Relativistic Chaos is Coordinate Invariant

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The noninvariance of Lyapunov exponents in general relativity has led to the conclusion that chaos depends on the choice of the space-time coordinates. Strikingly, we uncover the transformation laws of Lyapunov exponents under general space-time transformations and we find that chaos, as characterized by positive Lyapunov exponents, is coordinate invariant. As a result, the previous conclusion regarding the noninvariance of chaos in cosmology, a major claim about chaos in general relativity, necessarily involves the violation of hypotheses required for a proper definition of the Lyapunov exponents.

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Chaotic properties of dynamical systems with a reparametrizable time coordinate are important in physical theories without an absolute time, such as general relativity. The study of chaos in general relativity has followed two main lines. One considers the geodesic motion of test particles in a given gravitational field [1]. The other investigates the time evolution of the gravitational field itself [2,3], which is relevant in cosmology. While the former case has been studied with standard methods of the dynamical systems theory, an adequate characterization of chaos in the latter is currently an open problem. The difficulty comes from the dependence of the *shear* between nearby trajectories on their time parametrization. Accordingly, dynamical properties, such as mixing and initial-condition sensitivity, may depend on the time parametrization.

In classical physics, the study of dynamical systems concerns differential equations of the form

$$d\mathbf{x}/dt = \mathbf{F}(\mathbf{x}), \tag{1}$$

where t is a uniquely defined parameter that usually represents the time. Although a general definition of chaos is missing, it is widely accepted that it regards the dynamics of bounded orbits and that a chaotic system must present sensitive dependence on initial conditions [4]. Chaos can then be quantified in terms of Lyapunov exponents [5] insofar as the following conditions are satisfied: (i) the system is *autonomous*; (ii) the relevant part of the phase space is *bounded*; (iii) the invariant measure is *normalizable*; (iv) the domain of the time parameter is *infinite*. Such a characterization is convenient because it is invariant under space diffeomorphisms of the form $\mathbf{y} = \boldsymbol{\psi}(\mathbf{x})$. As a result, chaos is a property of the physical system and does not depend on the coordinates used to describe the system.

In general relativity, the nonexistence of an absolute time parameter forces us to consider Eq. (1) under space-time diffeomorphisms: $\mathbf{y} = \boldsymbol{\psi}(\mathbf{x}, t)$, $d\tau = \lambda(\mathbf{x}, t)dt$. A conceptual problem then arises because of the dependence of classical indicators of chaos, such as Lyapunov exponents and entropies, on the choice of the time parameter. This problem has attracted a great deal of attention since it was first identified in the mixmaster cosmological model [6], whose largest Lyapunov exponent was shown to be positive or zero for different choices of the coordinates [7]. In particular, numerous methods based on invariant curvature [8], symbolic dynamics [9], Painlevé analysis [10], and fractals [11,12] have been proposed toward an invariant characterization of chaos in cosmology. The problem, however, goes beyond relativistic cosmology since it has been argued that the same kind of noninvariance can be observed in a system as simple as the harmonic oscillator if time reparametrizations are allowed [13]. Moreover, it has been exhibited examples of systems whose nonmixing dynamics can be converted into a mixing one through a time reparametrization [14]. These results have led to the tacit assumption that chaos itself depends on the space-time coordinates. In general relativity, this noninvariance would imply that chaos is a property of the coordinate system rather than a property of the physical system (see Ref. [3], and references therein).

In this Letter, we investigate the transformation laws of Lyapunov exponents under space-time reparametrizations that preserve conditions (i)–(iv). To be specific, we consider a Euclidean phase space (not to be confused with the pseudo-Riemannian spacetime), where the relevant invariant measure is the natural measure. Our principal result is that Lyapunov exponents transform according to

$$h_\tau^i = h_t^i/\langle\lambda\rangle_t \qquad (i = 1, \ldots, N), \tag{2}$$

where $0 < \langle\lambda\rangle_t < \infty$ is the time average of $\lambda = d\tau/dt$ over typical trajectories and N is the phase-space dimension [15]. The *values* of the Lyapunov exponents are, of course, noninvariant because a simple reparametrization such as $(\mathbf{x}, t) \to (\mathbf{x}, \alpha t)$ transforms the exponents h^i into h^i/α. However, the *signs* of the Lyapunov exponents are invariant. In particular, if $h > 0$ is the largest Lyapunov exponent, the reparametrization only changes the characteristic time scale $\mathcal{T} \equiv 1/h$ for the manifestation of the chaotic behavior. The striking implication of our findings is that *chaos*, as characterized by positive Lyapunov exponents, is coordinate independent. As we show, the vanishing of Lyapunov exponents in the mixmaster cosmology, as well as in other examples previously considered in the literature, is due to the violation of at least one of the conditions (i)–(iv) above, which are required for the Lyapunov exponents to be meaningful indicators of initial-condition sensitivity and, hence, chaos. For instance, it has been frequently claimed that the exponential divergence of initially close trajectories that separate as $\delta\mathbf{x}(t) = \delta\mathbf{x}_0 \exp(ht)$ can be removed with a logarithmic reparametrization of the time [16]. The suggested transformation is defined as $t = \ln\tau$, so that $\delta\mathbf{x}(\tau) = \delta\mathbf{x}_0 \tau^h$. This reparametrization is then interpreted as converting a positive Lyapunov exponent into a zero Lyapunov exponent, and the inverse of this transformation has been used to support the claim that integrable systems can have positive Lyapunov exponents [13]. The problem with this argument is that, if the original system is autonomous, the reparametrized system is necessarily nonautonomous. Alternatively, if we increase the dimension of the phase space in order to eliminate the explicit dependence on time, the orbits become unbounded. In any case, Lyapunov exponents are not valid indicators of chaos. We show that a similar problem, although more subtle, is present in the mixmaster cosmological model.

The Lyapunov exponents of an invariant set of the phase space of system (1) are defined as

$$h_t^i = \lim_{t\to\infty} \frac{1}{t} \ln\frac{|\boldsymbol{\eta}_t^i(t)|}{|\boldsymbol{\eta}_{t0}^i|} \qquad (i = 1, \ldots, N), \tag{3}$$

$$\text{where } d\boldsymbol{\eta}_t^i(t)/dt = \mathbf{DF}(\mathbf{x}(t)) \cdot \boldsymbol{\eta}_t^i(t), \tag{4}$$

$\mathbf{x}(0)$ is a typical initial condition, and $\boldsymbol{\eta}_{t0}^i = \boldsymbol{\eta}_t^i(0)$ are tangent vectors at $\mathbf{x}(0)$. We assume that $\mathbf{F}$ is a continuously differentiable function of $\mathbf{x}$ and system (1) has N independent Lyapunov exponents. Behind definition (3) and (4) are the hypotheses (i)–(iv), namely that, with respect to $(\mathbf{x}, t)$, the function $\mathbf{F}$ does not depend explicitly on t, the dynamics is well defined for t in the interval $[0, \infty)$, and the invariant set is bounded and has finite natural measure [17]. Otherwise, positive Lyapunov exponent is not a well-defined criterion for chaos. To ensure that system (1) remains autonomous after the coordinate transformation $(\mathbf{x}, t) \to (\mathbf{y}, \tau)$, we consider that, when functions λ or $\boldsymbol{\psi}$ depend explicitly on t, the coordinates $\mathbf{x}$ and $\mathbf{y}$ are redefined to incorporate t as an additional dimension in the phase space [18]. As a result, the coordinate transformation is always reduced to a time-independent transformation of the following form:

$$\mathbf{y} = \boldsymbol{\psi}(\mathbf{x}), \qquad d\tau = \lambda(\mathbf{x})dt, \tag{5}$$

where λ is a strictly positive, continuously differentiable function, and $\boldsymbol{\psi}$ is a diffeomorphism. This is the general class of transformations for which integrability is coordinate invariant [19] in the sense that, if $\{I_1, I_2, \ldots\}$ are independent integrals of motion with respect to the coordinates $(\mathbf{x}, t)$, then $\{I_1 \circ \boldsymbol{\psi}^{-1}, I_2 \circ \boldsymbol{\psi}^{-1}, \ldots\}$ are independent integrals of motion with respect to $(\mathbf{y}, \tau)$.

Transformation (5) is composed of a time reparametrization followed by a space diffeomorphism. It is well known that the Lyapunov exponents are invariant under space diffeomorphisms [4]. Without loss of generality, we consider only transformations of the time parameter $(\mathbf{x}, t) \to (\mathbf{x}, \tau)$, where $d\tau = \lambda(\mathbf{x})dt$. All the orbits of the phase space are invariant under this kind of transformation as the velocity field of the reparametrized flow is parallel to the original one: $d\mathbf{x}/d\tau = \lambda^{-1}(\mathbf{x})\mathbf{F}(\mathbf{x})$. The Lyapunov exponents, however, may be different because Eqs. (3) and (4) become

$$h_\tau^i = \lim_{\tau\to\infty} \frac{1}{\tau} \ln\frac{|\boldsymbol{\eta}_\tau^i(\tau)|}{|\boldsymbol{\eta}_{\tau0}^i|}, \tag{6}$$

$$\text{and } d\boldsymbol{\eta}_\tau^i(\tau)/d\tau = \mathbf{D}[\lambda^{-1}\mathbf{F}](\mathbf{x}(\tau)) \cdot \boldsymbol{\eta}_\tau^i(\tau), \tag{7}$$

respectively. Incidentally, *mixing*, which is a property most often observed in chaotic systems, is not invariant under transformation (5) since it has been shown that an adequate time reparametrization of a nonchaotic irrational flow on a 3-torus is mixing [14].

For the same initial conditions, the defining relations of h_τ^i and h_t^i present two different factors. The first, associated with the time average of λ, comes from the difference between the two time parametrizations along the same orbit and is factored out when Eq. (6) is written as

$$h_\tau^i = \frac{1}{\langle\lambda\rangle_t} \lim_{t\to\infty} \frac{1}{t} \ln\frac{|\boldsymbol{\eta}_\tau^i(t)|}{|\boldsymbol{\eta}_{\tau0}^i|}, \tag{8}$$

where $\boldsymbol{\eta}_\tau^i(t) \equiv \boldsymbol{\eta}_\tau^i(\tau(t))$, $\tau(t) \equiv \int_0^t \lambda(\mathbf{x}(t))\, dt$, and $\langle\lambda\rangle_t \equiv \lim_{t\to\infty} \frac{1}{t} \int_0^t \lambda(\mathbf{x}(t))\, dt$. Condition $0 < \langle\lambda\rangle_t < \infty$ is a basic requirement for the natural measure to be well defined [17]. The second factor, due to the gradient of λ, is associated with the difference between the surfaces of simultaneous time for each parametrization and is separated out when $\boldsymbol{\eta}_\tau^i$ in Eq. (7) is parametrized in terms of t rather than τ:

$$\frac{d\boldsymbol{\eta}_\tau^i(t)}{dt} = \mathbf{DF}(\mathbf{x}(t)) \cdot \boldsymbol{\eta}_\tau^i(t) - [\mathbf{F} \cdot \nabla^\dagger \ln\lambda](\mathbf{x}(t)) \cdot \boldsymbol{\eta}_\tau^i(t). \tag{9}$$

The last term in this equation implies that the time evolution of vector $\boldsymbol{\eta}_\tau^i(t)$ in Eq. (8) is in general different from that of vector $\boldsymbol{\eta}_t^i(t)$ in Eq. (3). But the relevant question is: How large is this difference?

Here we show that the difference is subexponential, in the sense that $\boldsymbol{\eta}_t^i(t)$ and $\boldsymbol{\eta}_\tau^i(t)$ grow or shrink with the same exponential rate. This implies

$$\lim_{t\to\infty} \frac{1}{t} \ln\frac{|\boldsymbol{\eta}_\tau^i(t)|}{|\boldsymbol{\eta}_{\tau0}^i|} = \lim_{t\to\infty} \frac{1}{t} \ln\frac{|\boldsymbol{\eta}_t^i(t)|}{|\boldsymbol{\eta}_{t0}^i|}, \tag{10}$$

which in turn implies our main result (2).

First we analyze periodic orbits, which form the most fundamental building blocks of chaotic sets [20]. On a periodic orbit $\mathbf{x}^*$, it is convenient to adopt the explicit notation $\boldsymbol{\eta}_t^i(\mathbf{x}^*(t), t)$ and $\boldsymbol{\eta}_\tau^i(\mathbf{x}^*(t), t)$ for the solutions of the variational Eqs. (4) and (9), respectively. If $\mathbf{x}^*$ is a fixed point, we trivially have $\boldsymbol{\eta}_\tau^i(\mathbf{x}^*(t), t) = \boldsymbol{\eta}_t^i(\mathbf{x}^*(t), t)$ because the last term in Eq. (9) is zero. Now consider that $\mathbf{x}^*$ is a periodic orbit with least period $T > 0$ with respect to t, so that $\mathbf{x}^*(t+T) = \mathbf{x}^*(t)$. Let $h_t^i(\mathbf{x}^*)$ denote the *local* Lyapunov exponents for Eq. (3) on $\mathbf{x}^*$, and $\sigma_t^i(\mathbf{x}^*) \equiv \exp[h_t^i(\mathbf{x}^*)]$ denote the corresponding local Lyapunov numbers, where $i = 1, \ldots, N$. One of the Lyapunov numbers, say σ_t^N, is 1 because the bounded function $d\mathbf{x}^*/dt$ is a solution of Eq. (4), rendering zero to the corresponding Lyapunov exponent. The same is true for any parametrization. To study the other Lyapunov numbers, let π be the hyperplane orthogonal to $\mathbf{F}(\mathbf{x}^*(0))$ at $\mathbf{x}^*(0)$, and $\mathbf{M}: U \subset \pi \mapsto \pi$ be the first return map on this hyperplane, defined in a neighborhood U of $\mathbf{x}^*(0)$. This map does not depend on the time parametrization of the continuous flow, being exactly the same for both the original and reparametrized flow. It follows from standard results in Floquet theory [21] that the local Lyapunov numbers $\sigma_M^i(\mathbf{x}^*(0))$ of this map, defined as the magnitude of the eigenvalues of the Jacobian matrix of $\mathbf{M}$ at $\mathbf{x}^*(0)$, are the power T of the first $N-1$ local Lyapunov numbers of the flow, i.e., $\sigma_M^i(\mathbf{x}^*(0)) = \sigma_t^i(\mathbf{x}^*)^T$ for $i = 1, \ldots, N-1$. Geometrically, the local Lyapunov numbers of $\mathbf{M}$ can be defined as $\sigma_M^i(\mathbf{x}^*(0)) = \exp[h_M^i(\mathbf{x}^*)T]$, where $h_M^i(\mathbf{x}^*)$ is defined through Eq. (3) with $\boldsymbol{\eta}_t^i(\mathbf{x}^*(t), t)$ replaced with its orthogonal component to $\mathbf{F}(\mathbf{x}^*(t))$. From the identity $h_M^i(\mathbf{x}^*) = h_t^i(\mathbf{x}^*)$ then follows $h_M^i(\mathbf{x}^*) = \lim_{t\to\infty} \frac{1}{t} \ln[|\boldsymbol{\eta}_t^i(\mathbf{x}^*(t), t)|/|\boldsymbol{\eta}_t^i(\mathbf{x}^*(0), 0)|]$. The same is true for $\boldsymbol{\eta}_\tau^i(\mathbf{x}^*(t), t)$, so that, if $\tilde{h}_t^i$ denotes the limit in Eq. (8), then $h_M^i(\mathbf{x}^*) = \tilde{h}_t^i(\mathbf{x}^*)$ and $h_M^i(\mathbf{x}^*) = \lim_{t\to\infty} \frac{1}{t} \ln[|\boldsymbol{\eta}_\tau^i(\mathbf{x}^*(t), t)|/|\boldsymbol{\eta}_\tau^i(\mathbf{x}^*(0), 0)|]$. From these, it follows that Eq. (10) holds on periodic orbits.

But the same must be valid in general because the Lyapunov exponents of typical orbits are weighted averages over the local Lyapunov exponents of all periodic orbits in the respective ergodic component. The weight is the fraction of time spent by a typical orbit near the corresponding periodic orbit and is uniquely determined by the largest local Lyapunov exponent (assumed to be positive). The smaller this Lyapunov exponent, the longer it takes for the orbit to move away from that neighborhood. In other words, the largest local Lyapunov exponents define a natural measure, over which all the Lyapunov exponents are computed. Since $\tilde{h}_t^i(\mathbf{x}^*) = h_t^i(\mathbf{x}^*)$, both the measure and the local exponents are the same for h_t^i and $\tilde{h}_t^i$. Therefore $\tilde{h}_t^i = h_t^i$ on typical orbits, and this is equivalent to Eq. (10).

We now show that similar arguments can be extended directly to typical orbits. Equation (4) can be written as a map $M[\boldsymbol{\eta}_t^i(t)] = \boldsymbol{\eta}_t^i(t+\delta t)$, where $\boldsymbol{\eta}_t^i(t+\delta t) = \boldsymbol{\eta}_t^i(t) + \delta t \mathbf{DF}(\mathbf{x}(t)) \cdot \boldsymbol{\eta}_t^i(t)$. For the natural measure to be normalizable, $\mathbf{F}(\mathbf{x}(t))$ and $\nabla|\mathbf{F}(\mathbf{x}(t))|$ must not grow exponentially and $\mathbf{F}(\mathbf{x}(t))$ must not shrink exponentially along typical orbits. Since $d\mathbf{x}/dt$ is a solution of Eq. (4), the component $P(\boldsymbol{\eta}_t^i)$ of $\boldsymbol{\eta}_t^i$ parallel to the flow remains parallel in each iteration, and the Lyapunov exponent in this direction is zero. The orthogonal component $Q(\boldsymbol{\eta}_t^i)$ of $\boldsymbol{\eta}_t^i$ is mapped into two parts, one parallel and the other orthogonal to the flow. The parallel part is $PMQ[\boldsymbol{\eta}_t^i(t)] = \delta t[\mathbf{F} \cdot \nabla^\dagger \ln|\mathbf{F}|](\mathbf{x}(t)) \cdot Q[\boldsymbol{\eta}_t^i(t)]$, where neither $\mathbf{F}(\mathbf{x}(t))$ nor $\nabla \ln|\mathbf{F}(\mathbf{x}(t))|$ grows exponentially with t. If $h_t^i < 0$ ($h_t^i > 0$), each of the projections $P(\boldsymbol{\eta}_t^i)$ and $Q(\boldsymbol{\eta}_t^i)$ must shrink at least (grow at most) as $\exp(h_t^i t)$. But $Q(\boldsymbol{\eta}_t^i)$ cannot shrink faster (grow slower) than $P(\boldsymbol{\eta}_t^i)$ because otherwise the contribution of the orthogonal part to Eq. (3) would be negligible, resulting in $h_t^i = 0$, which violates the hypothesis that $h_t^i \neq 0$. Therefore, $Q(\boldsymbol{\eta}_t^i) \sim \exp(h_t^i t)$ for both $h_t^i < 0$ and $h_t^i > 0$. A similar relation is valid for $\boldsymbol{\eta}_\tau^i(t)$ because $\lambda^{-1} d\mathbf{x}/dt$ is a solution of Eq. (9). We then compare the solutions of Eq. (9) with those of Eq. (4) for identical initial conditions. The term involving matrix $-[\mathbf{F} \cdot \nabla^\dagger \ln\lambda](\mathbf{x}(t))$ is parallel to the flow and does not affect the orthogonal part in Eq. (9). The term that contributes to the orthogonal component, $\mathbf{DF}(\mathbf{x}(t))$, is the same as that in Eq. (4). Therefore, $Q[\boldsymbol{\eta}_t^i(t)] = Q[\boldsymbol{\eta}_\tau^i(t)]$, which again leads to Eq. (10).

An important implication of our findings is that positive Lyapunov exponents are necessarily mapped into positive Lyapunov exponents under time reparametrizations. This implies that the previous examples of noninvariant chaos in cosmology are based on the violation of hypotheses required for an interpretable computation of the Lyapunov exponents. Consider, for example, the mixmaster cosmological model [6], which is believed to describe generic cosmological singularities, and whose relevant hypotheses can be discussed explicitly. In the asymptotic limit (close to the big bang), the essential features of the continuous dynamics are represented in the Farey map, $F(u) = u - 1$ if $u \geq 1$ and $F(u) = u^{-1} - 1$ if $u < 1$, whose Lyapunov exponent is zero. This result is claimed to be in conflict with the corresponding result for the first return map on $[0, 1]$ (Gauss map), $G(v) = 1/v - [1/v]$, where $[1/v]$ is the integer part of $1/v$, whose Lyapunov exponent is positive [2]: $h = \pi^2/6 \ln 2$. The problem here is that, different from the first return map, the orbits of map F are typically unbounded. Map F can be compactified for $w = (u+1)^{-1}$, by defining $H:[0,1] \to [0,1]$, $H(w) = w/(1-w)$ if $w \leq 1/2$, and $H(w) = (1-w)/w$ if $w > 1/2$. The Lyapunov exponent is still zero. The problem now is that the invariant density, $\rho(w) = 1/w$, is not normalizable and all contributions to the Lyapunov exponent come from points that are arbitrarily close to $w = 0$. Similar problems are present in the continuous dynamics since in the usual coordinates the model is either nonautonomous or noncompact [6]. In addition, because the dynamics is limited by a cosmological singularity, the domain of frequently used time parameters, such as the cosmological time and the volume of the universe, is necessarily finite.

From the above, it appears that autonomous equations, bounded motions, and normalizable measure are properties mutually incompatible in the mixmaster dynamics when all the orbits are taken into account. We observe, however, that there are invariant bounded subsets of orbits in map F as well as invariant subsets with normalizable measure in map H which do have positive Lyapunov exponents. For example, map H has a nontrivial set of invariant orbits embedded in the interval $[\alpha, (1+\alpha)^{-1}]$, for every $\alpha \in (0, 1/3]$, which has normalizable measure. The invariant set is composed of all the orbits that never leave this interval and as such contains a countable number of periodic orbits and an uncountable number of nonperiodic orbits. The Lyapunov exponent of the invariant set, as computed along typical nonperiodic orbits, satisfies $h \geq 2\ln(1+\alpha)$. Since map H corresponds to the asymptotic behavior of one parametrization of the continuous mixmaster model, an invariant set of map H must correspond to an invariant set of the asymptotic dynamics for this particular parametrization, which therefore satisfies conditions (i)–(iv) and has a positive Lyapunov exponent. It follows then from our main result (2) that these asymptotically invariant sets must have positive Lyapunov exponents for any space-time reparametrization of the continuous dynamics that preserves these conditions on them. These invariant sets are therefore chaotic with respect to any coordinate system for which the Lyapunov exponents can be properly computed. In this sense, one can meaningfully say that the mixmaster cosmology exhibits coordinate independent chaotic behavior close to the big bang.

In summary, we have uncovered the transformation laws of the Lyapunov exponents for flows under space-time reparametrizations. Strikingly, systems exhibiting exponential separation of nearby orbits with respect to one choice of the time parameter will display exponential divergence with respect to any other time parameter that preserves conditions (i)–(iv). This implies that chaos is invariant under time reparametrizations, which is in sharp contrast with previous results in relativistic cosmology, where the apparent noninvariance of chaos has been the subject of an intensive debate. Our findings thus shed new light on the *conceptual* problem of chaos in cosmology [22].

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