Double-group theory on the half-shell and the two-level system. II. Optical polarization

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Relations are derived between several different descriptions of optical polarization by analogy to the theory of spin 1/2. The rotational slide rule developed in the preceding article (I) is used to (a) compute the final polarization state and phase of an optical beam given the optical wave matrix, initial state, and phase, (b) make conversions between various types of polarization parameters, and (c) find the output intensity for perfect polarizers. Other polarization problems and methods are discussed briefly.

I. INTRODUCTION

After the spin-$\frac{1}{2}$ states of the electron, the most well-known two-state system involves the two spin or polarization states of light, or the photon. Ever since the invention of cheap and handy polarizers, and certainly now with the discovery of lasers, liquid crystals, and so forth, many interesting experiments involving polarization have been created.[1] Indeed, most of the quantum-mechanical thought experiments can be realized using easily constructed optical polarization apparatuses.[2,3]

While the fundamental mathematical descriptions of spin-$\frac{1}{2}$ and optical polarization are practically the same, their physical interpretations are quite different. However, they compliment each other very nicely as we will show by comparing various developments in this article to those involving spin-$\frac{1}{2}$ in the preceding article I.

In the description of spin-$\frac{1}{2}$ we have a spin vector whose direction in ordinary three-dimensional Euclidean (xyz) space depends upon a state vector in a complex two-dimensional (spin-up, spin-down) spinor space. Since we live in the former space, we are naturally more familiar with the properties of vectors there than with the curious spinors. Therefore it may be a bit surprising to find out that the spinor space provides us with the simplest description of rotations. Indeed, we explain this in article I, where we develop a slide rule with which to compute rotations.

However, in the description of polarization the two Cartesian components x and y may play the same role which the spinor bases played while describing spin $\frac{1}{2}$. Now, to a certain extent, we can "live" in the fundamental space and understand spinors better.

In Secs. II–IV of this article we rederive many of the old results from the theory of polarization. However, we try to do it in a way which makes it simple and easy to remember, and we show the physical and mathematical connection with spin-$\frac{1}{2}$ theory which was discussed in the preceding article. We use the modern Dirac notation throughout, even when discussing classical quantities.

In Sec. V and VI we review the Jones calculus for polarization evolution.[4] We show how the rotational slide rule from article I can be made to perform Jones operations. As far as we know this instrument and its applications are entirely new.[5]

Finally in Sec. VII we briefly review Fano's density operator approach to polarization,[6] and the Mueller calculus.[7] We show how this relates to methods discussed in the preceding sections.

II. REVIEW OF BASIC DESCRIPTIONS OF POLARIZATION

We review here the basic mathematical description of the polarization states of a single photon, or what is practically the same thing, of a light beam containing many photons in the same state. The coherent beam of many equivalent photons is probably easier to imagine because it is a useful laser-beam model and we may describe its polarization behavior in terms of an electric vector $\mathbf{E}$.

One set of basic vectors for the photon or coherent photon beam is the plane or linear polarization basis $\{|x\rangle,|y\rangle\}$. Expanding a given polarization state vector $|\Psi\rangle$ using $|x\rangle$ and $|y\rangle$ we have

$$|\Psi\rangle = |x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle, \qquad (2.1)$$

where $\langle x|\Psi\rangle$ and $\langle y|\Psi\rangle$ are (complex in general) amplitudes of x and y polarization, respectively.

Another set of basic vectors uses the right and left circular bases given by

$$|r\rangle = (|x\rangle + i|y\rangle)/\sqrt{2}, \quad |l\rangle = (|x\rangle - i|y\rangle)/\sqrt{2} \qquad (2.2)$$

respectively, and we review the physical meaning of these states and their amplitudes now.

The state vector for a right circular polarized plane wave *en vacuo* is

$$|\Psi\rangle = \mathcal{A}e^{i(kz-\omega t)}|r\rangle, \qquad (2.3)$$

where $\omega/k = c - 3 \times 10^8$ m/sec is the speed of light. For the description of a coherent beam in this state we shall follow the convention that the *real parts* of the amplitudes $\langle x|\Psi\rangle$ and $\langle y|\Psi\rangle$ are the components of E_x and E_y, respectively, of the electric vector transverse to the beam. For

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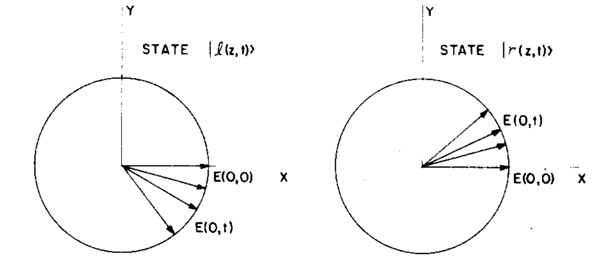


Fig. 1. Electric vector motion for circular polarization states.

the state described by Eq. (2.3) we have

$$\begin{aligned} E_x &= \mathrm{Re}\,\langle x|\Psi\rangle = \mathrm{Re}\,(\langle x|r\rangle\, e^{i(kz-\omega t)}) \\ &= (1/\sqrt{2})\cos(kz-\omega t), \\ E_y &= \mathrm{Re}\,\langle y|\Psi\rangle = \mathrm{Re}\,(\langle y|r\rangle\, e^{i(kz-\omega t)}) \\ &= -(1/\sqrt{2})\sin(kz-\omega t), \end{aligned} \tag{2.4}$$

where we set $\mathcal{A} = 1$, and used Eq. (2.2) assuming orthonormality of bases. ($\langle x|x\rangle = 1 = \langle y|y\rangle$ and $\langle x|y\rangle = 0 = \langle y|x\rangle$). Equation (2.4) gives a unit **E** vector undergoing a right-handed rotation as time advances at $z = 0$. This is shown on the right of Fig. 1. Similarily, replacing $|r\rangle$ by $|l\rangle$ in Eq. (2.3) would give opposite rotation as shown on the left-handed side of Fig. 1.

Note that the preceding interpretation of amplitudes puts no bound on $\mathcal{A}$ in Eq. (2.3) nor on any amplitude $\langle j|\Psi\rangle$. (The normalization $\langle i|j\rangle = \delta_{ij}$ of bases is just a convention.) However, if $\langle j|\Psi\rangle$ is interpreted as a quantum probability amplitude, i.e., if $|\langle j|\Psi\rangle|^2$ is the probability that a *single* photon in state $|\Psi\rangle$ will pick state j when forced to make the choice, then clearly $\mathcal{A}$ would have to be of unit magnitude in order for the sum of probabilities to be unity.

$$1 = \sum_j |\langle j|\Psi\rangle|^2 = \sum_j \langle\Psi|j\rangle\langle j|\Psi\rangle = \langle\Psi|\Psi\rangle. \tag{2.5}$$

However, until Sec. VI we shall use the classical interpertation of the amplitudes since it is nice to be able to keep track of the system with a real two-component vector such as was given by Eq. (2.4). We shall let the sum of the squares of the amplitudes be the intensity

$$I = \langle\Psi|\Psi\rangle = \sum_j |\langle j|\Psi\rangle|^2 \tag{2.6}$$

which may assume any real value. [For Eq. (2.3), $I = |\mathcal{A}|^2$.]

The general classical problem being studied here is that of the two-dimensional harmonic oscillator. We can imagine that the tip of the **E** vector traces the orbit of oscillating mass in the x-y plane. We shall show an elegant solution to the general conservative oscillator equations which uses the rotational slide rule to determine orbits numerically, given the initial conditions.

III. REVIEW OF THE MOTION OF POLARIZATION STATES

We consider briefly the two most well-known polarization motions or optical activity before reviewing the general case. The two common types of activity are Faraday rotation or circular dichroism, and birefringence.

Faraday rotation may occur when light passes through some materials in a direction of a magnetic field. Suppose that light in the state of circular polarization, such as the right-handed state described by Eq. (2.3), goes through an optical medium unchanged, but that the velocity $c_R = \omega/k_R$ of right-handed light was greater than the velocity $c_L = \omega/k_L$ of left-handed light. Then a general polarization state vector will be

$$\begin{aligned} |\Psi(z,t)\rangle &= \mathcal{R}e^{i(k_Rz-\omega t)}\,|r\rangle + \mathcal{L}e^{i(k_Lz-\omega t)}\,|l\rangle \\ &= (Re^{i\varphi_R}e^{ik_Rz}\,|r\rangle + Le^{i\varphi_L}e^{ik_Lz}\,|l\rangle)e^{-i\omega t}, \end{aligned} \tag{3.1}$$

where we assume initial phase shifts $\varphi_{R,L}$ in the amplitudes ($\mathcal{R} \equiv Re^{i\varphi_R}$, $\mathcal{L} \equiv Le^{i\varphi_L}$) and take R and L to be real. Rewriting this we have

$$|\Psi(z,t)\rangle = (Re^{-i\varphi}\,|r\rangle + Le^{i\varphi}\,|l\rangle)\,e^{-i\Phi}, \tag{3.2}$$

where

$$\begin{aligned} \varphi &= (\varphi_L - \varphi_R + k_Lz - k_Rz)/2, \\ \Phi &= \omega t - (\varphi_R + \varphi_L + k_Lz + k_Rz)/2. \end{aligned} \tag{3.3}$$

By substituting Eq. (2.2) relating ($|r\rangle$, $|l\rangle$) to plane bases ($|x\rangle$, $|y\rangle$) we have

$$\begin{aligned} |\Psi(z,t)\rangle = {} & (1/\sqrt{2})[(R+L)\cos\Phi\cos\varphi \\ & - (R-L)\sin\Phi\sin\varphi + i(\cdots]\,|x\rangle \\ & + (1/\sqrt{2})[(R+L)\cos\Phi\sin\varphi \\ & + (R-L)\sin\Phi\cos\varphi + i(\cdots]\,|y\rangle, \end{aligned}$$

where only the real parts

$$\begin{aligned} E_x(\varphi,t) &= \mathrm{Re}\,\langle x|\Psi\rangle = E_x(0,t)\cos\varphi - E_y(0,t)\sin\varphi, \\ E_x(0,t) &= [(R+L)/\sqrt{2}]\,(-\cos\Phi), \\ E_y(\varphi,t) &= \mathrm{Re}\,\langle y|\Psi\rangle = E_x(0,t)\sin\varphi + E_y(0,t)\cos\varphi, \\ E_y(0,t) &= [(R-L)/\sqrt{2}]\,(\sin\Phi), \end{aligned}$$

are kept, according to our convention for defining vector **E**.

Note that if φ is constant, then the **E** vector describes a closed ellipse with the major axis inclined at an angle φ at every point in the material. The relative magnitude of R and L amplitudes determines the handedness of rotation within the ellipse, and the axes

$$a = (R+L)/\sqrt{2}, \quad b = (R-L)/\sqrt{2} \tag{3.4}$$

of the polarization ellipse as shown in Fig. 2. If the right-handed part is faster than the left-handed part ($k_L > k_R$) then the ellipse rotates to the right rigidly as we move up the z axis, i.e., φ increases with z.

Birefringence occurs in solids that have "preferred axes." Suppose that light in the state of linear-x polarization or linear-y polarization goes through the medium unchanged, but that the velocity $c_x = \omega/k_x$ of the x-polarized wave is greater than that of the y-polarized wave. Then a general polarization state vector can be written

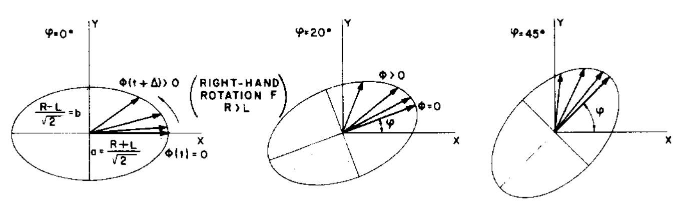


Fig. 2. Faraday rotation.

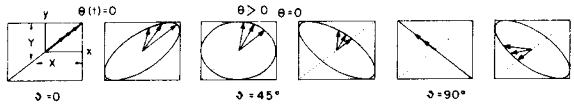


Fig. 3. Birefringence.

$$|\Psi\rangle = \mathscr{X}e^{i(k_xz-\omega t)}|x\rangle + \mathscr{Y}e^{i(k_yz-\omega t)}|y\rangle$$
$$= (Xe^{i\vartheta_x}e^{ik_xz}|x\rangle + Ye^{i\vartheta_y}e^{ik_yz}|y\rangle)e^{-i\omega t}, \quad (3.5)$$

where we assume initial phase shifts $\vartheta_{x,y}$ in the amplitudes

$$\mathscr{X} = e^{i\vartheta_x}X \qquad \mathscr{Y} = e^{i\vartheta_y}Y$$

and take X and Y to be real. Rewriting this we have

$$|\Psi(z,t)\rangle = (Xe^{-i\vartheta}|x\rangle + Ye^{i\vartheta}|y\rangle)e^{-i\theta}, \quad (3.6)$$

where

$$\vartheta = (\vartheta_y - \vartheta_x + k_yz - k_xz)/2,$$
$$\theta = \omega t - (\vartheta_x + \vartheta_y + k_xz + k_yz)/2. \quad (3.7)$$

The classical electric vector is then given by

$$E_x = \text{Re}\ \langle x|\Psi\rangle = X\cos(\vartheta + \theta),$$
$$E_y = \text{Re}\ \langle y|\Psi\rangle = Y\cos(\vartheta - \theta).$$

Now suppose that the initial phases are zero ($\vartheta_{x,y} = 0$) but that the x wave is faster than the y wave, ($k_y > k_x$) so that ϑ increases linearly with z. Then, at $z = 0$ or $\vartheta = 0$ the **E** vector oscillates in a plane or line of polarization coinciding with the diagonal of the X-Y rectangle as shown on the left of Fig. 3. Now, as z or ϑ increase an ellipse "grows" out of this line, becomes fatter, then shrinks back to the conjugate diagonal, and so on. Looking at this sequence you may get the illusion that there is a disc precessing around the y axis or else around the x axis. This illustion of disc rotation is particularily intense when we see an oscilloscope trace of the "Lissajous figure" made by two different x and y oscillators.

We shall see shortly that this illusion is not so farfetched as far as the underlying mathematical structure is concerned. All polarization changes such as the Faraday rotation or birefringence which we have discussed, or any combinations of them, can be related to some rotation.

IV. EVOLUTION OF POLARIZATION IN CONSERVATIVE MEDIA

We consider the propagation of a monochromatic plane wave through a polarizable media that is a combination of dichroic and birefringent material. We shall assume that the angular frequency ω (rad/sec) of the light is far enough away from material resonance frequencies that absorption is negligible, and we may assume a linear response for the polarization.

We shall use both plane and circular polarization bases to describe the general optical polarization state,

$$|\Psi\rangle = (Xe^{-i\vartheta}|x\rangle + Ye^{i\vartheta}|y\rangle)e^{-i\theta}$$
$$= (Re^{-i\varphi}|r\rangle + Le^{i\varphi}|l\rangle)e^{-i\Phi}, \quad (4.1)$$

i.e., we will study two different *representations* of states and operators. [Eq. (4.1) has the same form as Eqs. (3.2) and (3.6). However, as we will see, the relations in Eqs. (3.3) and (3.7) are correct only for the special cases treated there.]

We begin by defining a representation of the Pauli spinors as follows:

$$\mathscr{C}(\sigma_A) = \begin{pmatrix} \langle r|\sigma_A|r\rangle & \langle r|\sigma_A|l\rangle \\ \langle l|\sigma_A|r\rangle & \langle l|\sigma_A|l\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\mathscr{C}(\sigma_B) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \mathscr{C}(\sigma_C) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.2)$$

using the circular bases $|r\rangle$ and $|l\rangle$. We shall also need the same operators represented in the linear bases $|x\rangle$ and $|y\rangle$. A simple change of basis using Eq. (2.2) is needed.

$$\mathscr{L}(\sigma_\mu) = \begin{pmatrix} \langle x|\sigma_\mu|x\rangle & \langle x|\sigma_\mu|y\rangle \\ \langle y|\sigma_\mu|x\rangle & \langle y|\sigma_\mu|y\rangle \end{pmatrix}$$
$$= \begin{pmatrix} \langle x|r\rangle & \langle x|l\rangle \\ \langle y|r\rangle & \langle y|l\rangle \end{pmatrix} \begin{pmatrix} \langle r|\sigma_\mu|r\rangle & \langle r|\sigma_\mu|l\rangle \\ \langle l|\sigma_\mu|r\rangle & \langle l|\sigma_\mu|l\rangle \end{pmatrix}$$
$$\times \begin{pmatrix} \langle r|x\rangle & \langle r|y\rangle \\ \langle l|x\rangle & \langle l|y\rangle \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \mathscr{C}(\sigma_\mu) \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \quad (4.3)$$

Substituting each C matrix in turn from Eq. (4.2) into Eq. (4.3) gives the needed linear representations.

$$\mathscr{L}(\sigma_A) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathscr{L}(\sigma_B) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\mathscr{L}(\sigma_C) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (4.4)$$

We calculate the "expectation values" $\langle\psi|\sigma_\mu|\psi\rangle$ of each of the Pauli operators in the general state of Eq. (4.1). First each calculation is done using the circular representation, then the linear representation, and the two results are equated.

$$\langle\Psi|\sigma_\mu|\Psi\rangle = \sum_{c'=r,l}\sum_{c=r,l}\langle\Psi|c'\rangle\langle c'|\sigma_\mu|c\rangle\langle c|\Psi\rangle$$
$$= \sum_{p'=x,y}\sum_{p=x,y}\langle\Psi|p'\rangle\langle p'|\sigma_\mu|p\rangle\langle p|\Psi\rangle.$$

(Scalar products do not depend upon which basis is used.) For σ_A we have the following:

$$\langle\Psi|\sigma_A|\Psi\rangle = (Re^{i\vartheta}Le^{-i\vartheta})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} Re^{-i\varphi} \\ Le^{i\varphi} \end{pmatrix}$$
$$= (Xe^{i\vartheta}Ye^{-i\vartheta})\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} Xe^{-i\vartheta} \\ Ye^{i\vartheta} \end{pmatrix} \quad (4.5)$$
$$= 2RL\cos 2\varphi \quad = X^2 - Y^2.$$
$$= (a^2 - b^2)\cos 2\varphi.$$

In the last line we write the answer in terms of ellipse axes using Eq. (3.4) or

$$a^2 - b^2 = 2RL,$$
$$2ab = R^2 - L^2. \quad (4.6)$$

Similarily, for σ_B and σ_C we have the following:

$$\langle\Psi|\sigma_B|\Psi\rangle = 2RL\sin 2\varphi = 2XY\cos 2\vartheta \quad (4.7)$$
$$= (a^2 - b^2)\sin 2\varphi,$$
$$\langle\Psi|\sigma_C|\Psi\rangle = R^2 - L^2 = 2XY\sin 2\vartheta \quad (4.8)$$
$$= 2ab.$$

Finally, the intensity is given by

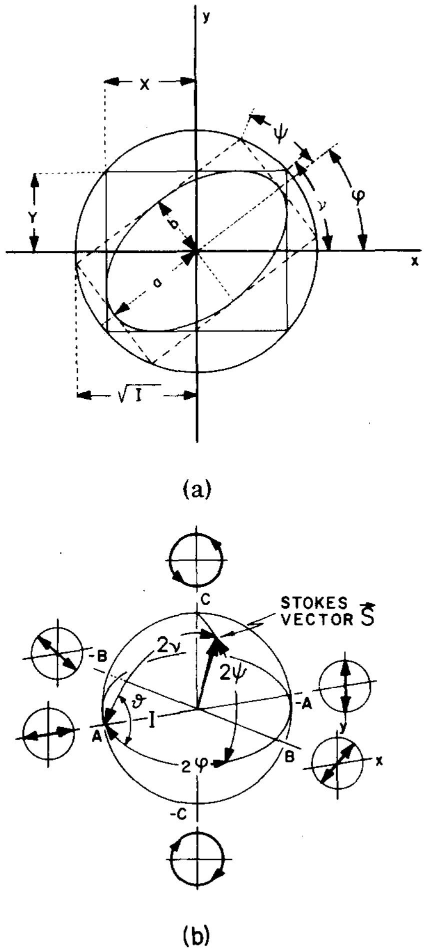


Fig. 4. Geometrical representations of polarization states. (a) The polarization ellipse. The ellipse is the locus of the electric vector. Any two independent parameters such as ϕ and ν, or ψ and ϕ, ··· etc. may be used to define its shape and orientation. (b) The Stokes vector. For each shape and orientation of the polarization ellipse there is a Stokes vector pointing in Stokes (ABC) space. One end of the Stokes vector corresponds to negative (clockwise) rotation of the electric vector and around the ellipse, while the opposite vector belongs to positive rotation. The Stokes vector is analogous to the spin vector of electron polarization.

$$
\begin{aligned}
I = \langle \Psi | \Psi \rangle &= R^2 + L^2 = X^2 + Y^2, \\
&= a^2 + b^2,
\end{aligned} \tag{4.9}
$$

where we have used Eqs. (2.5), (3.4), and (4.1).

Now it is possible to give geometric pictures of Eqs. (4.5)–(4.9). It is convenient to construct rectangular boxes which contain and define a polarization ellipse. Let one box have dimensions a by b and have its axes inclined at an angle φ so they are parallel to the axes of the ellipse. Let the other box be of dimension X by Y and center it on the x and y axes, as shown in Fig. 4(a). We let angles ψ and ν be the slope angles of the diagonals of these boxes, i.e., let

$$
\begin{aligned}
a &= \sqrt{I}\cos\psi, \quad b = \sqrt{I}\sin\psi, \\
X &= \sqrt{I}\cos\nu, \quad Y = \sqrt{I}\sin\nu.
\end{aligned} \tag{4.10}
$$

Finally, the last trick is to observe the following:

$$
\begin{aligned}
&\sin 2\psi = 2\sin\psi\cos\psi = 2ab/I, \\
&\cos 2\psi = \cos^2\psi - \sin^2\psi = (a^2 - b^2)I, \\
&\sin 2\nu = 2XY/I, \qquad \cos 2\nu = (X^2 - Y^2)/I
\end{aligned} \tag{4.11}
$$

and substitute these into Eqs. (4.5)–(4.9). We obtain the following:

$$
\begin{aligned}
\langle \psi | \sigma_A | \psi \rangle &= I\cos 2\psi \cos 2\varphi = I\cos 2\nu, \\
\langle \psi | \sigma_B | \psi \rangle &= I\cos 2\psi \sin 2\varphi = I \sin 2\nu \cos 2\vartheta, \\
\langle \psi | \sigma_C | \psi \rangle &= I\sin 2\psi = I\sin 2\nu \sin 2\vartheta.
\end{aligned} \tag{4.12}
$$

Equations (4.10)–(4.12) allow us to quickly derive explicit (though occasionally double-valued ··· caution!) relations for any parameter of polarization in terms of any of the others. In general there are three independent parameters counting intensity $I = a^2 + b^2 = R^2 + L^2 = X^2 + Y^2$. (Here we do not count the overall phase which disappeared when we took the scalar products. Later, we return to treat this too.) We may choose any set of three parameters such as (X,Y,ϑ), or (R,L,φ), or (I,ψ,ν), or (I,ψ,ϑ) ··· and so on, and all of these are easily related.

We see the geometric interpretation of some of these parameters as they appear on the polarization ellipse in Fig. 4(a) (Indeed, we could obtain all the equations by geometry alone but this turns out to be quite laborious.)

Also we may regard the three quantities in Eq. (4.12) as the components of a vector **S** in Cartesian ABC space. The first set of equations involving ψ and φ suggest that 2ψ is the polar elevation angle with respect to the C axis while 2φ is the azimuth with respect to A. The other set suggests an A-axis polar angle of 2ν and an azimuth from B of 2ϑ. The angles are indicated in Fig. 4(b). (Note that the arcs of length 2φ, 2ψ, and 2ν form a right spherical triangle of altitude 2ψ, hypotenuse 2ν, base 2φ, and base angle 2ϑ.)

The vector **S** is analogous to the expectation vector of angular momentum for the spin-½ state (compare with Eq. (4.6) in article I). and is called the Stokes vector after Stokes who first thought of using a vector description of optical polarization states in 1852.[8] Note that we use notation ABC for Stokes space to avoid confusion with "ordinary" xyz space. For optical polarization theory Stokes space plays the role held by "ordinary" xyz space in spin-½ theory, while at the same time "ordinary" components of polarization x,y take the place of spin-up and spin-down bases.

With "spin vector" description of states established, we seek now a "rotation vector" description of the optical medium which is analogous to the ω-vector for a spin-½ Hamiltonian. [See Eqs. (5.2) in article I.) This is possible, but its derivation is complicated by the fact that the equation of motion, i.e., Maxwell's equations

$$
\nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} - \frac{1}{c^2\epsilon_0}\frac{\partial^2\mathbf{P}}{\partial t^2} \tag{4.13}
$$

are of *second* order in *three* dimensions, and thus harder to deal with than the spin-½ Schrödinger equation [Eq. (5.1) in article I.]

However, let us assume that all field components parallel to the beam propagation direction (**z**) can be ignored, and that a Hermitian susceptibility tensor is given which defines the relation between polarization **P** and field **E** as follows:

$$
\begin{pmatrix} P_x/\epsilon_0 \\ P_y/\epsilon_0 \end{pmatrix} = \begin{pmatrix} \chi_{xx} & \chi_{xy} \\ \chi_{xy}^* & \chi_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}. \tag{4.14}
$$

We assume also that all quantities are functions of z and t only.

$$\mathbf{E}(z,t) = \mathbf{E}(z)e^{-i\omega t}; \qquad \mathbf{P}(z,t) = \mathbf{P}(z)e^{-i\omega t}. \quad (4.15)$$

Then Maxwell's equations take the following coupled form after substitution of Eqs. (4.14) and (4.15) into Eq. (4.13).

$$\frac{\partial^2}{\partial z^2}\begin{pmatrix}\langle x|\psi(z)\rangle\\ \langle y|\psi(z)\rangle\end{pmatrix} = -\frac{\omega^2}{c^2}\begin{pmatrix}1+\chi_{xx} & \chi_{xy}\\ \chi^*_{xy} & 1+\chi_{yy}\end{pmatrix}\begin{pmatrix}\langle x|\psi(z)\rangle\\ \langle y|\psi(z)\rangle\end{pmatrix}, \quad (4.16)$$

where the complex field vector

$$\mathbf{E}(z) = \mathrm{Re}\begin{pmatrix}\langle x|\psi(z)\rangle\\ \langle y|\psi(z)\rangle\end{pmatrix}$$

has been written in the $\{|x\rangle, |y\rangle\}$ basis or representation.

Now if we consider only wave solutions which are moving along in the positive z direction, then the matrix form of the solution has the same form as that of the first-order Schrödinger equation.

$$\begin{pmatrix}\langle x|\psi(z)\rangle\\ \langle y|\psi(z)\rangle\end{pmatrix} = \exp\left[iz\begin{pmatrix}K_{xx} & K_{xy}\\ K^*_{xy} & K_{yy}\end{pmatrix}\right]\begin{pmatrix}\langle x|\psi(0)\rangle\\ \langle y|\psi(0)\rangle\end{pmatrix}. \quad (4.17a)$$

The "wave matrix" K in the exponent is the "doubly positive" square root

$$K = (1+\chi)^{1/2}\omega/c,$$

$$\begin{pmatrix}K_{xx} & K_{xy}\\ K^*_{xy} & K_{yy}\end{pmatrix} = \frac{\omega}{c}\begin{pmatrix}1+\chi_{xx} & \chi_{xy}\\ \chi^*_{xy} & 1+\chi_{yy}\end{pmatrix}^{1/2} \quad (4.17b)$$

of the matrix in the equation of the motion. [Eq. (4.16)] We will explain how to compute the square roots of a matrix in a numerical example to follow.

We may expand K in terms of generations of rotation or Pauli spinors $\mathbf{J}_\mu = \sigma_\mu/2$ as was done in Eq. (5.2) in article I.

$$\begin{aligned}\begin{pmatrix}K_{xx} & K_{xy}\\ K^*_{xy} & K_{yy}\end{pmatrix} &= \frac{K_{xx}+K_{yy}}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \left[(K_{yy}-K_{xx})\begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix}\right.\\ &\quad \left. - (K_{xy}+K^*_{xy})\begin{pmatrix}0 & 1/2\\ 1/2 & 0\end{pmatrix} + (K_{xy}-K_{xy})\begin{pmatrix}0 & -1/2\\ 1/2 & 0\end{pmatrix}\right]\\ &= K_0\mathcal{L}(\mathbf{1}) - [(K_{yy}-K_{xx})\mathcal{L}(\mathbf{J}_A)\\ &\quad - 2\mathrm{Re}K_{xy}\mathcal{L}(\mathbf{J}_B) + 2\mathrm{Im}K_{xy}\mathcal{L}(\mathbf{J}_C)]. \quad (4.18)\end{aligned}$$

Then the solution to Eq. (4.17) takes the form

$$\begin{pmatrix}\langle x|\psi(z)\rangle\\ \langle y|\psi(z)\rangle\end{pmatrix} = e^{K_0 z/i}e^{\omega\cdot\mathcal{L}(\mathbf{J})z/i}\begin{pmatrix}\langle x|\psi(0)\rangle\\ \langle y|\psi(0)\rangle\end{pmatrix}, \quad (4.19)$$

where the components of the rotation axis vector ω are given by

$$\omega_A = K_{yy} - K_{xx}, \quad \omega_B = -2\mathrm{Re}K_{xy}, \quad \omega_C = 2\mathrm{Im}K_{xy}, \quad (4.20)$$

and have the dimensions (rad/m). We also let $K_0 = (K_{xx} + K_{yy})/2$.

This form of the solution is analogous to Eqs. (5.3) and (5.4) of article I. It makes the visualization and computation of polarization changes through an optical medium as easy

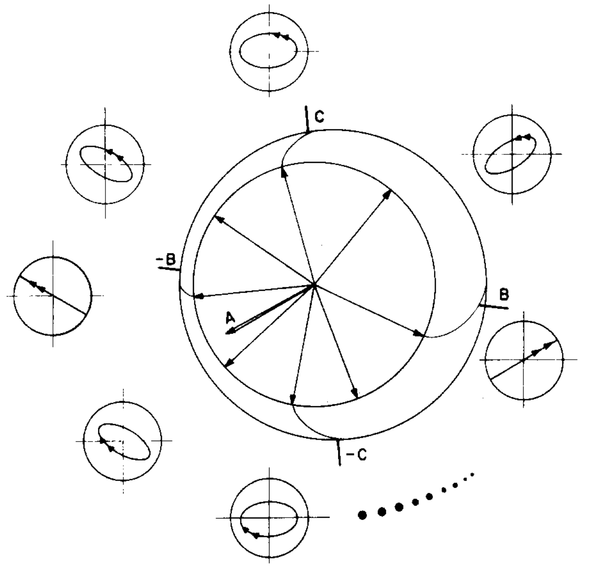


Fig. 5. Representing birefringence in Stokes space.

as ABC-space rotation. (Note that it is valid for only an entirely forward moving light wave is a nonabsorbing medium. Combinations of all four square roots of $1+\chi$ will be needed for a general solution, but then the intensity of the beam will vary with z.) For example, if we take $K_{yy} > K_{xx}$ and $K_{xy} = 0$, then Eq. (4.20) gives the following ω-vector components.

$$\omega_A = K_{yy} - K_{xx} > 0, \qquad \omega_B = 0, \qquad \omega_C = 0.$$

This corresponds to counter clockwise rotation around the A axis in ABC space by $\omega_A z$ radians as we advance z meters along the beam axis. By comparing the A-axis rotation in Stokes space shown in Fig. 5 to the sequence in Fig. 3 we see that $\omega = \omega_A\hat{A}$ corresponds to a birefringent medium. It gives the rotation around the x or y axes which we mentioned in Sec. III; the x and y state became the two ends of the A axis in Stokes space as shown in Fig. 4. Similarily, pure Faraday rotation (recall Fig. 2) corresponds to a rotation around the C axis in Fig. 4(b), i.e., to a pure imaginary off-diagonal component $K_{xy} = i|K_{xy}|$ in the wave matrix.

$$\omega_A = K_{yy} - K_{xx} = 0, \qquad \omega_B = -2\mathrm{Re}K_{xy} = 0,$$
$$\omega_C = 2\mathrm{Im}K_{xy} = 2K_{xy}.$$

General optical activity will correspond to a rotation around an arbitrary axis in ABC space. Consider, for example, a medium for which the susceptibility matrix gives

$$\frac{\omega^2}{c^2}(1+X) = \frac{1}{10}\begin{pmatrix}18 & -9+9i\\ -9-9i & 27\end{pmatrix}. \quad (4.21)$$

The desired K matrix is the positive square root of the preceding one, which we derive in Appendix A.

$$K = \frac{1}{\sqrt{10}}\begin{pmatrix}4 & -1+i\\ -1-i & 5\end{pmatrix}. \quad (4.22)$$

The rotation axis for this example has ABC components

$$\omega_A = 1/\sqrt{10}, \qquad \omega_B = 2/\sqrt{10}, \qquad \omega_C = 2/\sqrt{10}, \quad (4.23)$$

which gives a vector ω of length $|\omega| = 3/\sqrt{10}$ with the polar angles shown in Fig. 6.

At this point we can feed these angles into the rotation slide rule and calculate the evolution of any polarization state as explained in the following section. However, before doing this, it is instructive to see how this sort of calculation would be done using the well-known Jones calculus.[9]

Each optical medium of length z is assigned a Jones matrix. The Jones matrix for the medium described by Eqs. (4.21) and (4.22) would be (see Appendix A)

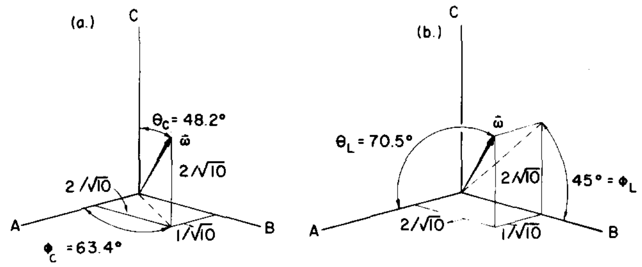


Fig. 6. ω vector for K matrix in Eq. (4.22). (a) Polar angles appropriate for use in circular polarization basis. (b) Polar angles appropriate for use in linear polarization basis.

$$J = e^{iKz} = \exp[i(9/10)^{1/2}z] \times \begin{pmatrix} 2 + \exp[i(9/10)^{1/2}z] & 1 - \exp[i(9/10)^{1/2}z] - i\{1 - \exp[i(9/10)^{1/2}z]\} \\ 1 - \exp[i(9/10)^{1/2}z] - i\{-1 + \exp[i(9/10)^{1/2}z]\} & 1 + 2\exp[i(9/10)^{1/2}z] \end{pmatrix} \quad (4.24)$$

Then the final polarization state vector if of light emerging from this medium would be the product of the Jones operator J_0 with the initial state vector

$$|f\rangle = J_0|i\rangle.$$

The Jones matrix for several consecutive media of length $z_1, z_2, \cdots$ is the consecutive matrix product of their respective Jones matrices.

$$J = J_n \cdots J_2 J_1 = e^{iK_nz_n} \cdots e^{iK_2z_2}e^{iK_1z_1}. \quad (4.25)$$

Note that one cannot write $e^Ae^B = e^{A+B}$ unless $AB = BA$.

V. POLARIZATION CALCULATIONS USING THE ROTATIONAL SLIDE RULE

If one has a series of optical elements and initial states, the Jones matrix algebra can become laborious and the results may be difficult to visualize. In order to visualize what happens to each state in each retarder, it may be better to use the rotational analogy. Furthermore, by using the rotational slide rule described in article I, one may quickly find desired polarization parameters, *including the overall phase*, with an accuracy of a degree or so. We discuss this procedure now.

First, the medium must be associated with a rotation $\mathbf{R}[\omega]$. For the particular K matrix in our example, we found the rotation axis vector ω shown in Fig. 6. The inputs required for the slide rule will be the rotation angle

$$|\omega|z = (3/\sqrt{10})z \quad (5.1)$$

and the polar angle of $\hat{\omega}$. If you want to work in the circular basis ($|r\rangle$, $|l\rangle$) then you need the polar angles shown in Fig. 6(a). For the linear basis ($|x\rangle$, $|y\rangle$) you need the ones in Fig. 6(b). The overall phase shift

$$K_0z = (K_{xx} + K_{yy})z/2 = 9z/(2\sqrt{10}) \quad (5.2)$$

should be noted also.

Next we associate the incoming polarization state $|\Psi\rangle$ with the rotation operator $\mathbf{R}(\alpha\beta\gamma)$ which produces $|\Psi\rangle$ when acting upon a chosen "origin" state $\sqrt{I}|1\rangle$ of the same intensity I.

$$\begin{aligned}|\Psi\rangle &= \mathbf{R}(\alpha\beta\gamma)\sqrt{I}\,|1\rangle \\ &= \sqrt{I}\,\mathbf{R}(\alpha 00)\mathbf{R}(0\beta 0)\mathbf{R}(00\gamma)|1\rangle. \quad (5.3)\end{aligned}$$

As in article I, we will find it more convenient to label states with Euler angles $(\alpha\beta\gamma)$, while optical operators are labeled more conveniently with axis angles $[\phi\theta\omega]$. It is easy to relate Euler angles to the phase angles in Fig. 4 and Eqs. (4.1)–(4.12). For example, if we want to work in the circular basis we would choose the original state $|1\rangle$ to be $|r\rangle$ in Eq. (5.3) (We use the representations given in Eqs. (3.2) or (3.3) of article I.)

$$\begin{aligned}\begin{pmatrix}\langle r|\Psi\rangle \\ \langle l|\Psi\rangle\end{pmatrix} &= \begin{pmatrix} e^{-i\alpha/2} & \cdot \\ \cdot & e^{i\alpha/2}\end{pmatrix} \times \begin{pmatrix}\cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2\end{pmatrix}\begin{pmatrix} e^{-i\gamma/2} & \cdot \\ \cdot & e^{i\gamma/2}\end{pmatrix}\begin{pmatrix}\sqrt{I} \\ 0\end{pmatrix} \\ &= e^{-i\gamma/2}\begin{pmatrix} e^{-i\alpha/2}\sqrt{I}\,\cos\beta/2 \\ e^{i\alpha/2}\sqrt{I}\,\sin\beta/2\end{pmatrix}, \quad (5.4)\end{aligned}$$

$$|\Psi\rangle = (e^{-i\alpha/2}\sqrt{I}\cos\beta/2|r\rangle + e^{i\alpha/2}\sqrt{I}\sin\beta/2|l\rangle)e^{-i\gamma/2}.$$

Comparing this with the original state definition on the right-hand side of Eq. (4.1) we have $\varphi = \alpha/2$, $\Phi = \gamma/2$, $R = \sqrt{I}\cos\beta/2$, and $L = \sqrt{I}\sin\beta/2$. Rewriting the amplitudes (R,L) in terms of ellipse axes using Eq. (3.4) and (4.10) we have

$$\begin{aligned}R &= (a+b)/\sqrt{2} = \sqrt{I}(1/\sqrt{2}\cos\psi + 1/\sqrt{2}\sin\psi) = \sqrt{I}\cos(\pi/4 - \psi), \\ L &= (a-b)/\sqrt{2} = \sqrt{I}(1/\sqrt{2}\cos\psi - 1/\sqrt{2}\sin\psi) = \sqrt{I}\sin(\pi/4 - \psi). \quad (5.5)\end{aligned}$$

This relates the second Euler angle β to the ellipse angle ψ: $\pi/2 - 2\psi$; β is therefore the C-axis polar angle in Fig. 4. To summarize: A polarization state is defined in the circular basis by a rotation of $|r\rangle$,

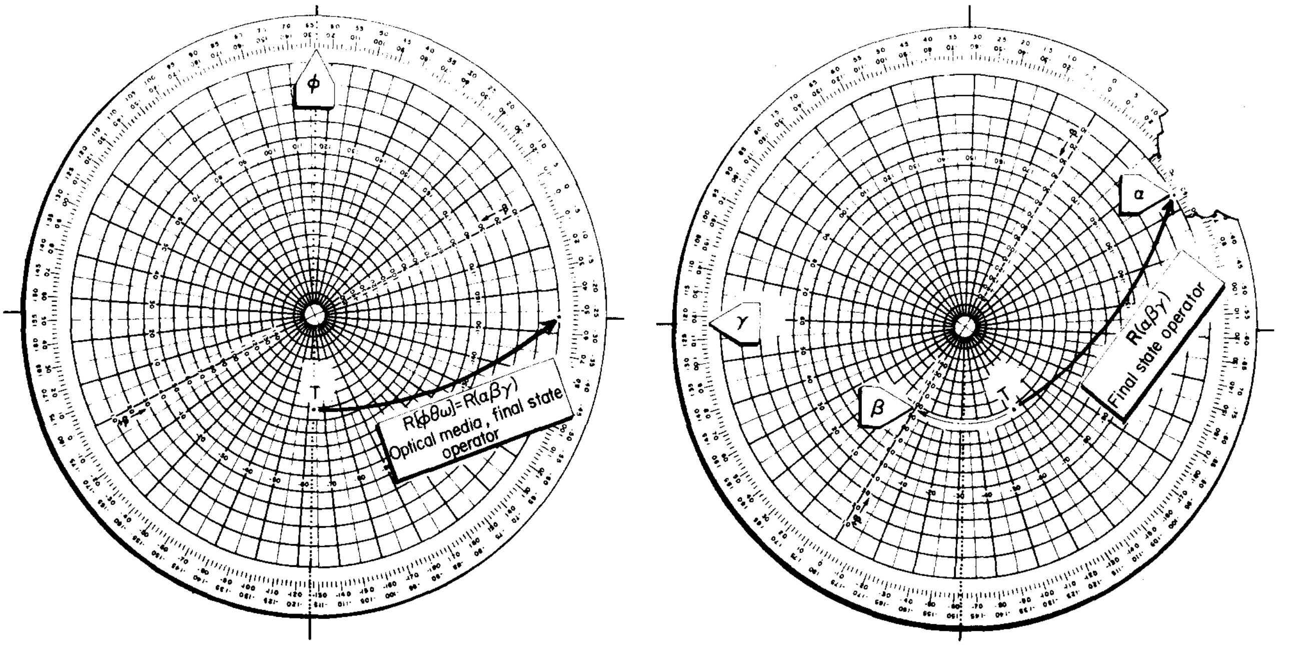
$$\begin{aligned}|\Psi\rangle &= (Re^{-i\varphi}|r\rangle + Le^{i\varphi}|l\rangle)e^{-i\Phi} = \mathbf{R}(\alpha\beta\gamma)|r\rangle\sqrt{I} \\ &= \sqrt{I}(\cos(\pi/4 - \psi)e^{-i\varphi}|r\rangle + \sin(\pi/4 - \psi)e^{i\varphi}|l\rangle)e^{-i\Phi}, \quad (5.6a)\end{aligned}$$

where the Euler angles and phases are related by

$$\begin{aligned}\alpha &= 2\varphi, & \varphi &= \alpha/2; \\ \beta &= \pi/2 - 2\psi, & \psi &= \pi/4 - \beta/2; \\ \gamma &= 2\Phi, & \Phi &= \gamma/2. \quad (5.6b)\end{aligned}$$

Similarily we can define a state in the linear basis by a rotation of $|x\rangle$

$$\begin{aligned}|\Psi\rangle &= (Xe^{-i\vartheta}|x\rangle + Ye^{i\vartheta}|y\rangle)e^{-i\theta} = \mathbf{R}(\alpha\beta\gamma)|x\rangle\sqrt{I} \\ &= \sqrt{I}(\cos\nu e^{-i\vartheta}|x\rangle + \sin\nu e^{i\vartheta}|y\rangle)e^{-i\theta}, \quad (5.7a)\end{aligned}$$



(a) (b)

Fig. 7. Evolution of circularly [R(000)] polarized beam in ω-active media. (a) Setting $[\phi\theta\omega]$ parameters of media rotation operator. (b) Reading $(\alpha\beta\gamma)$ coordinates of the final evolved beam.

where the Euler angles and phases are related by

$$\alpha = 2\vartheta, \qquad \beta = 2\nu, \qquad \gamma = 2\theta. \tag{5.7b}$$

[When we use the linear basis the A axis becomes the polar $(\alpha\gamma)$ axis and the C axis takes the place of the B axis for the β angle.]

Finally, the effect of passing the polarization state through the optical medium (K) is given by

$$\begin{aligned} e^{iKz}|\Psi\rangle &= e^{iKz}\mathbf{R}(\alpha\beta\gamma)|1\rangle, \\ e^{iKz}|\Psi\rangle &= e^{K_0 z/i}\mathbf{R}[\omega z]\mathbf{R}(\alpha\beta\gamma)|1\rangle, \\ e^{iKz}|\Psi\rangle &= \mathbf{R}(\alpha_z\beta_z\gamma_z)|1\rangle \\ &= \mathbf{R}[\omega z]\mathbf{R}(\alpha\beta\gamma - 2K_0 z)|1\rangle. \end{aligned} \tag{5.8}$$

The rotational slide rule is designed to allow the convenient calculation of the final Euler angles $(\alpha_z\beta_z\gamma_z)$ in Eq. (5.8) given $[\omega z]$ and $(\alpha\beta\gamma - 2K_0 z)$. These angles define the final polarization state and phase according to Eq. (5.6) and (5.7) depending on your choice of bases.

Before demonstrating a calculation we mention some shortcuts which may be helpful. First, if the input light is circularly polarized, then we may take $\alpha = \beta = \gamma = 0$ in Eq. (5.6) and the necessary ω polar angles are given by Fig. 6(a). The desired answers $(\alpha_z\beta_z\gamma_z)$ are read directly from the $\omega \rightarrow$ Euler angle conversion scales of the slide rule. (See example 1 below.) We may simplify operations in a similar way if the incoming light is linearly (say $|x\rangle$) polarized by switching to the $|x\rangle$, $|y\rangle$ basis and using the ω polar angles as given by Fig. 6(b). (See example 2 below.)

Finally, we find it is more convenient to write the general incoming polarization state as follows:

$$|\Psi\rangle = \mathbf{R}(\alpha\beta\gamma)|1\rangle = e^{-i(\alpha+\gamma)/2}\mathbf{R}(\alpha\beta - \alpha)|1\rangle \tag{5.9}$$

since the operator $\mathbf{R}(\alpha\beta - \alpha)$ is represented on the slide rule by a straight radial arrow of scale length β directed toward azimuth angle $\phi = \alpha$ where the ϕ-scale is the outermost ring on the top scale. [Or else $-\mathbf{R}(\alpha\beta - \alpha)$ is an arrow of length $2\pi - \beta$ in the other direction.] It involves less writing on the top scale and less chance for error if we represent the initial state using the lines that are already there. We can easily do the simple phase addition $(\alpha + \gamma)$ on paper or neglect it entirely if the overall phase is not important.

Example 1. Find the polarization state and phase of a circularily polarized beam that enters the medium described by Eq. (4.22) with zero phase and propagates $z = \sqrt{10}\pi/3$ units (i.e., $|\omega|z = \omega = 180°$)

Setting the slide rule on the azimuth $\phi_c = 63.4°$ [Fig. 6(a)] we draw an arrow along the $\theta_c = 48°$ scale line of length 180° (only the head and tail points of the arc are needed here) as shown in Fig. 7(a). Then we set the slide rule so the angle between the tail of the arrow (point T) and the $+\beta$ axis is bisected by the slide rule center line or θ scale as shown in Fig. 7(b). The Euler angle $\alpha_z = 63°$ is read at the head of the arrow on the α scale, $\beta_z = 96°$ on the $+\beta$ scale at the intersection with the θ line which intersects the tail, and $\gamma_z = 122°$ on the γ scale over the γ box. (See Fig. 5 in article I for an explanation of why this works.) This means the final polarization ellipse has angles $\varphi_z = \alpha_z/2 = 315°$, $\psi_z = 45° - \beta_z/2 = -3°$, and the final phase is $\Phi_z = (\alpha_z - z \text{ Trace } K_0)/2 = -14°$. This particular calculation (with $|\omega|z = 180°$) can be checked "visually," using geometry.

Example 2. Find polarization and phase of an X-polarized beam that enters the same medium as Example 1 with zero phase and propagates $z = \sqrt{10}\{50°\}/3$ units (i.e., $\omega = |\omega|z = 50°$)

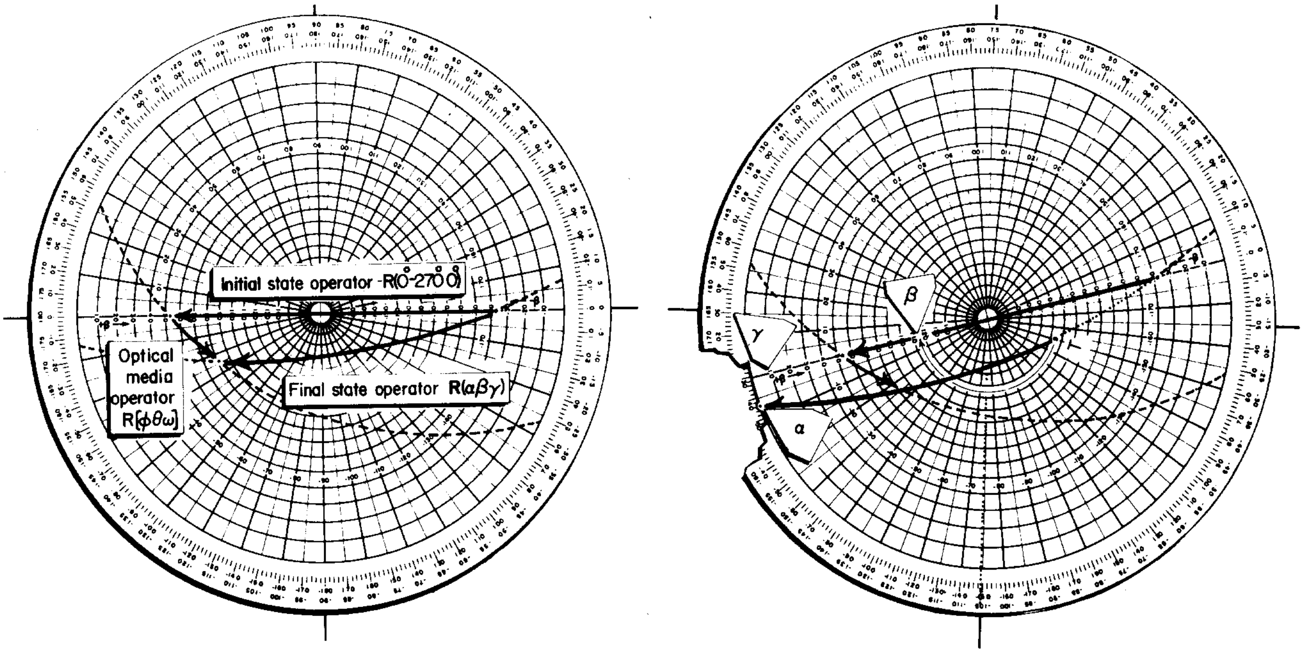


Fig. 8. Evolution of R(0° 90° 0°) polarized beam in ω-active media. (a) Vector addition gives resultant vector of final evolved beam. (b) $(\alpha\beta\gamma)$ coordinates of the evolved beam are read from the Euler scales.

We use the linear basis conversions, but the same procedure as given in Example 1. We convert a 50° arc at $\phi_L = 45°$ and $\theta_L = 71°$ (The polar angles came from Fig. 6(b)) to Euler angles $\alpha_z = -38°$, $\beta_z = 48°$, and $\gamma_z = 53°$. This implies that $\vartheta_z = \alpha_z/2 = -19°$, $\nu_z = \beta_z/2 = 24°$, and $\theta_z = (\gamma_z - zK_0)/2 = -49°$.

Example 3. Solve Example 2 in the circular basis

The parameters $2\varphi = \alpha = 0$, $2\psi = \pi/2 = \beta$, and $\Phi = \gamma/2 = 0$ of the incoming $|x\rangle$ state tell which rotation operator, namely, $\mathbf{R}(0\ 90°\ 0)$ transforms $|r\rangle$ into $|x\rangle$. The intersection of the $\phi = \alpha = 0$ line and the $(\theta_c = 48°, \phi_c = 63.4°)$ arc will be the terminus of the $\mathbf{R}(0\ 90°\ 0)$ arrow and the beginning of the 50° arc for $\mathbf{R}[\phi\theta\omega]$. It turns out we have to use the $-\mathbf{R}(0\ -270°\ 0)$ arrow pointing toward $\phi = 180°$, as shown in Fig. 8 since the $\mathbf{R}(0\ 90°\ 0)$ arrow would go off scale. (We have to put the -1 phase factor on at the end of the calculation now.) Finally the vector sum is found and converted to Euler angles which gives polarization angles $\varphi_z = 21°$ and $\psi_z = -13°$. Conversion using Eq. (4.12) shows these results to be consistent with those of Example 2. Two (-1) phase factors come up in this calculation one from the $-\mathbf{R}(0\ -270°\ 0)$ and one because the resultant arrow in Fig. 8(b) crosses the center line. (This detail is explained in Sec. IV of article I.) Therefore the final phase is just $\Phi_z = \gamma_z/2 = -61°$.

There exist many different types of polarization problems that can be visualized and solved on the slide rule. The ability to see graphically where a particular state or operator solution is going is the main advantage of the analog device. For this reason it may serve as a qualitative as well as a quantitative design tool. The peculiar topology of rotations is built into it.

Finally we note that Poincaré[10] developed an alternative description of polarization wherein he associated each state with the stereographic image of its Stokes vector in a plane tangent to the Stokes sphere shown in Fig. 4 of article I. (Sometimes this sphere is called Poincaré's sphere.) The fact that the upper scale of the slide rule is a Poincare plane may be useful for some other polarization problems such as those involving interference.

VI. PERFECT POLARIZER CALCULATIONS USING THE SLIDE RULE

A perfect polarizer is a medium which allows one type of polarization state $|p\rangle$ to pass through without any absorption while the orthogonal state $|b\rangle$ ($\langle p|b\rangle = 0$) is completely blocked or absorbed. Incoming light is state

$$|\Psi\rangle = |p\rangle\langle p|\Psi\rangle + |b\rangle\langle b|\Psi\rangle \tag{6.1}$$

comes out, if it comes out at all, entirely in state $|p\rangle$ with amplitude $\langle p|\Psi\rangle$ or intensity $|\langle p|\Psi\rangle|^2$.

It is convenient to represent such a perfect polarizer graphically by an axis through the Stokes sphere (See Fig. 4(b)) colinear with the oppositely pointing Stokes vectors belonging to $|p\rangle$ and $|b\rangle$, respectively. We shall call the $\mathbf{S}(p)$ and $\mathbf{S}(b)$ Stokes vectors the "passing" and "blocking" ends of the axis. Then it follows that the transmission probability or intensity for Ψ-polarized light through this polarizer is

$$|\langle p|\Psi\rangle|^2 = \cos^2\beta/2, \tag{6.2}$$

where β is the angle between the Stokes vector $\mathbf{S}(\Psi)$ of $|\Psi\rangle$ and t the vector $\mathbf{S}(p)$. If $|p\rangle$ is $|r\rangle$ or $|x\rangle$ this result follows directly from Eq. (5.6) or (5.7). It is easy to see that the proof for a general $|p\rangle$ would give the same thing.

The problem of finding β for any given polarizer and any given incoming state is easily solved on the slide rule. First the positions of $\mathbf{S}(\Psi)$ and $\mathbf{S}(p)$ are plotted using the ϕ and θ scales. [$-\mathbf{S}(\Psi)$ and/or $\mathbf{S}(b)$ may be plotted instead; whichever appears on the upper hemisphere of the Stokes sphere.] Then the slide rule is turned until an ω arc connects the two points, and the included angle can be read using the scale of this arc. We must divide the ω reading by 2 since its scale is twice the actual arc. The result is β if it is between $S(\Psi)$ and $S(p)$ or between $-S(\Psi)$ and $S(b)$. Otherwise it is the supplementary angle $(\pi - \beta)$ that is found.

For the sake of completeness we review some of the polarization problems which are not solved so easily using the geometrical methods. The first of these concerns the treatment of partially polarized beams.

Suppose someone makes a beam of light by mixing the outputs of many different perfect polarizers so the beam has a fraction f_a of photons in state $|a\rangle, f_b$ in $|b\rangle, f_c$ in $|c\rangle, \cdots$ and so on, where $f_a + f_b + f_c + \cdots = 1$. Now when we look at the beam, the probability of being hit by one of the a-polarized photons is f_a and so on for $b, c, \cdots$ but, if the mixing has been done well enough there is no way to make the separate $|a\rangle$, $|b\rangle$, or $|c\rangle$, beams come out again. If a perfect p polarizer is inserted into the beam, then all the different parts will, in general, contribute to the output intensity or probability

$$I(p) = f_a |\langle p|a\rangle|^2 + f_b |\langle p|b\rangle|^2 + \cdots. \quad (7.1)$$

Now there is a nice way to describe, mathematically, beams of light that are, shall we say, "messed up." This is an important thing since unless you always use expensive polaroid sunglasses most of the light you see falls into this category. Furthermore, the general formulation has become a very important tool for quantum theory. Rewriting Eq. (7.1) we have

$$\begin{aligned} I(p) = f_a \langle p|a\rangle\langle a|p\rangle + f_b \langle p|b\rangle\langle b|p\rangle \\ + f_c \langle p|c\rangle\langle c|p\rangle + \cdots = \langle p|\rho|p\rangle, \quad (7.2a) \end{aligned}$$

where

$$\rho = f_a|a\rangle\langle a| + f_b|b\rangle\langle b| + f_c|c\rangle\langle c| + \cdots \quad (7.2b)$$

is called the density operator. This operator describes the "state" of the beam, i.e., the condition of an ensemble of photons. Note if the beam is completely "pure", i.e., 100% polarized in state q, say, the operator is written as a single projection operator

$$\rho = |q\rangle\langle q|. \quad (7.3)$$

The operator description of a state incorporates the probabilities associated with a "messing up" along with the "unavoidable" quantum mechanical probabilities.

Operator ρ can be expanded in the usual ways in terms of elementary operators

$$\rho = \sum_{i=1}^{2} \sum_{j=1}^{2} |i\rangle\langle i|\rho|j\rangle\langle j| = \sum \sum \rho_{ij} |i\rangle\langle j|$$

using any basis $\{|1\rangle, |2\rangle\}$ such as $\{|r\rangle, |l\rangle\}$ or $\{|x\rangle, |y\rangle\}$, where the

$$\rho_{ij} = \langle i|\rho|j\rangle$$

are the components of the density matrix for that basis. Another way to write ρ, which is easier to interpret physically, involves an expansion in terms of the Pauli spinor operators [Eq. (4.2)].

$$\rho = (\rho_1 \mathbf{1} + \rho_A\sigma_A + \rho_B\sigma_B + \rho_C\sigma_C)/2.$$

Let us represent this expansion in the $\{|r\rangle, |l\rangle\}$ basis.

$$\begin{pmatrix} \rho_{rr} & \rho_{rl} \\ \rho_{lr} & \rho_{ll} \end{pmatrix} = \frac{\rho_1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\rho_A}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\rho_B}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{\rho_C}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For a pure state [Eq. (5.6)] represented in this basis

$$|\Psi\rangle = (e^{-i\alpha/2}\cos\beta/2|r\rangle + e^{i\alpha/2}\sin\beta/2|l\rangle)e^{-i\gamma/2}$$

the density matrix assumes the form:

$$|\Psi\rangle\langle\Psi| \rightarrow \begin{pmatrix} e^{-i\alpha/2}\cos\beta/2 \\ e^{i\alpha/2}\sin\beta/2 \end{pmatrix} \otimes (e^{i\alpha/2}\cos(\beta/2) e^{-i\alpha/2}\sin(\beta/2)) = \begin{pmatrix} \cos^2(\beta/2) & e^{-i\alpha}\cos(\beta/2)\sin(\beta/2) \\ e^{i\alpha}\cos(\beta/2)\sin(\beta/2) & \sin^2(\beta/2) \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1+\cos\beta}{2} & \dfrac{\cos\alpha\sin\beta - i\sin\alpha\sin\beta}{2} \\ \dfrac{\cos\alpha\sin\beta + i\sin\alpha\sin\beta}{2} & \dfrac{1-\cos\beta}{2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\cos\alpha\sin\beta}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\sin\alpha\sin\beta}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{\cos\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (7.4)$$

We see that the components ρ_A, ρ_B, and ρ_C are precisely those of the Stokes vector in ABC space for a normalized ($I = \frac{1}{2}$) state Ψ. [See Fig. 9(b).] Parameter $\rho_1 = \frac{1}{2}$ appears to be redundant here, but it will not be so for a more or less "messed up" beam. In fact one may define a messed up beam to be one for which

$$\rho_1^2 > \sum_{\mu=ABC} |\rho_\mu|^2.$$

For a completely random beam $\rho_\mu \rightarrow 0$ for $\mu = A, B$, and C. It is common to define the degree of polarization by

$$D = \sum_{\mu} |\rho_\mu|^2/\rho_1^2$$

and it varies between 0 and 1.

The manipulation of the four components ($\rho_1\ \rho_A\ \rho_B\ \rho_C$) of the density operator is known as Mueller calculus.[11] The 4×4 matrices which express the effect of polarizers and optical retarders are called Mueller matrices, and are used in very much the same way as Jones matrices. The two methods complement each other: Jones calculus cannot treat "messed up" beams quantitatively, while Mueller calculus contains no information about the overall phase of the state or beam.

Many other intriguing problems exist which are related to or are generalizations of the preceding analysis of polarization and the two-level system. One problem involves the effects of various kinds of damping, i.e., the effects arising from non-Hamiltonian susceptibility or K matrices. This is very important in various spin resonance processes, spin or photon echo effects, and self-induced transparency.[12] Last but not least, there is the theory of the three-level system, or for that matter, the n-level system or the n-

dimensional oscillator. Fano[13] has given an excellent physical picture for the $J = 1$ or three level system but it is still not clear how to generalize this for n levels.

APPENDIX A

In Sec. IV we require the square root and an exponential function of the matrix

$$\frac{M}{10} = \begin{pmatrix} 18 & -9+9i \\ -9-9i & 27 \end{pmatrix} \Big/ 10. \tag{A1}$$

These functions may be calculated using the spectral decomposition methods described in Appendix B of the preceding article I. The square root of matrix M is given by the following:

$$M^{1/2} = m_1^{1/2} P_1 + m_2^{1/2} P_2, \tag{A2}$$

where m_1 and m_2 are the eigenvalues of M, and $P_{1,2}$ are given by Eq. (B2) in article I. The secular equation of the matrix M in (A1) is

$$\det\begin{vmatrix} 18-m & -9+9i \\ -9-9i & 27-m \end{vmatrix} = 0 = m^2 - 45m + 324.$$

The eigenvalue solutions are $m_1 = 9$ and $m_2 = 36$. Substituting these in the formulas for P_i, we obtain for (A2) the following:

$$M^{1/2} = (9)^{1/2} \begin{pmatrix} 2 & 1-i \\ 1+i & 1 \end{pmatrix} \Big/ 3 + (36)^{1/2} \begin{pmatrix} 1 & -1+i \\ -1-i & 2 \end{pmatrix} \Big/ 3. \tag{A3}$$

For the pair of roots $(9^{1/2}, 36^{1/2})$ we could use $(\pm 3, \pm 9)$ or else $(\mp 3, \pm 9)$, i.e., there are four solutions for $M^{1/2}$. To obtain Eq. (4.22) we take all root signs to be positive.

To obtain the exponential form required in Eq. (4.24), we use the same P_i matrices. Positive roots are chosen here, too.

$$\begin{aligned} e^{iKz} &= \exp[i(M/10)^{1/2}z] = \exp[i(m_1/10)^{1/2}z]P_1 + \exp[i(m_2/10)^{1/2}z]P_2 \\ &= \exp[i3z/(10)^{1/2}] \begin{pmatrix} 2 & 1-i \\ 1+i & 1 \end{pmatrix} \Big/ 3 \\ &\quad + \exp[i6z/(10)^{1/2}] \begin{pmatrix} 1 & -1+i \\ -1-i & 2 \end{pmatrix} \Big/ 3. \end{aligned}$$

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[1]Many references exist that have been written since Edwin H. Land did his first pioneering work in this area. We quote two which contain extensive discussions and bibliographies for theory and experiment: W. A. Shurcliff, *Polarized Light* (Harvard U. P., Cambridge, MA 1966), and *Polarized Light-AAPT Selected Reprints* (American Institute of Physics, New York, 1963).

[2]G. Baym, *Lectures in Quantum Mechanics* (Benjamin, New York, 1970).

[3]U. Fano and L. Fano, *Physics of Atoms and Molecules* (University of Chicago, Chicago, 1972).

[4]R. C. Jones, J. Opt. Soc. Am. **46**, 126 (1956). See also Ref. 1.

[5]The circular slide rule and all applications described in this article are the sole invention of William G. Harter for whom patent applications are being made.

[6]U. Fano, J. Opt. Soc. Am. **39**, 859 (1949).

[7]H. Mueller, J. Opt. Soc. Am. **38**, 661 (1948).

[8]G. Stokes, Proc. R. Soc. (London) **11**, 545 (1862). See also Ref. 1.

[9]See Ref. 4.

[10]H. Poincaré, *Theorie Mathematique de la Lumiere* (Gauthiers Villars, Paris, 1892). See also Ref. 1.

[11]See Ref. 1.

[12]J. I. Steinfeld, *Molecules and Radiation* (Harper and Row, New York, 1974), pp. 251–296.

[13]U. Fano, in *Spectroscopic and Group Theoretical Methods in Physics: Racah Memorial Volume*, edited by F. Bloch *et al.* (North-Holland, Amsterdam, 1968), p. 153.