A simpler derivation of Feigenbaum's renormalization group equation for the period-doubling bifurcation sequence

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One interesting and important property of nonlinear dynamical systems is that they can exhibit universality—behavior that is quantitatively identical for a broad class of systems. The first and most famous example of universality in a dynamical system was identified by Feigenbaum [M. J. Feigenbaum, J. Stat. Phys. **19**, 25–52 (1978), **21**, 669–706 (1979)] in the period-doubling route to chaos. This note presents a new derivation of Feigenbaum's renormalization group equation, used to understand this universality. The argument, designed for incorporation into an undergraduate dynamical systems course, is simpler than those in standard textbooks.

I. INTRODUCTION

The subject of dynamical systems is attractive to teach to undergraduates because students with only a moderate background in physics and mathematics can explore and understand many interesting phenomena.[1] One important unifying concept in the subject, also important in many other areas of physics such as statistical mechanics and particle physics, is that of universality, whereby a whole class of different systems have certain properties which are *quantitatively* identical.[2] Renormalization group equations can be used to understand why universality arises. This note outlines a derivation of the normalization group equation for the period-doubling route to chaos which is substantially simpler than standard treatments.[3–5] The derivation of the renormalization group equation requires only elementary mathematics and is suitable for incorporation into an undergraduate course on dynamical systems.

Section II contains review material that is covered elsewhere[1,3–5] and is included here to make the discussion reasonably self-contained. Readers familiar with this material may skip to Sec. III.

II. BACKGROUND

A dynamical system is a rule that, given a set of variables describing a configuration at some time, gives the values of these variables at a later time. (One example of such rules is Newton's laws, which enable the determination of the velocities and positions of a set of particles, given their initial values and the force law.) Here the focus is on dynamical systems that are one-dimensional maps, defined by equations of the form:

$$x_{j+1}=f_\lambda(x_j). \tag{1}$$

The subscript on f denotes that it depends on a control parameter λ. Given an initial value x_0 of the variable x, one uses Eq. (1) to generate the sequence of values $\{x_1,x_2,x_3,\ldots\}$. It is natural to interpret the index j as discrete time. We will be studying the properties of these sequences at long times, after initial transients have decayed, and examine how the behavior changes as λ is varied.

Our specific example is the logistic map, which has $f_\lambda(x)=\lambda x(1-x)$, so that the map is defined by

$$x_{j+1}=\lambda x_j(1-x_j). \tag{2}$$

The logistic map (as well as many other dynamical systems) has an infinite sequence of period doublings as the control parameter λ is increased. We first review what period doubling is, and then discuss its universal properties.

For $\lambda<3$, at long times the sequence generated by the logistic map settles down to a single value x^*. Since the value of x repeats each time j is incremented $[f_\lambda(x^*)=x^*]$, the orbit has period one. When $3<\lambda<1+\sqrt{6}\approx 3.449$, at long times x alternates between two values x_1 and x_2 $[f_\lambda(f_\lambda(x_1))\equiv f_\lambda^2(x_1)=x_1;\ f_\lambda^2(x_2)=x_2]$; the period of the orbit has doubled and is now two. At $\lambda=1+\sqrt{6}$ the period of the orbit doubles again, to four. As one continues to increase λ, the period of the observed orbit doubles over and over. An infinite number of period doublings occur as λ is increased up to the value $\lambda=\lambda_\infty=3.569\,934\,669\ldots$; at $\lambda=\lambda_\infty$ the orbit has infinite period, and for λ just above λ_∞ the motion is chaotic (aperiodic, and very sensitive to small perturbations).

Feigenbaum[6] observed that the range in λ over which an orbit of length 2^n is observed, $\Delta\lambda_n$, shrinks geometrically with n when n is large:

$$\frac{\Delta\lambda_{n-1}}{\Delta\lambda_n}=\delta, \tag{3}$$

with $\delta=4.6692\ldots$. Moreover, he examined several different map functions and found for all of them that the range of control parameter as a function of n obeys Eq. (3) with the *same* value of δ.

Feigenbaum also showed that the values of x on the orbits have universal properties.[6] Consider the orbits of length 2^n in the period-doubling sequence which include the value $x=\frac{1}{2}$ (at which the function reaches its maximum value). Let λ_n be the parameter value where such an orbit of length 2^n is observed. Given a cycle of length 2^n starting at $x_0=\frac{1}{2}$, then the value of x exactly halfway through the cycle, $x_{2^{n-1}}\equiv f_{\lambda_n}^{2^{n-1}}(x=\frac{1}{2})$ (which is closer to $\frac{1}{2}$ than any other point on the orbit), also converges geometrically with n. Specifically,

$$\frac{\frac{1}{2}-f_{\lambda_n}^{2^{n-1}}(x=\frac{1}{2})}{\frac{1}{2}-f_{\lambda_{n+1}}^{2^n}(x=\frac{1}{2})}\approx-\alpha, \tag{4}$$

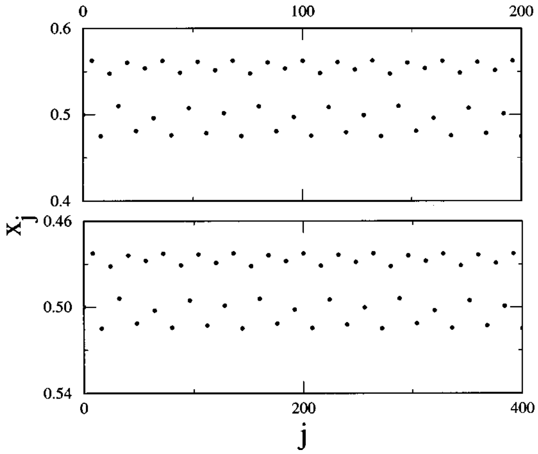

Fig. 1. Two plots of the time series of x values for the logistic map ($x_{j+1}=\lambda x_j(1-x_j)$) with the parameter value $\lambda=\lambda_\infty=3.569\,945\,669...$, starting with $x_{j=0}=\frac{1}{2}$. Note that the axes have different scales in the lower panel than in the upper panel: the horizontal axis is compressed by a factor of 2, and the vertical axis is inverted and expanded by a factor of 2.5.

where $\alpha=2.502\,907\,875...$. Once again, many different map functions display this scaling with exactly the same value of α.

Feigenbaum[7] demonstrated that the fact that the two exponents α and δ are universal over a large class of systems[8,9] is intimately related to the existence of a universal function $g(z)$ which satisfies

$$g(z)=-\alpha g(g(z/\alpha)). \tag{5}$$

The universality of the exponents follows because whenever g has a quadratic maximum $[g(z)=a_0-a_2z^2+\cdots]$, this equation determines α and δ uniquely.

Feigenbaum derived Eq. (5) by defining an infinite sequence of functions, with $g(x)$ defined as a limit of this sequence. In his method, the function g is obtained as the limit of an infinite number of functional compositions. This method, which is that presented in most standard texts,[3–5] is complicated by the need to introduce an infinite set of auxiliary functions.

III. DERIVATION

Our derivation of Eq. (5) uses just the properties of the time series generated by iterating the map at $\lambda=\lambda_\infty$:

$$x_{j+1}=\lambda_\infty x_j(1-x_j), \tag{6}$$

with $x_0=\frac{1}{2}$.

Figure 1 shows two different graphs of the same sequence of $\{x_j\}$'s generated by the logistic map with $\lambda\cong\lambda_\infty$, starting at $x_0=\frac{1}{2}$. Note that the axes on the two graphs are different: Compared to Fig. 1(a), the (horizontal) j axis of Fig. 1(b) is compressed by a factor of 2, and the (vertical) x axis is inverted and magnified by a factor of roughly 2.5.

Figure 1(b) looks a lot like Fig. 1(a). In fact, if the j axis of the time series plot is compressed by a factor of 2, and if the x_j axis of the plot is inverted about $x=\frac{1}{2}$ and then blown up by the factor $\alpha\sim 2.502\,907\,875...$, then all the points on the rescaled plot can be superimposed directly onto those in the original graph.[10] In other words, if one labels the points which appear in Fig. 1(a) by an index k (i.e., ignoring those iterates which are too far from $x=\frac{1}{2}$ to appear on the graph), so that the plot is of a sequence $\{x_{k=0},x_{k=1},x_{k=2},\ldots\}$, then every x_k in the sequence satisfies:

$$-\alpha(x_{2k}-\tfrac{1}{2})=x_k-\tfrac{1}{2}. \tag{7}$$

This equation embodies the empirical observation that this time series looks the same when it is rescaled appropriately.

The renormalization group equation is obtained by looking for a function g that generates a time series that has the self-similarity property embodied in Eq. (7). We define $z_k=x_k-\frac{1}{2}$ and rewrite Eq. (7) as

$$-\alpha z_{2k}=z_k. \tag{8}$$

Equation (8) holds for all k. In particular, it holds if we replace k with $k+1$:

$$-\alpha z_{2(k+1)}=z_{k+1}. \tag{9}$$

We are looking for a mapping function g that generates this sequence via $z_{k+1}=g(z_k)$. Thus, $z_{2k+2}=g(g(z_{2k}))$, and we can rewrite Eq. (9) as

$$-\alpha g(g(z_{2k}))=g(z_k). \tag{10}$$

Using Eq. (8), we have

$$-\alpha g(g(-z_k/\alpha))=g(z_k). \tag{11}$$

Thus the function g must satisfy

$$-\alpha g(g(-z/\alpha))=g(z). \tag{5$'$}$$

Note that the value of α is never specified in the derivation of the renormalization group equation (5′). Solving Eq. (5′) under quite general conditions determines α and δ; this fact underlies universality.

IV. SOLVING THE RENORMALIZATION GROUP EQUATION

Reference 3 presents, at a level appropriate for undergraduates, methods for solving the renormalization group (RG) equation (5′) and obtaining g, α, and δ. The standard method for obtaining g and α[7,11] is to expand g in a Taylor series about its maximum at $z=0$ and equate coefficients (the assumption that a Taylor series expansion exists is where the restriction is made to functions with quadratic maxima). The standard method of obtaining δ[7,11,3] involves solving a functional eigenvalue equation, a method not accessible to most undergraduates. Hilborn[3] presents a treatment that retains the spirit of the standard treatment, and which does not assume previous knowledge of linear algebra. Nonetheless, calculating δ is substantially more advanced than the rest of the presentation.[12]

V. REMARKS

This note presents a derivation of Feigenbaum's renormalization group equation for the period-doubling route to chaos that is considerably simpler than that in standard texts. The spirit of the derivation presented here is identical to that used to derive renormalization group equations for second-order phase transitions.[13] In fact, the method presented here for period-doubling in $1-d$ maps is directly analogous to the decimation renormalization group for the one-dimensional Ising model.[14]

The similarity of our derivation and standard applications of RG methods in statistical mechanics clarifies the connection between the renormalization group as applied to dynamical systems and to statistical mechanics. Since one does not need to have a background in statistical mechanics to study the universal behavior of iterated maps, even students with no knowledge of statistical mechanics can learn the fundamentals of universality and renormalization.

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[1]An extensive and very useful list of resources available for teaching courses in nonlinear dynamics is Robert C. Hilborn and Nicholas B. Tufillaro, ''Resource Letter ND-1: Nonlinear dynamics,'' Am. J. Phys. **65**, 822–834 (1997).

[2]Universality is an important concept in other areas of physics, including particle physics and statistical mechanics. See, e.g., Michael E. Peskin, *An Introduction to Quantum Field Theory* (Addison–Wesley, Reading, MA, 1995); Nigel Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group*, Frontiers in Physics Vol. 85 (Addison–Wesley, Reading, MA, 1992); Shang-Keng Ma, *Modern Theory of Critical Phenomena*, Frontiers in Physics Vol. 46 (Addison–Wesley, Reading, MA, 1976).

[3]R. C. Hilborn, *Chaos and Nonlinear Dynamics: An Introduction for Scientists and Engineers* (Oxford U.P., New York, 1994).

[4]E. Ott, *Chaos in Dynamical Systems* (Cambridge U.P., New York, 1993).

[5]S. H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications in Physics, Biology, Chemistry, and Engineering* (Addison–Wesley, Reading, MA, 1994).

[6]M. J. Feigenbaum, ''Quantitative universality for a class of nonlinear transformations,'' J. Stat. Phys. **19**, 25–52 (1978).

[7]M. J. Feigenbaum, ''The universal metric properties of nonlinear transformations,'' J. Stat. Phys. **21**, 669–706 (1979).

[8]The exponents α and δ are the same for all sequences $\{x_j\}$ generated by the rule $x_{j+1}=f(x_j)$, so long as $f(z)$ has a quadratic maximum about which it can be expanded in a Taylor series. Moreover, the exponents α and δ are the same for an even broader class of dynamical systems, including experimental systems such as some Rayleigh–Benard convection cells.

[9]See, e.g., A. Libchaber, C. Laroche, and S. Fauve, ''Period doubling cascade in mercury, a quantitative measurement,'' J. Phys. Lett. **43**, L211–216 (1982); M. Giglio, S. Musazzi, and U. Perini, ''Transition to chaotic behavior via a reproducible sequence of period-doubling bifurcations,'' Phys. Rev. Lett. **47**, 243–246 (1981).

[10]It is a recommended exercise that students verify this claim for themselves. Strictly speaking, scale invariance holds only in the limit $x\rightarrow\frac{1}{2}$, but empirically one finds that the two graphs in Fig. 1 superimpose quite accurately.

[11]M. J. Feigenbaum, ''Universal behavior in nonlinear systems,'' Los Alamos Science **1**, 4–27 (1980), reprinted in P. Cvitanovic, ed., *Universality in Chaos* (Hilger, New York, 1989), 2nd ed.

[12]In the undergraduate nonlinear dynamics course taught at the University of Chicago, the determination of α is a homework problem, whereas determining δ is a longer-term student project.

[13]K. G. Wilson, ''Problems in physics with many scales of length,'' Sci. Am. **241**, 158–179 (August, 1979).

[14]D. Nelson and M. Fisher, ''Soluble renormalization groups and scaling fields for low-dimensional Ising systems,'' Ann. Phys. (N.Y.) **91**, 226–274 (1975); Humphrey J. Maris and Leo P. Kadanoff, ''Teaching the renormalization group,'' Am. J. Phys. **46**, 652–657 (1978).

FUNDING RESEARCH

The problem facing science is not... that the reductionist imperative is putting the rest of science at risk. Few if any of us who are interested in the search for the laws of nature doubt the validity of the other motives for research. (I suspect that eventually I will come to feel that research on cancer or heart disease is more important than anything else.) The problem is that some people, including some scientists, deny that the search for the final laws of nature has its own special sort of value, a value that *also* should be taken into account in deciding how to fund research.

Steven Weinberg, ''Reductionism Redux,'' The New York Review of Books, 5 October 1995, pp. 39–42.

ETHICAL SCRUTINY

The notion that science should be called to account for itself ethically has become commonplace in the biological sciences; during the last two decades we have seen a proliferation of bioethics centers. If biological science should be subject to ethical scrutiny, why not physics?

Margaret Wertheim, *Pythagoras' Trousers—God, Physics, and the Gender Wars* (Random House, New York, 1995), p. 251.