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ON A PROOF BY PETROV OF THE STABILITY OF PLANE COUETTE FLOW AND PLANE POISEUILLE FLOW*

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Summary. A proof by Petrov [10] that plane Couette flow and plane Poiseuille flow are always stable with respect to infinitesimal disturbances is examined and found to be based on a set of incomplete functions, thus rendering the proof invalid.

1. Introduction. Petrov [10] gave a proof of the stability for all Reynolds numbers of both plane Couette flow and plane Poiseuille flow with respect to infinitesimal disturbances. Since then both problems have received considerable attention, and while these more recent results tend to confirm the belief that plane Couette flow is always stable (Grohne [5], Gallagher and Mercer [3], [4], Deardorff [2]), regions of instability have been found for the plane Poiseuille problem (Lin [6], Thomas [12], Grohne [5], Shen [11], Nachtsheim [8]). However, Pekeris [9] also concluded that plane Poiseuille flow is always stable, but it is believed that his proof is not valid because it employs a series representation which does not apply to unstable motions. In view of these results it was concluded that Petrov's proof is defective. This was indeed found to be the case. For ease of reference we use the same notation as Petrov.

As is well known, the problem of the stability of plane parallel flows of a viscous and incompressible fluid with respect to infinitesimal disturbances of the form $\phi(y) \exp \{i\alpha(x - ct)\}$ is reduced to the problem of finding the eigenvalues of the Orr–Sommerfeld equation

$$\phi^{(iv)} - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha R[(u - c)(\phi'' - \alpha^2\phi) - u''\phi] \tag{1}$$

with $\phi(0) = \phi'(0) = \phi(1) = \phi'(1) = 0$, where $u(y)$ is the velocity profile and $c = c_r + ic_i$ is the eigenvalue, R being the Reynolds number. If $c_i < 0$, the disturbance is damped and amplified if $c_i > 0$; while if $c_i = 0$, the motion is said to be neutrally stable.

2. The method of Petrov. Putting $f = \phi'' - \alpha^2\phi$ in (1), the equation becomes

$$f'' - \alpha^2 f = i\alpha R\{(u - c)f - u''\phi\}. \tag{2}$$

Essentially, Petrov's method is to expand the function f in a series of orthogonal functions f_s. These are defined as follows.

Let $\phi_s, \mu_s, s = 1, 2, 3, \cdots$, be the eigenvectors and eigenvalues of the differential equation

$$\phi^{(iv)} - 2\alpha^2\phi'' + \alpha^4\phi + \mu(\phi'' - \alpha^2\phi) = 0$$

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with $\phi(0) = \phi'(0) = \phi(1) = \phi'(1) = 0$. These are found to be

$$\phi_s = (\cos k_s y - \cosh \alpha y) + \beta_s\left(\sin k_s y - \frac{k_s}{\alpha}\sinh \alpha y\right), \tag{3}$$

where

$$\beta_s = \frac{-\alpha(\cos k_s - \cosh \alpha)}{\alpha \sin k_s - k_s \sinh \alpha}, \qquad k_s = \sqrt{\mu_s - \alpha^2}.$$

It can be shown that $\mu_s \geqq 0$. The k_s are determined from the equation

$$\sin k_s = \frac{2\alpha}{k_s}\left[\frac{\cosh \alpha \cos k_s - 1}{(\alpha^2/k_s^2 - 1)\sinh \alpha}\right].$$

The functions f_s are defined by $f_s = \phi_s'' - \alpha^2\phi_s$. These are found to satisfy the following relationships:

$$\begin{aligned} &\int_0^1 \phi_s f_k \, dy = \int_0^1 f_s f_k \, dy = \int_0^1 f_s(f_k'' - \alpha^2 f_k)\, dy = 0, \qquad s \neq k, \\ &\int_0^1 f_s(f_s'' - \alpha^2 f_s)\, dy = -\mu_s \| f_s\|^2, \end{aligned} \tag{4}$$

where $\| f_s\|^2 = \int_0^1 f_s^2 \, dy$.

Multiplying (2) by $\bar{f}$, integrating and making the substitution $f = \sum_{s=1}^{\infty} b_s f_s$, one obtains

$$-\sum_{s=1}^{\infty} \mu_s \cdot \| f_s\|^2 \cdot |b_s|^2 = \alpha R c_i \int_0^1 f\bar{f}\, dy + i\alpha R \int_0^1 \frac{u'''}{2}(\bar{\phi}'\phi - \phi'\bar{\phi})\, dy \tag{5}$$

on integrating by parts and taking the real part. For plane Couette flow and plane Poiseuille flow $u''' = 0$ so that (5) yields $c_i < 0$ for all values of α and R.

It is obvious that the above formal procedure can only be justified if the $\{f_s\}$ form a complete set in the space under discussion. We shall show in the next section that the set is incomplete in the complex Hilbert space $L^2(0, 1)$, where $L^2(0, 1)$ denotes the space of square summable functions (in the Lebesgue sense) defined on the interval (0, 1). This of course means that any proof which uses these functions is invalid unless it is proved that they are complete with respect to the subspace of $L^2(0, 1)$ under discussion. Thus it is possible that the set is complete with respect to the subspace generated by the solutions of (1) in the plane Couette flow case. In order to explain the discrepancy discussed in § 1 it would be more satisfactory to prove that the set of functions is incomplete with respect to the subspace of $L^2(0, 1)$ generated by the solutions of (1) in the plane Poiseuille problem. Both of these more difficult problems are under investigation.

3. Incompleteness of the set $\{f_s\}$. A necessary and sufficient condition for the completeness of an orthonormal set in $L^2(0, 1)$ is given by Parseval's equality [1, p. 16],

$$\sum_{s=1}^{\infty} \left[\int_0^1 \frac{f_s \bar{g}}{\| f_s \|} \, dy \right]^2 = \int_0^1 |g|^2 \, dy \tag{6}$$

for all functions g in $L^2(0, 1)$. Here $\| f_s \| = \left[\int_0^1 f_s^2 \, dy \right]^{1/2}$ is the norm of f_s (note that the f_s are real).

If $\alpha = 0$, we take $g(y) = 1, 0 \leqq y \leqq 1$, and obtain the following necessary condition for completeness:

$$\sum_{s=1}^{\infty} \frac{\left[\int_0^1 f_s \, dy \right]^2}{\| f_s \|^2} = 1. \tag{7}$$

But $f_s = \phi_s''$ for $\alpha = 0$, so that $\int_0^1 f_s \, dy = \phi_s'(1) - \phi_s'(0) = 0$ by the end conditions on ϕ_s. Hence (7) is not satisfied in this case.

If $\alpha \neq 0$, we take g as any of the ϕ_k. Using (4) we obtain, from (6),

$$\frac{\left[\int_0^1 f_k \phi_k \, dy \right]^2}{\| f_s \|^2} = \int_0^1 \phi_k^2 \, dy.$$

Hence

$$\left[\int_0^1 f_k \phi_k \, dy \right]^2 = \int_0^1 f_k^2 \, dy \int_0^1 \phi_k^2 \, dy.$$

This is Schwarz's inequality with equality sign, and so f_k and ϕ_k are linearly dependent [1, p. 2]. Hence $f_k = c\phi_k$ where c is a nonzero constant. Replacing f_k by $\phi_k'' - \alpha^2 \phi_k$ we obtain $\phi_k'' = (c + \alpha^2)\phi_k$, which from (3) easily yields the contradiction that c is zero.

Hence the set $\{f_s\}$ is incomplete in $L^2(0, 1)$ for all values of α.

REFERENCES

[1] N. I. AKHIEZER AND I. M. GLAZMAN, *Theory of Linear Operators in Hilbert Space*, Frederick Ungar, New York, 1961.

[2] J. W. DEARDORFF, *On the stability of viscous plane Couette flow*, J. Fluid Mech., 15 (1963), pp. 623–631.

[3] A. P. GALLAGHER AND A. McD. MERCER, *On the behaviour of small disturbances in plane Couette flow*, Ibid., 13 (1962), pp. 91–100.

[4] ———, *On the behaviour of small disturbances in plane Couette flow. Part 2. The higher eigenvalues*, Ibid., 18 (1964), pp. 350–352.

[5] D. GROHNE, *Über das Spektrum bei Eigenschwingungen ebener Laminarströmungen*, Z. Angew. Math. Mech., 34 (1954), pp. 344–357.

[6] C. C. LIN, *On the stability of two-dimensional parallel flows*, Proc. Nat. Acad. Sci. U.S.A., 30 (1944), pp. 316–323.

[7] ———, *The Theory of Hydrodynamic Stability*, Cambridge, at the University Press, 1955.

[8] P. R. NACHTSHEIM, *An initial value method for the numerical treatment of the Orr-Sommerfeld equation for the case of plane Poiseuille flow*, Tech. Note D-2414, National Aeronautics and Space Administration, 1964.

[9] C. L. PEKERIS, *Stability of the laminar parabolic flow of a viscous fluid between parallel fixed walls*, Phys. Rev. (2), 74 (1948), pp. 191–199.

[10] G. I. PETROV, *The application of Galerkin's method to the problem of the steadiness of the flow of a viscous liquid*, Prikl. Mat. Meh., 4 (1940), pp. 3–11.

[11] S. F. SHEN, *Calculated amplified oscillations in plane Poiseuille and Blasius flows*, J. Aeronaut. Sci., 21 (1954), pp. 62–64.

[12] L. H. THOMAS, *On the stability of plane Poiseuille flow*, Phys. Rev. (2), 91 (1953), pp. 780–783.