5.2. Definition of a Lie Group; With Examples

A Lie group is a special kind of continuous group. The group elements $R(a)$ are labelled by r real parameters $a^1, a^2, \ldots, a^r$,

$$R(a) = R(a^1, a^2, \ldots, a^r) . \tag{5-19}$$

The parameters a^ρ may vary over a finite or an infinite range. The space of the r parameters is called the *group-parameter space.* A group G is called a Lie group of *order* r if $R(a)$ obeys the following five postulates:

1. The identity element $R(a_0)$ exists, that is,

$$R(a_0)R(a) = R(a)R(a_0) = R(a), \quad \text{for any } R(a) \in G . \tag{5-20}$$

The parameters a_0 of the identity element are usually taken as zero, that is, $R(a_0) = R(0)$.

2. For any a we can find $\overline{a}$ such that

$$R(\overline{a})R(a) = R(a)R(\overline{a}) = R(0) ,$$

i.e., for every $R(a)$ an inverse exists:

$$R(\overline{a}) = R^{-1}(a) . \tag{5-21}$$

3. For given parameters a and b, we can find c in the set of parameters such that

$$R(c) = R(b)R(a) , \tag{5-22}$$

where the parameters c are real functions of the real parameters a and b,

$$c = \varphi(a, b) . \tag{5-23}$$

Equation (5-23) is called the *combination law of group parameters* and tells us that the group is closed.

4. Associativity.

$$R(a)[R(b)R(c)] = [R(a)R(b)]R(c) ,$$
$$\varphi(\varphi(c, b), a) = \varphi(c, \varphi(b, a)) . \tag{5-24}$$

5. The parameters c in (5-23) are analytic functions of a and b, and the $\overline{a}$ in (5-21) are analytic functions of a.

A Lie group is said to be *compact* if its parameters are bounded.

Example 1: The real linear transformation group $GL(2, R)$ in two-dimensional space

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad R(a) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} . \tag{5-25}$$

The collection of all 2×2 nonsingular matrices $R(a)$ forms a real linear transformation group under matrix multiplication. Its elements are labelled by four real parameters $(a_{11}, a_{12}, a_{21}, a_{22})$. The order of $GL(2, R)$ is therefore four. If we restrict ourselves to the transformations with $\det(R(a)) = 1$, we obtain a subgroup of $GL(2, R)$, called the *special real linear transformation group* of dimension two, and denoted by $SL(2, R)$.

Example 2: The complex linear transformation group $GL(2, C)$ in two-dimensional space. If the parameters a in (5-25) are allowed to be complex, $R(a)$ form a complex linear transformation group of dimension 2. Let $a_{kl} = b_{kl} + ic_{kl}$, where b_{kl} and c_{kl} are real. Therefore its elements are characterized by eight real parameters, $a^1 = b_{11}, \ldots, a^8 = c_{22}$, and the order of $GL(2, C)$ is therefore eight.

Example 3: The group SU_2. If the matrices in (5-25) are unitary, that is,

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} , \tag{5-26a}$$

$$U^\dagger U = 1 , \tag{5-27}$$

then the collection of matrices (5-26a) forms the *unitary group* U_2. If we further restrict the matrices to those satisfying

$$\det(U) = 1 , \tag{5-28}$$

the corresponding group is called the *special unitary group* SU_2.

From (5-26a) and (5-28) we have

$$U^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} . \tag{5-26b}$$

Moreover from (5-27), (5-26a) and (5-28), we have

$$d = a^*, \quad c = -b^* , \quad |a|^2 + |b|^2 = 1 .$$

Therefore the most general form of the group elements of SU_2 is

$$U = \begin{pmatrix} e^{i\xi}\cos\eta & -e^{i\zeta}\sin\eta \\ e^{i\zeta}\sin\eta & e^{-i\xi}\cos\eta \end{pmatrix} . \tag{5-29}$$

It contains three real parameters ξ, η and ζ. Thus the order of SU_2 is three.

Example 4: The 2–dimensional rotation group R_2. A point $P(x, y)$ in the x–y plane goes to a point $P'(x', y')$ after a rotation through angle φ about the z–axis. From Fig. 5.2, we have

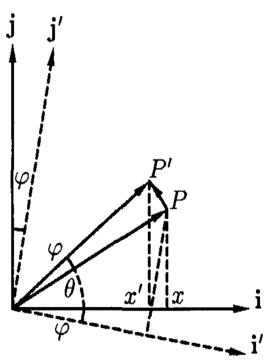

Fig. 5.2. Rotations of points and axes.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} r\cos(\varphi + \theta) \\ r\sin(\varphi + \theta) \end{pmatrix} = R_z(\varphi)\begin{pmatrix} x \\ y \end{pmatrix} , \tag{5-30a}$$

$$R_z(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} . \tag{5-30b}$$

The matrices $R_z(\varphi)(0 \le \varphi < 2\pi)$ constitute the 2–dimensional rotation group R_2. $R_z(\varphi)$ in (5-30b) is identical to the rep $D(\varphi)$ of (2-60b) carried by the basis $\varphi_1(x) = x$ and $\varphi_2(x) = y$.

There is only one real parameter, thus $r = 1$. The matrix $R_z(\varphi)$ of (5-30b) is orthogonal, that is,

$$R_z(\varphi)\tilde{R}_z(\varphi) = I\ , \quad \tilde{R}_z^{-1}(\varphi) = R_z(\varphi)\ , \tag{5-31}$$

so R_2 is also called the *special orthogonal group* of dimension 2, or SO_2.

Comparing (5-27) with (5-31) it is seen that if the unitary transformations are restricted to be real, the unitary group degenerates to the orthogonal group. For example, (5-29) goes over to (5-30b) when $\xi = \zeta = 0$, and $\eta = \varphi$.

From Fig. 5.2 it is seen that if the point P is kept fixed and the coordinates axes are rotated through the angle $-\varphi$, then the same relation (5-30) holds between the coordinates x', y' of the same point P in the new axes i' and j', and its old coordinates x, y.

It is easy to see that the hierarchies of the groups so far mentioned are $GL(2,C) \supset \overset{S}{\cancel{R}}L(2,R) \supset R_2$, and $GL(2,C) \supset U_2 \supset SU_2 \supset R_2$.

Example 5: The 3–dimensional rotation group R_3. In analogy with (5-31), the transformation matrices for rotations through angles α, β, γ about the x, y, z axes, respectively, are

$$\begin{gathered}
R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}\ , \quad R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}\ , \\
R_z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\ .
\end{gathered} \tag{5-32}$$

Alternatively, we can first rotate the point P through angle γ about the z axis, then rotate through angle β about the y axis, and finally rotate through angle α about the z axis. The set of angles (α, β, γ) are the Euler angles (Rose 1957). As a result of these three rotations, the point P goes to the point P'. The relation between the coordinates of P and P' is

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathcal{D}(\alpha\beta\gamma) \begin{pmatrix} x \\ y \\ z \end{pmatrix}\ , \tag{5-33}$$

$$\begin{aligned}
\mathcal{D}(\alpha,\beta,\gamma) &= R(\alpha,\beta,\gamma) = R_z(\alpha)R_y(\beta)R_z(\gamma) \\
&= \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma, & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma, & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma, & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma, & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma, & \sin\beta\sin\gamma, & \cos\beta \end{pmatrix}\ .
\end{aligned} \tag{5-34}$$

$\mathcal{D}$ is an orthogonal matrix $\mathcal{D}^{-1} = \tilde{\mathcal{D}}$. The Lie group R_3 is also called the special orthogonal group SO_3 of dimension three. In Chapter 6 we discuss this rotation group in more detail.

Let us now consider a new coordinate system $\overline{xyz}$ which is obtained from rotating successively the original coordinate system through angles γ, β and α about the axes z, y, and z, respectively (see the diagram in Bohr 1969, p. 76). From the discussion in Example 4 we know that the relation between the old and new coordinates is given by

$$\begin{pmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{pmatrix} = \mathcal{D}^{-1}(\alpha\beta\gamma) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \tilde{\mathcal{D}}(\alpha\beta\gamma) \begin{pmatrix} x \\ y \\ z \end{pmatrix}\ . \tag{5-35}$$

5.3. Lie Algebras

One great contribution of Sophus Lie to the theory of Lie groups was to consider those elements which differ infinitesimally from the identity, and to show that from them one can obtain most of the properties of the Lie group.

We begin with the Taylor expansion of the group elements $R(a)$,

$$R(a) = R(0) + a^\rho X_\rho + \dots , \tag{5-36}$$

$$X_\rho = \left(\frac{\partial R(a)}{\partial a^\rho}\right)_{a=0} , \tag{5-37}$$

are called *infinitesimal generators* or simply *generators of the Lie group* . For a Lie group of order r, there are r linearly independent generators. To explore the neighborhood of the identity, we only need retain terms linear in a in (5-36), that is,

$$R(a) = 1 + a^\rho X_\rho . \tag{5-38}$$

The inverse element is

$$R^{-1}(a) = 1 - a^\rho X_\rho . \tag{5-39}$$

Suppose that there are two infinitesimal elements and each has only one non-vanishing parameter,

$$R(a) = 1 + \varepsilon X_\rho , \quad R(b) = 1 + \varepsilon X_\sigma . \tag{5-40}$$

According to the definition of the Lie group,

$$\begin{aligned} R(a)R(b) &= R(c) = 1 + C^\tau X_\tau , \\ R(b)R(a) &= R(c') = 1 + C'^\tau X_\tau , \\ [R(a),\ R(b)] &= \varepsilon^2 C^\tau_{\rho\sigma} X_\tau , \end{aligned}$$

$$C^\tau_{\rho\sigma} = (C^\tau - C'^\tau)/\varepsilon^2 . \tag{5-41}$$

On the other hand from (5-40) we have

$$[R(a),\ R(b)] = \varepsilon^2 [X_\rho, X_\sigma] . \tag{5-42}$$

Comparing (5-41) with (5-42), we get an important relation:

$$[X_\rho, X_\sigma] = C^\tau_{\rho\sigma} X_\tau , \tag{5-43}$$

namely, the commutator of two generators is a linear combination of the r generators. The coefficients $C^\tau_{\rho\sigma}$ are called the *structure constants* of the Lie group. They have the following two properties.

1. They are anti-symmetric with respect to the subscripts.

$$C^\tau_{\rho\sigma} = -C^\tau_{\sigma\rho} . \tag{5-44}$$

2. According to the Jacobi identity

$$[[X_\rho, X_\sigma], X_\tau] + [[X_\sigma, X_\tau], X_\rho] + [[X_\tau, X_\rho], X_\sigma] = 0 , \tag{5-45a}$$

we have

$$C^\mu_{\rho\sigma} C^\nu_{\mu\tau} + C^\mu_{\sigma\tau} C^\nu_{\mu\rho} + C^\mu_{\tau\rho} C^\nu_{\mu\sigma} = 0 . \tag{5-45b}$$

The r generators span a real r-dimensional vector space $\mathcal{L}_r$. Any vector in the space can be expressed as $a^\rho X_\rho$. The product of two basis vectors in the space is defined by their commutator (5-43). The set $\{X_\rho\}$ is thus closed under linear combinations and multiplications defined by (5-43), that is, $\{X_\rho\}$ constitutes an algebra and is called the *Lie algebra* corresponding to the given Lie group. If a^ρ are real, it is called a real algebra, otherwise it is a complex Lie algebra.

Results obtained by Lie reduce the searching for irreps of the Lie group with an infinite number of elements, to a search for irreps of the Lie algebra with a finite number of elements. Having found irreps of the Lie algebra, the irreps of the Lie group are also known. Therefore the Lie algebra plays a crucial role in the theory of Lie groups. For a given Lie group, we always first find its corresponding Lie algebra. In physical problems, it often occurs that a certain kind of Lie algebra emerges naturally; nevertheless, the corresponding Lie group does not have a simple physical meaning. In such cases, we only deal with the Lie algebra and do not bother about the related Lie group at all.

In the space $\mathcal{L}_r = \{X_\rho : \rho = 1, 2, \ldots r\}$, any vector can be expressed as

$$X = a^\rho X_\rho \,, \tag{5-46}$$

where a^ρ can be thought of as the coordinates of an abstract vector X. According to (5-2), the basis vectors and the coordinates transform in the following ways:

$$X'_\rho = B_\rho{}^\sigma X_\sigma \,, \quad a'^\rho = A^\rho{}_\sigma a^\sigma, \quad A = \tilde{B}^{-1} \,. \tag{5-47a,b,c}$$

In the new coordinate system with the basis $\{X'_\rho\}$, the structure constants are $C'^\tau_{\rho\sigma}$,

$$[X'_\rho, X'_\sigma] = C'^\tau_{\rho\sigma} X'_\tau \,. \tag{5-48}$$

From (5-9b), the relation between the new and old structure constants is

$$C'^\tau_{\rho\sigma} = B_\rho{}^\mu B_\sigma{}^\nu A^\tau{}_\lambda C^\lambda_{\mu\nu} \,. \tag{5-49}$$

Equation (5-48) shows that the Lie algebra of the same Lie group may take different forms due to the different choices of the group parameters. This point merits special attention when we are dealing with the classification of Lie algebras.

Example 1: The group $GL(2, R)$. Using (5-25) and (5-37) we get the four generators

$$\begin{aligned} X_1 &= e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad X_2 = e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ X_3 &= e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad X_4 = e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{5-50}$$

It can be shown that those generators obey the following commutation relations:

$$[e_{\alpha\beta}, e_{\gamma\delta}] = \delta_{\beta\gamma} e_{\alpha\delta} - \delta_{\alpha\delta} e_{\gamma\beta} \,. \tag{5-51}$$

Example 2: The group SO_2. From (5-30b) and (5-37) we obtain

$$X_\varphi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5-52}$$

Example 3: The group SO_3. From (5-32) and (5-37) we have

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5-53a}$$

They obey the commutation relations

$$[X_1, X_2] = X_3 \quad \text{cyclic in 1,2,3} \,. \tag{5-53b}$$

5.4. Finite Transformations

Equation (5-38) is the expression for infinitesimal transformations. Now let us find the expression for finite transformations.

Consider first the single parameter group SO_2. The counterparts of (5-36) and (5-37) are

$$R(\delta\varphi) = 1 + \delta\varphi X_\varphi \, , \quad X_\varphi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \, . \tag{5-54}$$

Let the infinitesimal angle $\delta\varphi = \varphi/N$, where N is an arbitrarily large number. Therefore

$$R(\delta\varphi) \cong \left(1 + \frac{\varphi}{N} X_\varphi\right) \, .$$

Applying $R(\delta\varphi)$ N times, we obtain the finite rotation

$$\begin{aligned} R(\varphi) &\cong \left(1 + \frac{\varphi}{N} X_\varphi\right)^N = \sum_{n=0}^{N} \binom{N}{n} \left(\frac{\varphi}{N} X_\varphi\right)^n \\ &\underset{N\to\infty}{\longrightarrow} 1 + \varphi X_\varphi + \frac{\varphi^2}{2!} X_\varphi^2 + \frac{\varphi^3}{3!} X_\varphi^3 + \ldots \\ &= \cos\varphi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\varphi \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \, . \end{aligned} \tag{5-55}$$

This is the familiar result (5-30b). Equation (5-55) can be written formally as

$$R(\varphi) = e^{\varphi X_\varphi} \, . \tag{5-56}$$

In the above discussion we ignored the unchanged z–component. If the z–component is included, then the generator X_φ in (5-52) goes over to X_3 in (5-53a). Letting $X_3 = -iJ_z$, we get the representative matrix of the operator J_z in the Cartesian coordinate system as shown in (5-58b). The group elements of SO_2 thus take the well–known form

$$R_z(\varphi) = e^{-i\varphi J_z} \, . \tag{5-57}$$

Analogously, we introduce for SO_3

$$X_1 = -iJ_x, \quad X_2 = -iJ_y, \quad X_3 = -iJ_z \, . \tag{5-58a}$$

From (5-53) we have

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \quad J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} , \quad J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \, . \tag{5-58b}$$

$$[J_x, J_y] = iJ_z, \quad \text{cyclic in } x, y, z \, . \tag{5-59}$$

$J_{x,y,z}$ are the three components of angular momentum . Equation (5-58b) is their matrix representation in the 3–dimensional Cartesian basis.

The rotation operators corresponding to (5-32) are

$$R_x(\alpha) = e^{-i\alpha J_x}, \quad R_y(\beta) = e^{-i\beta J_y}, \quad R_z(\gamma) = e^{-i\gamma J_z} \, . \tag{5-60}$$

The operator for a rotation through angle φ about an axis $\mathbf{n}$ with orientation angle (θ', φ') can be expressed as

$$R_{\mathbf{n}}(\varphi) = e^{-i\varphi\mathbf{n}\cdot\mathbf{J}} = \exp\left[-i\varphi(J_x \sin\theta' \cos\varphi' + J_y \sin\theta' \sin\varphi' + J_z \cos\theta')\right] \, . \tag{5-61}$$

Such a rotation can be written as a product of three rotations

$$R_{\mathbf{n}}(\varphi) = R(\varphi', \theta', 0)R(\varphi, 0, 0)R(0, -\theta', -\varphi') = R(\varphi', \theta', 0)R(\varphi, -\theta', -\varphi') \,, \tag{5-62}$$

namely, first rotate the $\mathbf{n}$–axis onto the z–axis, then rotate through angle φ about the z–axis, and finally bring the z–axis back to the $\mathbf{n}$–axis. Using (5-62) and (5-34) we can get the matrix form of the rotation $R_{\mathbf{n}}(\varphi)$ in the 3–dimensional space x, y and z.

The transition from the infinitesimal transformation (5-54) to the finite transformation can be extended to the more general case

$$R(\delta a) \cong 1 + a^\rho X_\rho, \quad R(a) = \exp\left(a^\rho X_\rho\right) . \tag{5-63a, b}$$

It should be mentioned that it is not always possible to write the finite transformation in the form (5-63b). If the transformation can be put in this form, then the group parameters a^ρ are said to be *canonical.* For example in (5-61) $a_x = \varphi \sin\theta' \cos\varphi', a_y = \varphi \sin\theta' \sin\varphi', a_z = \varphi\cos\theta'$ are canonical parameters. If we choose the Euler angles α, β and γ as the group parameters of SO_3, from (5-34) and (5-60), we have

$$R(\alpha, \beta, \gamma) = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} \,. \tag{5-64}$$

Since J_y and J_z do not commute,

$$R(\alpha, \beta, \gamma) \neq e^{-i(\alpha J_z + \beta J_y + \gamma J_z)} \,.$$

Therefore the Euler angle α, β and γ are not canonical parameters.

5.5. Correspondence between Lie Groups and Lie Algebras

The classifications of Lie groups and Lie algebras are in one-to-one correspondence. This correspondence is based on the two relations (5-65) and (5-66) which follow. Let R_ρ, R_σ be two infinitesimal elements. Making an expansion of (5-63b) and retaining terms up to ε^2, we obtain

$$R_\rho \cong 1 + \varepsilon X_\rho + \frac{\varepsilon^2}{2!} X_\rho^2, \quad R_\sigma \cong 1 + \varepsilon X_\sigma + \frac{\varepsilon^2}{2!} X_\sigma^2 \,.$$

Therefore

$$[R_\rho, R_\sigma] = \varepsilon^2 [X_\rho \,, X_\sigma] = \varepsilon^2 C^\tau_{\rho\sigma} X_\tau \,. \tag{5-65}$$

$$R_\rho R_\sigma R_\rho^{-1} R_\sigma^{-1} = 1 + \varepsilon^2 [X_\rho, X_\sigma] = 1 + \varepsilon^2 C^\tau_{\rho\sigma} X_\tau \,. \tag{5-66}$$

According to the above two relations it is easy to establish the following correspondences:

Lie groups	Lie algebras
1a. Abelian Lie groups $$[R_\rho, R_\sigma] = 0 \,, \quad \rho, \sigma = 1, 2, \ldots, r \,. \tag{5-67a}$$	1b. Abelian Lie algebras $$[X_\rho, X_\sigma] = 0 \,, \quad \rho, \sigma = 1, 2, \ldots, r \,. \tag{5-67b}$$
2a. Subgroups G_s of a Lie group G. Let $X_i, X_j, \ldots, X_k$ be the generators of G_s. Let $R_i = 1 + \varepsilon X_i, R_j = 1 + \varepsilon X_j$. Therefore	2b. Subalgebras A_s of a Lie algebra A. By (5-68a), $[R_i, R_j]$ is an element of the group algebra of G_s. Using (5-65) we know that $X_i, X_j, \ldots, X_k$ form a subalgebra A_s of A, that is,

$$R_i R_j \in G_s \,. \tag{5-68a}$$

3a. Invariant subgroups.
If the elements $R_i, R_j, \ldots, R_k$ belong to an invariant subgroup G_s, one has from (1-28)

$$R_\rho R_i R_\rho^{-1} \in G_s \,, R_i \in G_s \,, \quad \rho = 1, 2, \ldots, r$$

Thus

$$R_\rho R_i R_\rho^{-1} R_i^{-1} \in G_s \,. \tag{5-69a}$$

4a. Simple Lie group.
A Lie group which has no invariant subgroups is a simple Lie group.

5a. Semi–simple Lie group.
The Lie group which has no Abelian invariant subgroups is a semi–simple Lie group.

$$[X_i, X_j] \in A_s \,. \tag{5-68b}$$

3b. Invariant subalgebras.
From (5-69a) and (5-66) it is known that

$$[X_a, X_\rho] = C_{a\rho}^b X_b \,, \quad a, b = i, j, \ldots, k \,, \ \rho = 1, 2, \ldots, r \,. \tag{5-69b}$$

The algebra $X_i, \ldots, X_k$ is called the invariant subalgebra of A.

4b. Simple Lie algebra
A Lie algebra which has no invariant subalgebra is a simple Lie algebra.

5b. Semi–simple Lie algebra.
The Lie algebra which has no Abelian invariant subalgebras is a semi–simple Lie algebra.

6a. **Theorem 5.1:** A semi–simple Lie group is a direct product of a set of simple Lie groups,

$$G = G_1 \times G_2 \times \ldots \times G_n \,, \tag{5-70a}$$

where G_i are simple and $[G_i, G_j] = 0$.

6b. **Theorem 5.1′:** A semi–simple Lie algebra is a direct sum of a set of simple Lie algebras,

$$A = A_1 \oplus A_2 \oplus \ldots \oplus A_n \,, \tag{5-70b}$$

where A_i are simple, $[A_i, A_j] = 0$ and the intersections between any A_i and A_j are zeroes.

7. A *compact Lie algebra* is one corresponding to a compact Lie group.

It is important to distinguish between the semi–simple and non–semi–simple Lie groups, since Abelian invariant subgroups, though apparently the easiest to deal with, can actually be the most troublesome from the point of view of representations. Fortunately, in most physical applications we deal only with semi–simple Lie groups. Below we mainly concern ourselves with semi–simple Lie groups. (The criteria for semi–simple Lie groups is given in Sec. 5.13.)

In a semi–simple Lie algebra the maximum number of linearly independent generators, denoted $H_1, \ldots, H_l$, which commute with one another, is called the *rank* of the Lie algebra or the rank of the corresponding Lie group, designated by l. (An equivalent definition of rank is given in Sec. 5.18.) . The set of operators $H_1, \ldots H_l$ form a subalgebra, called the *Cartan subalgebra* .

Naturally, any Lie group must be of at least rank 1.

Example 1: For SO_2, there is only one generator J_z. Naturally J_z commutes with itself. Therefore SO_2 is an Abelian group with rank $l = 1$.

Example 2: For SO_3, there are three generators J_x, J_y and J_z. Each of them only commutes with itself. SO_3 is a non–Abelian group of rank 1.

SO_2 and SO_3 are both simple.

5.6 Linear Transformation Groups

In Secs. 5.2 and 5.3, we gave the general definitions of Lie groups and Lie algebras. In Sec. 5.2 we also gave some simple examples. We will now extend these examples to the general linear transformation groups. These groups are the most useful ones in physics. Assume $R(a) = R(a^1, a^2, \ldots, a^r)$ is an n-dimensional linear transformation,

$$x \xrightarrow{R(a)} x' = R(a)x \,, \tag{5-71a}$$

or equivalently

$$x'^{\alpha} = R_{\alpha\beta}(a)x^{\beta}, \quad \alpha = 1, 2, \ldots, n \,. \tag{5-71b}$$

Here x may be real or complex. The set of all $n \times n$ matrices $R(a)$ forms a linear transformation group in n-dimensional space. It can be further classified into the following categories:

1. $GL(n, C) \equiv GL(n)$, the *general complex linear transformation group.* The matrix elements $R_{\alpha\beta}(a)$ are complex numbers. The group contains $2n^2$ real parameters; therefore the order is $r = 2n^2$.

2. $GL(n, R)$, the *general real linear transformation group.* The matrix elements are restricted to real numbers. There are n^2 real parameters. The order is $r = n^2$.

3. $SL(n, C), SL(n, R)$, the *special linear transformation groups.* These two groups are obtained from $GL(n, C)$ and $GL(n, R)$ by requiring that the determinants of the transformations be unity. Their orders are equal to $2n^2 - 2$ and $n^2 - 1$, respectively. Obviously we have

$$GL(n, C) \supset SL(n, C) \supset SL(n, R), \quad GL(n, R) \supset SL(n, R).$$

4. U_n and SU_n, the *unitary group* and *unimodular unitary group* in n dimensions. Restricting matrices $R(a)$ to be unitary, that is,

$$R(a)R^{\dagger}(a) = R^{\dagger}(a)R(a) = I \,, \tag{5-72a}$$

we get the unitary group U_n of order $r = n^2$. The unitary group is compact, since by (5-72a) the matrix elements $|R_{\alpha\beta}(a)| \le 1$. The condition (5-72a) also stipulates that

$$\det R(a) = \exp(i\varphi) \,. \tag{5-72b}$$

Demanding that the determinants of $R(a)$ equal unity, we obtain the unimodular unitary group SU_n of order $r = n^2 - 1$. The unitarity (5-72a) ensures that the quantity $\sum_{\alpha=1}^{n} |x^{\alpha}|^2$ is an invariant under the unitary transformation,

$$\sum_{\alpha=1}^{n} |x^{\alpha}|^2 = \sum_{\alpha=1}^{n} |x'^{\alpha}|^2 \,. \tag{5-73}$$

The fundamental role of unitary groups in quantum mechanics is easily understood when one realizes that the probabilistic nature of quantum theory requires a preservation of squares of absolute values of various inner product of wave functions.

5. The group $U(n, m)$. All the linear transformations which keep the quantity

$$\sum_{\alpha=1}^{n} |x^{\alpha}|^2 - \sum_{\beta=n+1}^{n+m} |x^{\beta}|^2 \tag{5-74}$$

invariant form the group $U(n, m)$ with order $r = (n + m)^2$. $U(n, m)$ is a noncompact group. Obviously, $U_n = U(n, 0) = U(0, n)$, $GL(n, m) \supset U(n, m)$. Similarly we can define the group $SU(n, m)$, with order $r = (n + m)^2 - 1$.

6. The complex orthogonal group $O(n,C)$. All the complex linear transformations which leave $\sum_{\alpha=1}^{n}(x^\alpha)^2$ invariant form the complex orthogonal group. From

$$\sum_{\alpha=1}^{n}(x'^\alpha)^2 = \sum_{\alpha\beta\beta'} R_{\alpha\beta}R_{\alpha\beta'}x^\beta x^{\beta'} = \sum_{\beta=1}^{n}(x^\beta)^2 \,, \tag{5-75}$$

we have

$$\sum_{\alpha} R_{\alpha\beta}R_{\alpha\beta'} = \delta_{\beta\beta'} \,. \tag{5-76a}$$

Thus $R(a)$ are orthogonal matrices,

$$\tilde{R}(a)R(a) = 1 \,. \tag{5-76b}$$

$O(n,C)$ has $n(n-1)/2$ complex parameters (see Sec. 5.8), therefore it is of order $r = n(n-1)$. From (5-76b) we have

$$\det\big(\tilde{R}(a)\big)\det\big(R(a)\big) = 1 \,, \quad \det\big(R(a)\big) = \pm 1 \,. \tag{5-76c}$$

The transformation matrices of $O(n,C)$ can be divided into two sets, one is associated with $\det\big(R(a)\big) = +1$, and the other with $\det\big(R(a)\big) = -1$. The set with determinant $+1$ forms a subgroup respreseting *proper rotations*, the *unimodular complex orthogonal group* $SO(n,C)$. We can decompose the group $O(n,C)$ into cosets with respect to the subgroup $SO(n,C)$, that is,

$$O(n,C) = SO(n,C) \oplus SO(n,C) \times I \,, \tag{5-77}$$

where I is the space inversion operator.

The quotient group $O(n,C)/SO(n,C)$ is a group of order 2. The set with determinant -1 represents rotation–reflections. Any element of $SO(n,C)$ can be reached from the identity via continuous paths in parameter space, while the elements with $\det\big(R(a)\big) = -1$ cannot. In other words, the group $O(n,C)$ consists of two disconnected parts and we cannot go from one part to the other continuously.

7. The real orthogonal group O_n. Restricting the matrices of $O(n,C)$ to be real leads to the real orthogonal group, denoted by O_n or $O(n)$, which is of order $r = \frac{1}{2}n(n-1)$. By further requiring $\det\big(R(a)\big) = 1$, we get the unimodular orthogonal group SO_n. It is still of order $\frac{1}{2}n(n-1)$. Similarly, the group O_n also consists of two disconnected parts. Obviously we have

$$O(n,C) \supset SO(n,C) \supset SO_n \,, \quad O_n \supset SO_n \,.$$

8. The group $O(n,m)$.

All the real linear transformation which leave the quantity

$$\sum_{\alpha=1}^{n}(x^\alpha)^2 - \sum_{\beta=n+1}^{n+m}(x^\beta)^2 \tag{5-78}$$

invariant form the group $O(n,m)$ with order $r = \frac{1}{2}[n(n-1) + m(m-1)] + nm$. $O(n,m)$ is a noncompact group. The Lorentz group $O(3,1)$ is a special case of $O(n,m)$.

Obviously, we have $O_n = O(n,0) = O(0,n)$.

9. *Complex symplectic, real symplectic* and *unitary symplectic groups* $Sp(2n,C)$, $Sp(2n,R)$, and Sp_{2n}.

Suppose $\mathbf{x} = (x^1,\ldots,x^n;x^{-1},\ldots,x^{-n})$ and $\mathbf{y} = (y^1,\ldots,y^n;y^{-1},\ldots y^{-n})$ are two column vectors with dimension $2n$ and $R(a)$ are $2n\times 2n$ matrices, which transform $\mathbf{x}$ and $\mathbf{y}$ into $\mathbf{x}'$ and $\mathbf{y}'$:

$$\mathbf{x}' = R(a)\mathbf{x}, \quad \mathbf{y}' = R(a)\mathbf{y} \,. \tag{5-79a}$$

The symplectic group is the set of all $2n \times 2n$ linear transformations $R(a)$ which leave the skew-symmetric bilinear form

$$\sum_{\alpha=1}^{n}(x^{\alpha}y^{-\alpha} - x^{-\alpha}y^{\alpha}) \tag{5-79b}$$

invariant. If the $2n \times 2n$ matrices $R(a)$ are complex (real), it is called the *complex* (real) *symplectic group* of order $2n(2n+1)\big(n(2n+1)\big)$. If the complex matrices $R(a)$ are unitary, the group is called the *unitary symplectic group* Sp_{2n} . We have

$$GL(2n, C) \supset Sp(2n, C) \supset Sp(2n, R), \quad Sp(2n, C) \supset Sp_{2n}, \quad SU_{2n} \supset Sp_{2n} .$$

The groups $Sp(2n, C)$ and $Sp(2n, R)$ are noncompact, while Sp_{2n} is compact.

5.7. Infinitesimal Operators for Linear Transformation Groups

Consider subjecting $x^1, x^2, \dots, x^n$ to an infinitesimal transformation

$$x' = R(a)x, \quad R(a) = 1 + \mathcal{A}(a) , \tag{5-80a}$$

$$\mathcal{A}(a) = \sum_{\alpha\beta} a_{\alpha\beta}e_{\alpha\beta} , \tag{5-81}$$

where $a_{\alpha\beta}$ are infinitesimal quantities and $e_{\alpha\beta}$ are the $n \times n$ matrices defined in (2-4). The $e_{\alpha\beta}$ obey the commutator (5-51) and the following relation

$$e_{\alpha\beta}e_{\gamma\delta} = \delta_{\beta\gamma}e_{\alpha\delta} . \tag{5-82}$$

Equation (5-80a) can also be rewritten as

$$x'^{\alpha} = x^{\alpha} + a^{\alpha}{}_{\beta}x^{\beta} . \tag{5-80b}$$

Under the transformation (5-80b), an arbitrary function $\psi(x)$ goes over to

$$\psi'(x) = \psi(x') = \psi(x^{\alpha} + a^{\alpha}{}_{\beta}x^{\beta}) = \psi(x) + a^{\alpha}{}_{\beta}\, x^{\beta}\frac{\partial}{\partial x^{\alpha}}\psi(x) . \tag{5-83}$$

Defining the infinitesimal operators

$$E_{\beta\alpha} = x^{\beta}\frac{\partial}{\partial x^{\alpha}} , \tag{5-84}$$

Eq. (5-83) reads

$$\psi'(x) = (1 + a^{\alpha}{}_{\beta}E_{\beta\alpha})\psi(x) = (1 + a^{\rho}X_{\rho})\psi(x) , \tag{5-85}$$

and (5-80b) can be expressed as

$$x'^{\alpha} = \Big(1 + \sum_{\beta} a^{\alpha}{}_{\beta}E_{\beta\alpha}\Big)x^{\alpha} . \tag{5-80c}$$

From this we obtain a simple method for finding the infinitesimal operators of the linear transformation group:

1. First find the infinitesimal matrix $\mathcal{A}$ in the infinitesimal transformations (5-80a), that is,

$$\mathcal{A} = \sum_{\alpha\beta} a^{\alpha}{}_{\beta}e_{\alpha\beta} . \tag{5-86a}$$

Notice that not all the parameters $a_{\alpha\beta}$ are independent, except for the group $GL(n, R)$ or $GL(n, C)$.