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ON THE EFFECTIVE SPACE OF CIRCULAR COUETTE FLOW AND THE STRUCTURE OF ITS ATTRACTORS

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Abstract The experimental data from three space points are analyzed in terms of the qualitative theory of differential equations. Attractors were found which, within the experimental resolution, could be represented as a direct product of a "fast" one-dimensional torus corresponding to azimuthal waves and a "slow" two-dimensional torus corresponding to small interactions between them. The slow tori observed have rational values (13/31 and 3/7) of the rotation number.

A method for experimentally determining the coefficients in the phenomenological equation describing the interaction of azimuthal waves is proposed. The influence of instrumental noise on the uncertainty in the determination is estimated.

INTRODUCTION

It is now accepted that the simple Landau-Hopf model of the onset of turbulence does not correspond to reality in detail and that the concept of the stochastic attractor in the phase space of an appropriate dynamic system qualitative insight into the nature of

stochasticity in a hydrodynamic flow. Some experimental evidence for this hypothesis was found on the basis of a spectral analysis of one-point measurements in circular Couette flow with a rotating inner cylinder.[2,4]
In this paper, we report on the next two steps in the study of the nature of stochasticity in this flow. In the first step, using the multichannel ion-current technique and careful long-time measurements (for details, see reference 1), we analyzed the topology of those sets which attract the phase trajectories of hydrodynamic flow and found attractors that look like windings a round a three-dimensional torus. In the second, we propose a new method to analyze the complete experimental time series, allowing us to establish the correspondence between a given flow and a dynamic model (chosen a priori from a class of ordinary differential equations) and to determine the coefficients and their dependences on Reynolds number.

OBSERVATIONS OF RESONANT TORI

We used a cell with inner diameter D1 = 35 mm, outer diameter D2 = 55 mm, and height H = 282 mm between fixed end plates. The number $Re = \omega_m d_1 \times (d_2 - d_1)/\nu$, where ω_m is the angular speed of the inner cylinder and ν is the viscosity, was maintained within ±0.015% during the few hours required for the flow to become steady state and the data to be acquired.

The following transitions were observed. (1) At Re =75, the Taylor vortices fills the annular space from one end of the cylinder to the ofher. This process is not a difurcation in the strict sense.[3,5] The primary flow contains 14 pairs of vortices. (2) At Re_c =1030, the first difurcation occurs; it developes travelling azimuthal waves that produce fast oscillations (f_1 = 1.6 Hz) of hydrodynamic variables in a fixed system of reference. The magnitude of waves grows gradually in accordance with the Landau law. In the phase space of our flow, this process is described by a Hopf bifurcation from a fixed point to a limit cycle, i.e., to a one-dimensional torus T_1.

To study the following bifurcations, we obtained the envelopes of main frequency of signals from the three neighboring pairs of Taylor vortices. (The ion current probes were placed at 90,110, and 130 mm from the upper end plate.) This procedure corresponds to averaging Poincare maps along the fast limit cycle and gives "slow" variables. This technique shows that, at Re = 1051, these variables begin to oscillate with frequency f_2 = 0.009 Hz, so the next difurcation is the appearance of a new limit cycle in slow variables. Taking the fast motion into account, we can say that the observed attractor is a two-dimensional torus with f = (1.6, 0.009) Hz. The ratio f_1/f_2 is about 200, so, in this case, the question of the existence of phase locking requires a much more precise apparatus.

This attractor is destroyed at $Re = 1058$. During a time interval of the order of 1000 s, each signal, taken separately, seens to be chaotic. But the composition of any two signals plotted on a graphic display during more that 10000 s develops singularities that are typical of torus projection onto a plane. In particular, with a proper choice of projective plane P, this attractor has a hole. The points of intersection of the experimental trajectories with the plane $P_1 \perp P$ are shown in figure 1,

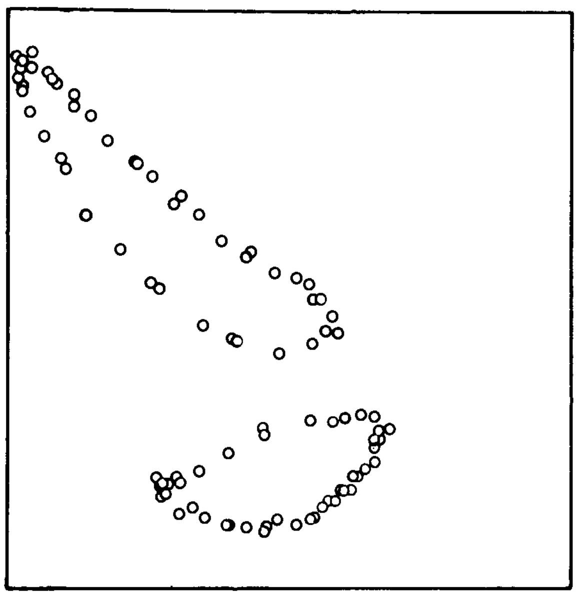

FIGURE 1. The cross-section of a slow torus.

which shows that the motion in slow variables occupies a surface of a two-dimensional torus. Consequently, with fast motion taken into account, the flow studied here has an attractor on a three-dimensional torus. If we construct the period-one angle map $\theta(n+1) = F(\theta(n))$ (figure 2), it becomes clear that the mapping

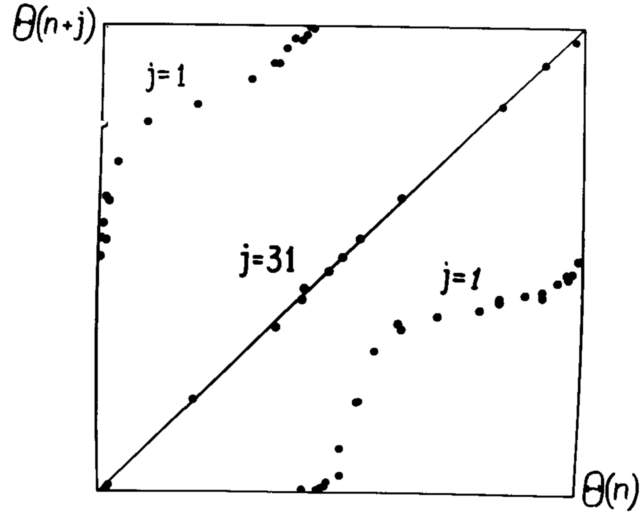

FIGURE 2. Plots of θ_{n+j} versus θ_n with $j = 1$ and $j = 31$.

of the torus meridian into itself is a single-valued invertible function. Any integer power of this function exhibits the same character, but there is a surprising feature if the power is equal to 31: all the experimental points occupy a diagonal of a square. This means that the phase trajectory is locked after 31 circuits requires a much more precise apparatus.

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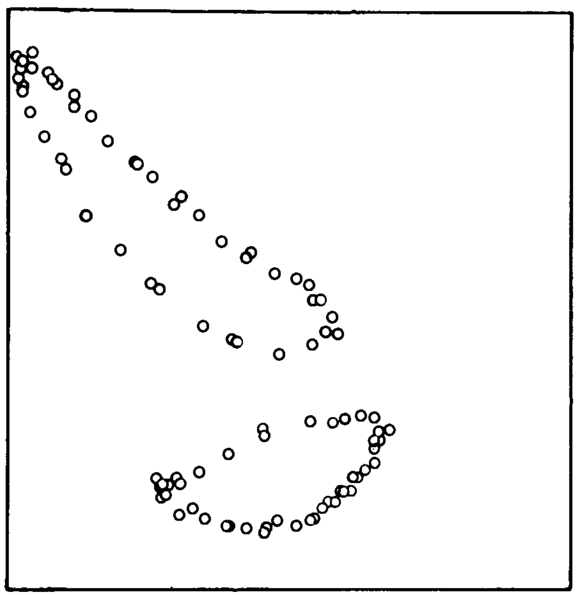

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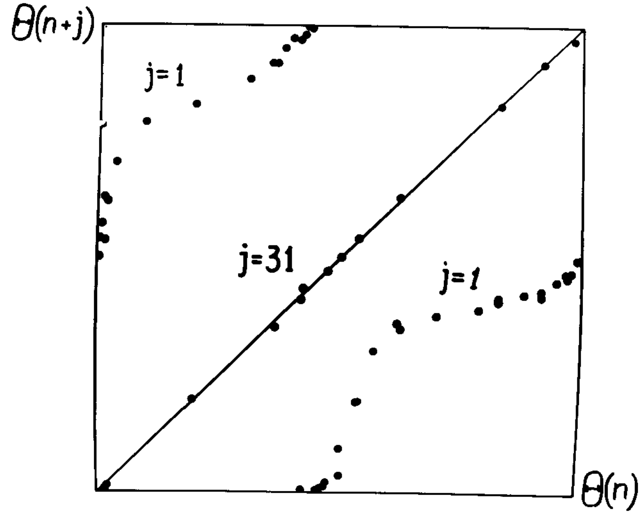

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$\Lambda_{ik} = \overline{(c_i - \bar{c}_i)(c_k^* - \bar{c}_k^*)}$ must be equal to zero. The diagonalization of Λ using an unitary matrix U gives the direction cosines of a hyperplane, which allow us to calculate the coefficients of the governing equation (2). In real situations, it can merely be assumed that the attractor will occupy a flat hyperellipsoid and that one of its dimensions will be comparatively small. In this connection, the question arises, Does instrumental noise have anyinfluence on the uniqueness of the solution of the inverse problem? To answer this question and to study the efficiency of the proposed method, we have made a series of numerical simulations. Equations (2) were solved by the use of a computer with various values of the parameters. An additional noise was then superimposed on the solutions obtained and the algorithm described above was applied to restore the coefficients.

We found that the uncertainty of the determination of the coefficients can be better than 10^{-2} if there is no noise and the time interval contains not fewer than 1000 characteristic periods of motion. For example, at $B = 0.8$, and $\Gamma = \gamma/a = 0.05$ (i.e., in the region of a two-dimensional torus), the vector of eigenvalues λ_i was equal to (3.03, 0.093, 0.03, 4×10^{-7}) and the determined coefficients were $\theta^* = 9.996$ and $B^* = 0.798$. The next interval of the same duration gives $\lambda_i = (3.87,$ $0.09, 0.03, 10^{-7})$ and $\theta^* = 10.14$ and

$B^* = 0.795$. It is seen that the statistics obtained are not rich enough to give a representative estimation of the correlation matrix, but that the coefficients θ and B were determined quite well. This can be explained by the circumstance that the thickness of the hyperellipsoid $\lambda_4 = 10^{-7}$ is much less than its transverse size $\lambda_3 = 3\times10^{-3}$.

As γ increases motion on the attractor becomes more complicated and less correlated, the transverse size grows, and the uncertainty diminishes. Thus, at $\theta = 10$, $B = 0.8$, and $\Gamma = 0.15$ in the regime of the developed chaotic attractor, $\lambda_i = (2.11, 1.11, 0.78, 10^{-4})$ and $\theta^* = 9.975$ and $B^* = 0.7995$.

Let a small Gaussian noise now be added,

$$\tilde{A}_n = A_n(t) + f_n(t), \quad \langle f_n f_{n'}^* \rangle = f^2(\tau)\delta_{nn'}$$

$$f^2(0) = \varepsilon \langle |A_n|^2 \rangle, \quad \varepsilon \ll 1$$

a noisy matrix Λ'_{ik} will be added to the correlation matrix Λ_{ik}. In general, Λ' is not diagonalized by the same transformation as Λ. It is clear that information about the diagonalization of the matrix in question completely disappears if ε is of the same order as the minimum transverse size of the attractor. Therefore, the relative uncertainty can be estimated by the formula $\Delta\theta/\theta \approx \Delta B/B \sim \varepsilon/\lambda_3$ which has been confirmed by numerical simulation. It is essential that the increase in Re lead to an increase in the transverse size and a decrease in the uncertainty.

The application of this method to laboratory experiment faces an additional difficulty. This difficulty is conditioned by the circumstance that the measured quantities are, in general, not the same as shose involved in the geverning equation (2); thus, to closure the equations, it is necessary to extend the number of measured variables. We hope that this difficulty will be circumvented.

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