary map f_i suffices to distinguish the branches of H_i^{-1}. Accordingly, we propose a partition for the CML that is the product of the partitions of the local maps. More precisely,

$$\mathcal{P}_{\text{CML}} = \bigvee_{i=1}^{N} \mathcal{P}^i = \mathcal{P}^1 \vee \mathcal{P}^2 \vee \ldots \vee \mathcal{P}^N, \tag{5}$$

where $\mathcal{P} \vee \mathcal{P}' = \{P_k \cap P_l' : 0 \leq k \leq |\mathcal{P}| - 1, 0 \leq l \leq |\mathcal{P}'| - 1\}$ is the mathematical *join* of $\mathcal{P}$ and $\mathcal{P}'$. The form of (4) implies that the pre-images of $H(\mathbf{x_0})$ are uniquely labeled under this choice of partition. We conjecture, then, that $\mathcal{P}_{\rm CML}$ is generating for chaotic behavior in the CML. Our reasoning is that, if the ergodicity is preserved by the coupling, and given uniform contraction of the inverse map away from the critical curves, the set of pre-images $\{H^{-n}(\mathbf{x_0}) : n \in \mathbb{N}\}$, each labeled differently under $\mathcal{P}_{\rm CML}$, covers the chaotic attractor and thus no two points on the attractor can be represented by the same semi-infinite symbol sequence.

As an example, consider the 2-element lattice $H : I^2 \rightarrow I^2$ written as

$$\begin{aligned} x^1_{n+1} &= (1-\epsilon) f(x^1_n) + \epsilon f(x^2_n), \\ x^2_{n+1} &= \epsilon f(x^1_n) + (1-\epsilon) f(x^2_n), \end{aligned} \tag{6}$$

where $f(x) = 1 - 2|x|$ is the tent map on the interval $I = [-1, 1]$. The tent map in this form has a generating partition consisting of the regions $[-1, 0)$ and $(0, 1]$ which we shall denote as $\{P^1_0, P^1_1\}$, respectively, for coordinate x^1, and as $\{P^2_0, P^2_1\}$ for x^2. Applying (5) the partition for the coupled system is

$$\begin{aligned} \mathcal{P}_{\rm CML} &= \mathcal{P}^2 \vee \mathcal{P}^1 \qquad (7) \\ &= \{P^2_0 \cap P^1_0, P^2_0 \cap P^1_1, P^2_1 \cap P^1_0, P^2_1 \cap P^1_1\}. \end{aligned}$$

Reading the subscripts of each term as a binary number, we relabel the regions as $\{P_0, P_1, P_2, P_3\}$ and assign to them the symbols 0, 1, 2, and 3. These four quadrants of I^2 are illustrated in Fig. 1(a).

The interesting dynamics of (6) occur for $\epsilon \in [0, 0.5]$. When the coupling is very strong ($0.25 < \epsilon \leq 0.5$) the synchronization manifold is stable and the long-term behavior is that of a solitary tent map, for which $\mathcal{P}_{\rm CML}$ is known to be generating. At weaker coupling strengths ($0 < \epsilon < 0.25$) the dynamics are considerably more complicated. In this regime H is everywhere expanding and thus the inverse mapping (3) converges to a set of zero measure. Fig. 1 illustrates the refining of I^2 into increasingly smaller regions for $\epsilon = 0.1$. When this process is extended indefinitely, each point in the square receives a semi-infinite symbolic label different from every other point.

The most powerful feature of the symbolic picture is that time evolution is reduced to a simple shift in sequence space. We denote the shift operator as σ and define its action on a symbol sequence as $\sigma(s_0 s_1 s_2 \ldots) = s_1 s_2 s_3 \ldots$, so that the effect of σ is to discard the leading symbol. If we set $\mathbf{s} = \pi(\mathbf{x_0})$, then it follows from (2) that $\sigma^n(\mathbf{s}) = \pi(\mathbf{x_n})$. Consequently, the

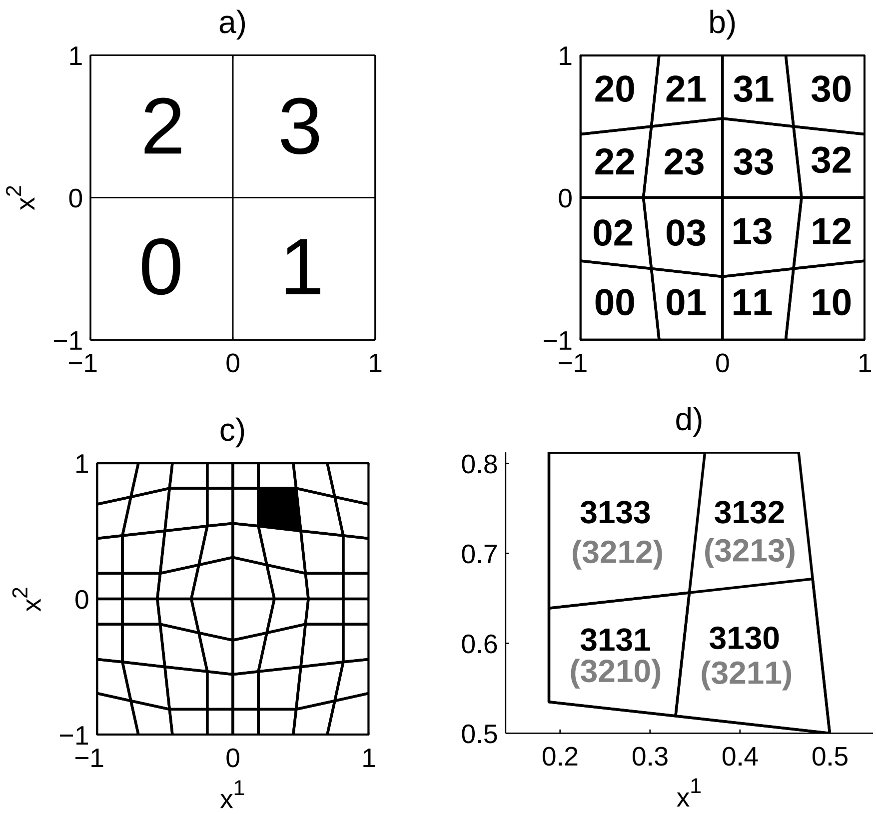

FIG. 1: Refinements of the square under a two element full tent map lattice with coupling strength $\epsilon = 0.1$. a) The primary partitioning of the square into four symbols, b) the second refinement into 2-block sequences, c) 3-block sequences (unlabeled) with region 313 darkened, and d) region 313 subdivided into its 4-block words. The equivalent sequences under the inverse Gray transformation are shown parenthetically.

entire evolution of $\mathbf{x_0}$ is contained in its symbolic representation. A point on a period-m orbit, for example, is conjugate to a sequence formed by infinitely repeating m symbols. Finding all such sequences within the given alphabet is a matter of combinatorics. The caveat here is that not all sequences are necessarily allowed by the dynamics of H. We now turn our attention to tabulating these grammar restrictions.

In unimodal maps the symbolic sequence associated with the image of the critical point, referred to as the *kneading sequence*, plays a special role in determining the admissibility of all other sequences [17–20]. When interpreted as a Gray code [21], the kneading sequence is the largest of all allowable sequences. Any sequence that orders above the kneading sequence is forbidden while those that order below it are admissible provided they do not contain subblocks that are inadmissible. The order of a sequence can be found by applying an inverse Gray transformation, G^{-1}, and interpreting the result as binary number. Given a symbol sequence $\mathbf{s} = s_1 s_2 s_3 \ldots, s_n \in \{0, 1\}$, the bitwise elements of $G^{-1}(\mathbf{s})$, denoted $b_1 b_2 b_3 \ldots$, are determined by setting $b_1 = s_1$ and

$$b_i = b_{i-1} \otimes s_i, \text{ for } i > 1, \tag{8}$$

where $\otimes$ is the *exclusive-OR* operator.

To accommodate lattices, we propose an extension of the Gray ordering to multiple symbols by defining the $\otimes$ operator to act bitwise on the binary representation of the symbols. For example: $2 \otimes 3 = 10 \otimes 11 = 01 = 1$. Thus the word 313 is mapped by (8) to 321 via the steps $b_1 = 3$, $b_2 = 3 \otimes 1 = 2$, and $b_3 = 3 \otimes 1 \otimes 3 = 1$. Applying G^{-1} onto the symbol sequences produced by lattices of coupled unimodal maps gives spatial information about their location relative to other sequences. In this notation the subregion labels appear in the same spatial relationship as they do in the primary partitioning: the lower left "quad" of region 321 is labeled 3210, the lower right "quad" is 3211 and so on (see Fig. 1(d)).

In the CML the critical point is generalized to a critical curve. It can be shown that this curve is the boundaries of the generating partition. The image of the critical curve produces the associated kneading curve, seen as the diamond shape in Fig. 2(a), within which the chaotic attractor of (6) is contained [22]. The symbolic sequences associated with this kneading curve are maximal in the sense that all sequences that lie outside of the region it encloses are forbidden. In this aspect, our work has strong relationship to previous work [23] as type of high-dimensional pruning front.

An algorithm for finding m-block forbidden sequences can be arrived at by comparing consecutive refinements of the kneading curve. Consider the set comprised of all the m-block subregions of the $(m-1)$th refinement of the kneading curve. The members of this set that lie strictly outside the mth refinement of the kneading curve are the m-block forbidden words. These spatial relationships can be resolved entirely within the symbolic representation via G^{-1}. The result of this procedure for $\epsilon = 0.2$ of the 2-element lattice (6) is listed in Table I and shown graphically in Fig. 2(b) for words up to $m = 5$.

To see the utility of Table I, consider the problem of locating all period-2 orbits of (6) for $\epsilon = 0.2$. In sequence space points belonging to period-m orbits are represented by sequences comprised of infinitely repeating m-block words. We use a bar to denote infinite repetition,

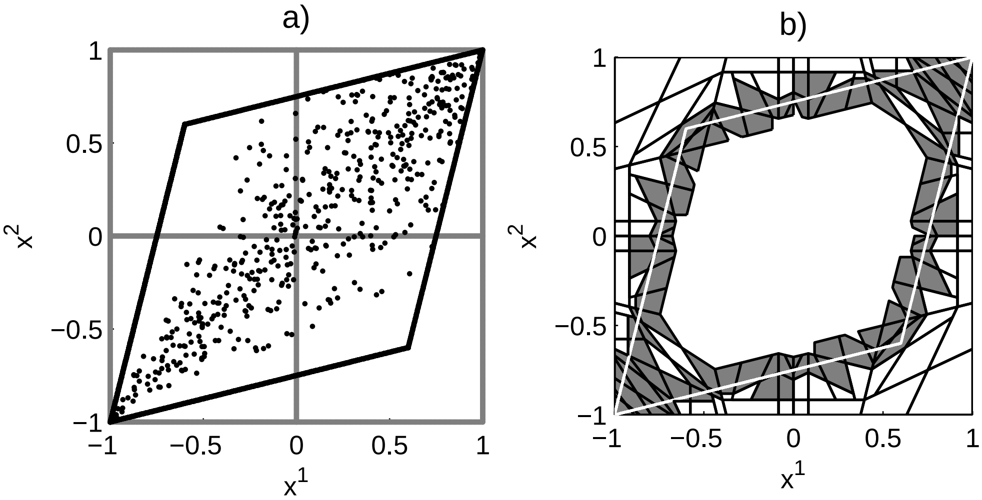

FIG. 2: The image of the critical curve and its fifth refinement for $\epsilon = 0.2$. a) The kneading curve (heavy black lines) is the image of the boundaries of the generating partition (gray). The chaotic attractor, represented by 500 points, lies within this curve. b) Here the kneading curve is shown in white and its fifth refinement by the gray-shaded regions. The area outside the kneading curve is tiled in by the 3-, 4-, and 5-block forbidden regions listed in Table I.

so that $\overline{01}$ and its shift $\overline{10}$ represent one possible period-2 orbit. In a four symbol alphabet there are exactly six such pairs: $(\overline{01}, \overline{10})$, $(\overline{02}, \overline{20})$, $(\overline{03}, \overline{30})$, $(\overline{12}, \overline{21})$, $(\overline{13}, \overline{31})$, and $(\overline{23}, \overline{32})$. The first two are inadmissible because they contain the forbidden words 010, 101, 020, and 202 listed in line 3 of the table. The other period-2 orbits do not contain any of the forbidden words shown. Because our table is truncated at $m = 5$, we must also check that the orbits are not members of the 5th refinement of the kneading curve; the admissibility of these regions is still unresolved at this stage of refinement. It can be shown that the remaining orbits satisfy this requirement. Fig. 3 displays the locations of the allowable period-2 orbits computed with the first seven terms of (3).

The results presented here generalize in an obvious manner to larger arrays. Clearly, for

TABLE I: Forbidden m-block words for $\epsilon = 0.2$.

m	forbidden m-block words
1	none
2	none
3	011, 010, 022, 020, 100, 101, 110, 111, 102, 120, 122, 200, 201, 202, 210, 211, 220, 222, 310, 311, 320, 322
4	0012, 0021, 1032, 1033, 1031, 1132, 1231, 2031, 2033, 2032, 2132, 2231, 3012, 3021
5	00132, 00133, 00231, 00233, 01331, 02332, 11230, 11330, 11331, 12130, 12330, 12332, 22130, 22330, 22332, 21230, 21330, 21331, 30231, 30233, 30132, 30133, 32332, 31331

large N one must look to symmetry to reduce the problem to a manageable size. Once the inverse Gray transformation is applied, sequence groups that are related by axial and diagonal symmetries can be generated by exchanging the corresponding symbols. In the example case, reflection about the diagonals is effected by an exchange of symbols 0 and 3, and 1 and 2, respectively. Consider the forbidden word 1132 and its inverse Gray form (1031). Applying the $0 \leftrightarrow 3$ exchange to (1031) gives (1301), which then yield the additional sequences (2032) and (2302) upon exchanging symbols 1 and 2. Finally, transforming back to Gray ordering, the forbidden 4-block words 1231, 2132, and 2231 are arrived at. Thus the search for forbidden sequences need not be carried out over the entire sequence space.

In summary, the spatio-temporal chaotic behavior of a broad class of CMLs can be efficiently described using the methods of symbolics dynamics. For coupled unimodal maps the appropriate partition is the join of the individual generating partitions, the sequences produced follow a generalized Gray order, and a kneading curve can be defined that delimits forbidden sequences. From this one can rigorously catalog all possible motions within the natural dynamics. Accordingly, we claim that symbolic dynamics provides a natural coordinate system for understanding complex spatio-temporal behavior in these systems.

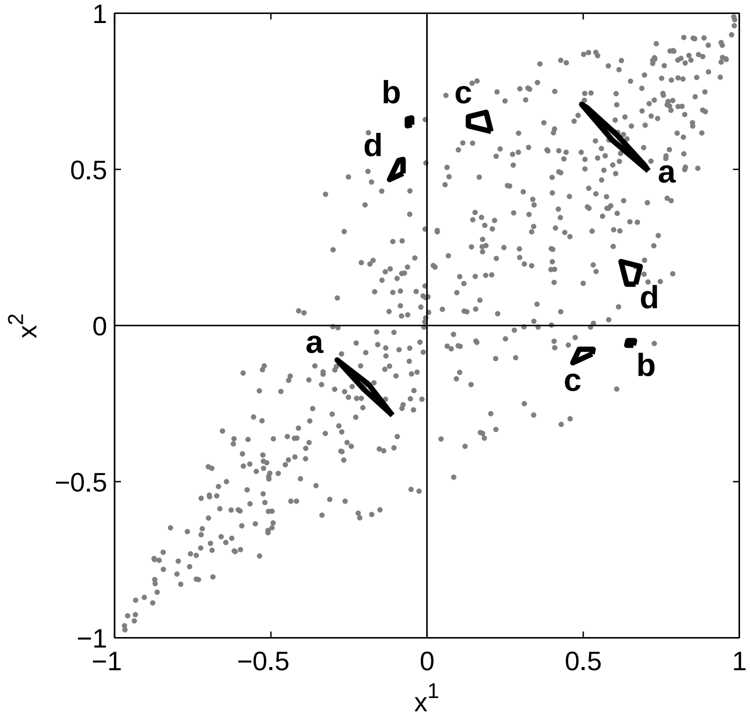

FIG. 3: The locations of the period-2 orbits resolved to 7 symbols are overlaid on the attractor (shown in gray). Regions labeled a, b, c, and d correspond to sequences 0303030, 1212121, 1313131, and 2323232, and their shifts, respectively.

[1] K. Kaneko, *Formation, Dynamics, and Statistics of Patterns*, edited by K. Kawasaki *et al.* (World Scientific, Singapore, 1990).

[2] B. P. Kitchens, *Symbolic Dynamics*, (Springer, NY, 1998); B.-L. Hao and W.-M. Zheng, *Applied Symbolic Dynamics and Chaos*, (World Scientific, Singapore 1998).

[3] R. Badii and A. Politi, *Complexity: Hierarchical structures and scaling in physics*, (Cambridge University Press, UK, 1997).

[4] L. A. Bunimovich and Ya. G. Sinai, Nonlinearity **1** (1988) 491-516.

[5] B. Fernandez and P. Guiraud, Dynamical Systems-Series B **4** (2004); W. Just, J. Stat. Phys. **90**, 727 (1998) and J. Stat. Phys. **105**, 133 (2001).

[6] L. A. Buminovich, Physica D **103**, 1 (1997).

[7] Y. B. Pesin and Y. G. Sinai, Advances in Soviet Mathematics **3**, 165-198, 1991.

[8] V. M. Gundlach, D. A. Rand, Nonlinearity **6**, 165-230 (1993).

[9] E. Bollt and E. Kostelich, Phys. Lett. A **245**, 399 (1998).

[10] N. J. Corron, S. D. Pethel, and K. Myneni, Phys. Rev. E **66**, 036204 (2002).

[11] N. J. Corron and S. D. Pethel, Phys. Lett. A **313**, 192 (2003).

[12] S. D. Pethel, N. J. Corron, Q. R. Underwood, and K. Myneni, Phys. Rev. Lett. **90**, 254101 (2003).

[13] S. Hayes, C. Grebogi, and E. Ott, Phys. Rev. Lett. **70**, 3031 (1993).

[14] E. Bollt, Y.-C. Lai, and C. Grebogi, Phys. Rev. Lett. **79**, 3787 (1997).

[15] E. Bollt, Int. J. Bif. and Chaos **13**, 269 (2003).

[16] K. Kaneko, Prog. Theor. Phys. **74,** 1033 (1985).

[17] J. Milnor and W. Thurston, *On iterated maps of the interval I and II* Princeton University Press (Princeton, 1977).

[18] P. Collet and J.-P. Eckmann, *Iterated Maps On The Interval As Dynamical Systems*, (Birkhauser Boston, 1980).

[19] W. de Melo and S. van Strein, *One-Dimensional Dynamics*, Springer-Verlag (New York, 1992).

[20] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems* II ed., (Addison-Wesley, NY 1989).

[21] W. Press, S. Teukolsky, W. Vetterling, and B. Flannery, *Numerical Recipes in C*, (Cambridge University Press, NY, 1992). See section 20.2.

[22] A very readable introduction to critical curves can be found in R. H. Abraham, L. Gardini, and C. Mira, *Chaos in Discrete Dynamical Systems*, (Springer-Verlag, NY 1997).

[23] P. Cvitanovic, G. H. Gunaratne, and I. Procaccia, Phys. Rev. A **38**, 1503 (1988) ; K. Hansen, Chaos **2**, 71 (1988) ; R. Hagiwara and A. Shudo, J. Phys. A: Math. Gen. **37**, 10545 (2004); K. T. Hansen and S. Güttler, J. Phys. A: Math. Gen. **30**, 3421 (1997); H. R. Dullin, J. D. Meiss and D. G. Sterling, arXiv:nlin.CD/0408015 (2004).