BRIEF COMMUNICATIONS

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On the role of linear mechanisms in transition to turbulence

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(Received 18 May 1993; accepted 2 November 1993)

Recent work has shown that linear mechanisms can lead to substantial transient growth in the energy of small disturbances in incompressible flows even when the Reynolds number is below the critical value predicted by linear stability (eigenvalue) analysis. In this note it is shown that linear growth mechanisms are necessary for transition in flows governed by the incompressible Navier–Stokes equations and that non-normality of the linearized Navier–Stokes operator is a necessary condition for subcritical transition.

A first step in studying the stability of a flow is to investigate the evolution of small disturbances by linearizing the Navier–Stokes (NS) equations.[1] Stability is then determined by examining the eigenvalues of the linearized problem. The flow is said to be *linearly stable* if it does not have any solutions that grow exponentially as time increases.

The results of linear stability analysis are mixed. For example, for the Bénard convection problem, a stationary flow between flat plates held at different temperatures, linear stability analysis agrees with experiment. On the other hand, for plane Poiseuille and Couette flows, transition to turbulence can occur at subcritical Reynolds numbers (below the critical value predicted by linear stability analysis).[2–5] This discrepancy has led to much work on nonlinear analysis, with varying degrees of success at predicting transition.[6–8] However, transition to turbulence is still not completely understood.

In recent years, there has been a reexamination of the linearized NS equations. It has been shown that there can be growth in the energy of small disturbances in incompressible shear flows, even when the flow is linearly stable. For inviscid channel flows, algebraic growth of the disturbance energy of the form $\mathscr{E}=O(t)$ as $t\rightarrow\infty$ is possible.[9,10] Physically, the growth is due to the lift-up mechanism,[11] where disturbances in the form of streamwise vortices move fluid from regions of high velocity to regions of low velocity, creating streamwise streaks that grow in length with time. For viscous flows at subcritical Reynolds numbers the disturbance energy of solutions to the linearized NS equations decays to 0 as $t\rightarrow\infty$ because of dissipative effects, but there can be substantial energy growth before the decay.[12–19] For example, at Reynolds number 4000 growth in the energy by a factor of $\approx 18\,000$ is possible for plane Couette flow. The potential for transient growth was recognized by Orr at the beginning of the century.[20] Nonlinear simulations show that the linear growth mechanisms play a fundamental role in transition to turbulence.[21,22]

The potential for transient behavior exists if the governing linear operator is *non-normal.* A normal matrix or operator, $\mathscr{L}$, satisfies $\mathscr{L}^+\mathscr{L}=\mathscr{L}\mathscr{L}^+$, where $\mathscr{L}^+$ is the adjoint.[23,24] In many applications, normal operators whose spectra are identical behave similarly. For a non-normal operator, information in addition to the spectrum may be needed to determine the behavior. For example, one may examine the resolvent, or equivalently the pseudospectra.[23,25]

The purpose of this Brief Communication is to make the following statements about the role played by linear mechanisms in transition to turbulence:

(i) Linear mechanisms are necessary for transition to turbulence.

(ii) Non-normality of the linearized Navier–Stokes operator is a necessary condition for subcritical transition.

We verify these statements for steady laminar flows that satisfy the incompressible NS equations in bounded or periodic domains.

A connection between linear and nonlinear stability for flows governed by the NS equations has been established previously.[26] If a flow is linearly stable then it is nonlinearly stable for all sufficiently small initial disturbances. On the other hand, if a flow is linearly unstable, then it is nonlinearly unstable to arbitrarily small initial disturbances.

The fact that nonlinear mechanisms are crucial for transition does not contradict statement (i). By (i), we mean that the linearized NS equations must have either exponentially growing solutions, algebraically growing solutions, or transiently growing solutions for transition to occur. Statement (ii) gives a necessary condition for subcritical transition. However, non-normality of the linearized NS operator is not sufficient for subcritical transition.

 1070-6631/94/6(3)/1396/3/$6.00

Statement (ii) is similar to a result proved by Galdi and Straughan.[27]

We begin with the notation. Let $\Omega \in \mathbf{R}^3$ be a closed domain with boundary $\partial\Omega$. If Ω is unbounded, then we assume that it is periodic in the directions that it is unbounded. We assume that $\partial\Omega$ is sufficiently smooth so that we will not deal with regularity issues. Let $\mathbf{x}=(x_1, x_2, x_3)$ denote a point in $\mathbf{R}^3$, $\mathbf{u}(\mathbf{x},t)=(u_1, u_2, u_3)$ the velocity, and $p(\mathbf{x},t)$ the pressure. The nondimensionalized NS equations for an incompressible flow are

$$\frac{\partial \mathbf{u}}{\partial t}=-\mathbf{u}\cdot\nabla\mathbf{u}-\nabla p+\frac{1}{R}\Delta\mathbf{u}+\mathbf{f}, \tag{1}$$

with $\nabla\cdot\mathbf{u}=0$, $\mathbf{u}(t=0)=\mathbf{u}_0$, and $\mathbf{u}=\mathbf{u}_s(\mathbf{x},t)$ for $x\in\partial\Omega$. Here $\mathbf{f}$ is the body force per unit mass, R is the Reynolds number, and $\mathbf{u}_s(\mathbf{x},t)$ is the velocity of the surface $\partial\Omega$. We assume that the motion of the surface is such that Ω remains fixed.

Assume that (1) admits the time-independent solution $(\mathbf{U}, P)$. To study the stability of $(\mathbf{U}, P)$ let $\mathbf{u}=\mathbf{U}+\mathbf{v}$ and $p=P+q$ and substitute these expressions into (1). We obtain

$$\frac{\partial \mathbf{v}}{\partial t}=-\mathbf{v}\cdot\nabla\mathbf{v}-\mathbf{U}\cdot\nabla\mathbf{v}-\mathbf{v}\cdot\nabla\mathbf{U}-\nabla q+\frac{1}{R}\Delta\mathbf{v}, \tag{2}$$

with $\nabla\cdot\mathbf{v}=0$, $\mathbf{v}(\mathbf{x},0)=\mathbf{v}_0$, and homogeneous boundary conditions. Equation (2) is valid for arbitrary amplitude perturbations.

The linearized NS equations are obtained by dropping the term $\mathbf{v}\cdot\nabla\mathbf{v}$ in (2). We have

$$\frac{\partial \mathbf{w}}{\partial t}=-\mathbf{U}\cdot\nabla\mathbf{w}-\mathbf{w}\cdot\nabla\mathbf{U}-\nabla r+\frac{1}{R}\Delta\mathbf{w}, \tag{3}$$

with $\nabla\cdot\mathbf{w}=0$, $\mathbf{w}(t=0)=\mathbf{w}_0$, and homogeneous boundary conditions.

Rewriting (3) in operator notation, we have

$$\frac{\partial \mathbf{w}}{\partial t}=\mathscr{L}\mathbf{w}, \quad \mathbf{w}(t=0)=\mathbf{w}_0, \tag{4}$$

where

$$\mathscr{L}\mathbf{w}=\Pi[-\mathbf{U}\cdot\nabla\mathbf{w}-\mathbf{w}\cdot\nabla\mathbf{U}+(1/R)\Delta\mathbf{w}]. \tag{5}$$

The operator Π projects onto the space of divergence free functions which are parallel to the boundary $\partial\Omega$.[28,29] We can choose the underlying Hilbert space, $\mathscr{H}$, to be the space of divergence-free functions in $L^2(\Omega)$ satisfying homogeneous (or periodic) boundary conditions on $\partial\Omega$ with inner product

$$(\mathbf{u},\mathbf{v})=\int \mathbf{u}\cdot\mathbf{v}\,d\mathbf{x}, \tag{6}$$

where the region of integration is Ω (or one period).

We define the disturbance energy of $\mathbf{v}$ to be $\mathscr{E}=\|\mathbf{v}\|^2/2=(\mathbf{v},\mathbf{v})/2$. The flow is said to be unstable if there is an initial disturbance $\mathbf{v}_0$ such that $\mathscr{E}(t)$ does not decay to 0 as $t\to\infty$. Similarly, let us define $\mathscr{F}=(\mathbf{w},\mathbf{w})/2$ as the energy for the linear problem.

We first consider the linear problem. Energy growth is said to occur if for some $t>0$, $\mathscr{F}(t)>\mathscr{F}(0)$. Taking the dot product of (3) with $\mathbf{w}$ and integrating, we obtain the Reynolds–Orr equation, the starting point for the energy method:[30]

$$\frac{d\mathscr{F}}{dT}=-\int \mathbf{w}\cdot D\mathbf{w}\,dx-\frac{1}{R}\int(\nabla\mathbf{w}):(\nabla\mathbf{w})d\mathbf{x}. \tag{7}$$

Here we have $(D\mathbf{w})_i=(U_{i,j}+U_{j,i})w_j/2$ and $(\nabla\mathbf{w}):(\nabla\mathbf{w})=w_{i,j}w_{i,j}$, where $w_{i,j}=\partial w_i/\partial x_j$ and the summation convention is used. The first term in (7) measures the exchange of energy between the disturbance $\mathbf{w}$ and the mean flow $\mathbf{U}$. The second term is related to energy dissipation and is always negative.

If $dF/dt<0$ for all $\mathbf{w}\in\mathscr{H}$, then the disturbance energy $\mathscr{F}(t)$ cannot grow. Letting R_g' be the largest Reynolds number at which this occurs, it can be shown that

$$\frac{1}{R_g'}=\sup_{\mathbf{w}}\frac{-\int\mathbf{w}\cdot D\mathbf{w}\,d\mathbf{x}}{\int(\nabla\mathbf{w}):(\nabla\mathbf{w})d\mathbf{x}}, \tag{8}$$

where the supremum is taken over $\mathbf{w}\in\mathscr{H}$. For flows in bounded domains, Serrin proved that R_g' is positive.[31]

If $R<R_g'$, then $\mathscr{F}(t)$ decays monotonically to 0. If $R>R_g'$, then $\mathscr{F}(t)$ may grow for certain initial velocity fields. In general, $R_g'\leqslant R_c$, where R_c is the critical Reynolds number for the existence of exponentially growing solutions for the linearized NS equations.

The evolution equation for $\mathscr{E}$ can be derived in the same manner as that for $\mathscr{F}$. We obtain

$$\frac{d\mathscr{E}}{dt}=-\int \mathbf{v}\cdot D\mathbf{v}\,d\mathbf{x}-\frac{1}{R}\int(\nabla\mathbf{v}):(\nabla\mathbf{v})d\mathbf{x}, \tag{9}$$

using the fact that $\int\mathbf{v}\cdot(\mathbf{v}\cdot\nabla)\mathbf{v}\,d\mathbf{x}=0$ (by the divergence theorem).

Let R_g denote the largest Reynolds number such that $d\mathscr{E}/dt\leqslant 0$ for all $\mathbf{v}_0\in\mathscr{H}$. Since the expressions for $d\mathscr{E}/dt$ and $d\mathscr{F}/dt$ are the same, it follows that the expression for $1/R_g$ is the same as that for $1/R_g'$. The supremum is again taken over $\mathscr{H}$.[32] Hence, it follows that $R_g=R_g'$.

Transition cannot occur for $R<R_g$, and there is no growth in the disturbance energy for the linear problem unless $R\geqslant R_g$. This means that transition to turbulence cannot occur unless there is the potential for disturbance energy growth for the linear problem (3), establishing (i).

A necessary and sufficient condition for growth for the linear problem (3) can be given in terms of the resolvent. The spectrum of $\mathscr{L}$ is essentially the set of points $z\in\mathbf{C}$ where $\|(zI-\mathscr{L})^{-1}\|$ is infinite. Roughly speaking, however, operator behavior depends on the region where $\|(zI-\mathscr{L})^{-1}\|$ is "large."[25]

For a normal operator $\mathscr{A}$, the resolvent satisfies[23]

$$\|(zI-\mathscr{A})^{-1}\|=[1/\mathrm{dist}(z,\Lambda)]\quad\forall z\notin\Lambda. \tag{10}$$

Here Λ denotes the spectrum of $\mathscr{A}$ and $\mathrm{dist}(z,\Lambda)$ is the distance of z to the spectrum. For a non-normal operator, the equality in (10) is replaced by $\geqslant$, and $\|(zI-\mathscr{A})^{-1}\|$ may be large even if z is relatively far from the spectrum.

The Hille–Yosida theorem[33] states that $\mathscr{F}(t)\leqslant e^{\omega t}\mathscr{F}(0)$ for some $\omega\in\mathbf{R}$ for all initial disturbances $\mathbf{w}_0\in\mathscr{H}$ if and only if

$$\|(zI-\mathscr{L})^{-1}\|\leqslant[1/(\mathrm{Re}\,z-\omega)]\ \forall\ \mathrm{Re}\,z>\omega. \tag{11}$$

TABLE I. Critical parameter values for stability predicted by energy methods and linear stability analysis. For the Bénard problem the parameter is a Rayleigh number. For the other flows, the parameter is a Reynolds number. The final column indicates if the linearized NS operator for the flow is normal (N) or non-normal (NN).

Flow	R_g	R_t	R_c	$\mathscr{L}$
Hagen–Poiseuille	81.5	≈ 2000	∞	NN
Plane Poiseuille	49.6	≈ 1000	5772	NN
Plane Couette	20.7	≈ 360	∞	NN
Bénard	1708	≈ 1700	1708	N
Rigid rotation	∞		∞	NN

Suppose that $\mathscr{L}$ is normal for all R and that $R<R_c$. Then Λ lies in the open left half-plane, and (11) will hold for some $\omega<0$ by (10), where $\omega > \mathrm{Re}(\lambda_0)$ and λ_0 is the least stable eigenmode. Hence there is no energy growth. The results in the last section imply that the flow is nonlinearly stable, establishing (ii).

We have shown that there is no energy growth if $R<R_c$ for the linear problem if the operator $\mathscr{L}$ is normal. Thus statement (i) implies $R_g=R_c$ if $\mathscr{L}$ is normal.

Table I presents results on the transitional Reynolds numbers, R_t, found in experiments and the values R_g and R_c for various flows.[1,30] Plane Poiseuille, plane Couette flow, and Hagen–Poiseuille flow are well-known examples where R_g and R_c differ, and the linearized NS operators for these flows can be shown to be non-normal. The Bénard problem is a canonical example for which the linearized operator is normal. Although the NS equation must be modified, our main results apply. In general the linearized NS operators for flows between two infinite concentric rotating cylinders are non-normal, and R_c and R_g differ. Rigidly rotating flow, flow between cylinders rotating with the same angular frequency, is nonlinearly stable for all Reynolds numbers, since the energy exchange term in (7) is 0. Here $R_g=R_c$ even though the linearized NS operator is non-normal.

The value of R_g depends on the choice of the norm. The fact that R_g and R_c differ greatly in most cases for the energy norm has led to continuing work on energy methods. For some flows a new norm can be chosen so that $R_g=R_c$.[30,34]

ACKNOWLEDGMENTS

We thank Gunilla Kreiss, Jian-Guo Liu, Enrique Thomann, Nick Trefethen, and the referee for comments on the paper.

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