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Advanced Mathematical Methods for Scientists and Engineers I

Asymptotic Methods and Perturbation Theory

With 148 Figures

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CHAPTER

SEVEN

PERTURBATION SERIES

You have erred perhaps in attempting to put colour and life into each of your statements instead of confining yourself to the task of placing upon record that severe reasoning from cause to effect which is really the only notable feature about the thing. You have degraded what should have been a course of lectures into a series of tales.

—Sherlock Holmes, *The Adventure of the Copper Beeches*
Sir Arthur Conan Doyle

(E) 7.1 PERTURBATION THEORY

Perturbation theory is a large collection of iterative methods for obtaining approximate solutions to problems involving a small parameter ε. These methods are so powerful that sometimes it is actually advisable to introduce a parameter ε temporarily into a difficult problem having no small parameter, and then finally to set $\varepsilon = 1$ to recover the original problem. This apparently artificial conversion to a perturbation problem may be the only way to make progress.

The thematic approach of perturbation theory is to decompose a tough problem into an infinite number of relatively easy ones. Hence, perturbation theory is most useful when the first few steps reveal the important features of the solution and the remaining ones give small corrections.

Here is an elementary example to introduce the ideas of perturbation theory.

Example 1 *Roots of a cubic polynomial.* Let us find approximations to the roots of

$$x^3 - 4.001x + 0.002 = 0. \tag{7.1.1}$$

As it stands, this problem is not a perturbation problem because there is no small parameter ε. It may not be easy to convert a particular problem into a tractable perturbation problem, but in the present case the necessary trick is almost obvious. Instead of the single equation (7.1.1) we consider the *one-parameter family* of polynomial equations

$$x^3 - (4 + \varepsilon)x + 2\varepsilon = 0. \tag{7.1.2}$$

When $\varepsilon = 0.001$, the original equation (7.1.1) is reproduced.

It may seem a bit surprising at first, but it is easier to compute the approximate roots of the family of polynomials (7.1.2) than it is to solve just the one equation with $\varepsilon = 0.001$. The reason

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for this is that if we consider the roots to be functions of ε, then we may further assume a perturbation series in powers of ε:

$$x(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n. \tag{7.1.3}$$

To obtain the first term in this series, we set $\varepsilon = 0$ in (7.1.2) and solve

$$x^3 - 4x = 0. \tag{7.1.4}$$

This expression is easy to factor and we obtain in *zeroth-order* perturbation theory $x(0) = a_0 = -2, 0, 2$.

A *second-order* perturbation approximation to the first of these roots consists of writing (7.1.3) as $x_1 = -2 + a_1 \varepsilon + a_2 \varepsilon^2 + O(\varepsilon^3)$ $(\varepsilon \to 0)$, substituting this expression into (7.1.2), and neglecting powers of ε beyond ε^2. The result is

$$(-8 + 8) + (12a_1 - 4a_1 + 2 + 2)\varepsilon + (12a_2 - a_1 - 6a_1^2 - 4a_2)\varepsilon^2 = O(\varepsilon^3), \qquad \varepsilon \to 0. \tag{7.1.5}$$

It is at this step that we realize the power of generalizing the original problem to a family of problems (7.1.2) with variable ε. It is because ε is *variable* that we can conclude that the coefficient of each power of ε in (7.1.5) is *separately* equal to zero. This gives a sequence of equations for the expansion coefficients $a_1, a_2, \ldots$:

$$\varepsilon^1: \quad 8a_1 + 4 = 0; \qquad \varepsilon^2: \quad 8a_2 - a_1 - 6a_1^2 = 0;$$

and so on. The solutions to the equations are $a_1 = -\frac{1}{2}$, $a_2 = \frac{1}{8}$, ... Therefore, the perturbation expansion for the root x_1 is

$$x_1 = -2 - \tfrac{1}{2}\varepsilon + \tfrac{1}{8}\varepsilon^2 + \cdots. \tag{7.1.6}$$

If we now set $\varepsilon = 0.001$, we obtain x_1 from (7.1.6) accurate to better than one part in 10^9. The same procedure gives

$$x_2 = 0 + \tfrac{1}{2}\varepsilon - \tfrac{1}{8}\varepsilon^2 + O(\varepsilon^3), \qquad x_3 = 2 + 0 \cdot \varepsilon + 0 \cdot \varepsilon^2 + O(\varepsilon^3), \qquad \varepsilon \to 0.$$

(Successive coefficients in the perturbation series for x_3 all vanish because $x_3 = 2$ is the exact solution for all ε.) All three perturbation series for the roots converge for $\varepsilon = 0.001$. Can you prove that they converge for $|\varepsilon| < 1$? (See Prob. 7.6.)

This example illustrates the three steps of perturbative analysis:

1. *Convert the original problem into a perturbation problem by introducing the small parameter ε.*
2. *Assume an expression for the answer in the form of a perturbation series and compute the coefficients of that series.*
3. *Recover the answer to the original problem by summing the perturbation series for the appropriate value of ε.*

Step (1) is sometimes ambiguous because there may be many ways to introduce an ε. However, it is preferable to introduce ε in such a way that the *zeroth-order* solution (the leading term in the perturbation series) is obtainable as a closed-form analytic expression. Perturbation problems generally take the form of a soluble equation [such as (7.1.4)] whose solution is altered slightly by a perturbing term [such as $(2 - x)\varepsilon$]. Of course, step (1) may be omitted when the original problem already has a small parameter if a perturbation series can be developed in powers of that parameter.

Step (2) is frequently a routine iterative procedure for determining successive coefficients in the perturbation series. A *zeroth-order* solution consists of finding the leading term in the perturbation series. In Example 1 this involves solving the *unperturbed problem*, the problem obtained by setting $\varepsilon = 0$ in the perturbation problem. A *first-order* solution consists of finding the first two terms in the perturbation series, and so on. In Example 1 each of the coefficients in the perturbation series is determined in terms of the previous coefficients by a simple linear equation, even though the original problem was a nonlinear (cubic) equation.

Generally it is the existence of a closed-form zeroth-order solution which ensures that the higher-order terms may also be determined as closed-form analytical expressions.

Step (3) may or may not be easy. If the perturbation series converges, its sum is the desired answer. If there are several ways to reduce a problem to a perturbation problem, one chooses the way that is the best compromise between difficulty of calculation of the perturbation series coefficients and rapidity of convergence of the series itself. However, many series converge so slowly that their utility is impaired. Also, we will shortly see that perturbation series are frequently divergent. This is not necessarily bad because many of these divergent perturbation series are asymptotic. In such cases, one obtains a good approximation to the answer when ε is very small by summing the first few terms according to the optimal truncation rule (see Sec. 3.5). When ε is not small, it may still be possible to obtain a good approximation to the answer from a slowly converging or divergent series using the summation methods discussed in Chap. 8.

Let us now apply these three rules of perturbation theory to a slightly more sophisticated example.

Example 2 *Approximate solution of an initial-value problem.* Consider the initial-value problem

$$y'' = f(x)y, \qquad y(0) = 1,\ y'(0) = 1, \tag{7.1.7}$$

where $f(x)$ is continuous. This problem has no closed-form solution except for very special choices for $f(x)$. Nevertheless, it can be solved perturbatively.

First, we introduce an ε in such a way that the unperturbed problem is solvable:

$$y'' = \varepsilon f(x)y, \qquad y(0) = 1,\ y'(0) = 1. \tag{7.1.8}$$

Second, we assume a perturbation expansion for $y(x)$ of the form

$$y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x), \tag{7.1.9}$$

where $y_0(0) = 1$, $y_0'(0) = 1$, and $y_n(0) = 0$, $y_n'(0) = 0$ $(n \geq 1)$.

The zeroth-order problem $y'' = 0$ is obtained by setting $\varepsilon = 0$, and the solution which satisfies the initial conditions is $y_0 = 1 + x$. The nth-order problem $(n \geq 1)$ is obtained by substituting (7.1.9) into (7.1.8) and setting the coefficient of ε^n $(n \geq 1)$ equal to 0. The result is

$$y_n'' = y_{n-1} f(x), \qquad y_n(0) = y_n'(0) = 0. \tag{7.1.10}$$

Observe that perturbation theory has replaced the intractable differential equation (7.1.7) with a sequence of inhomogeneous equations (7.1.10). In general, any inhomogeneous equation may be solved routinely by the method of variation of parameters whenever the solution of the associated homogeneous equation is known (Sec. 1.5). Here the homogeneous equation is precisely the unperturbed equation. Thus, it is clear why it is so crucial that the unperturbed equation be soluble.

The solution to (7.1.10) is

$$y_n = \int_0^x dt \int_0^t ds\, f(s) y_{n-1}(s), \qquad n \geq 1. \tag{7.1.11}$$

Equation (7.1.11) gives a simple iterative procedure for calculating successive terms in the perturbation series (7.1.9):

$$y(x) = 1 + x + \varepsilon \int_0^x dt \int_0^t ds (1+s) f(s)$$
$$+ \varepsilon^2 \int_0^x dt \int_0^t ds\, f(s) \int_0^s dv \int_0^v du\, (1+u) f(u) + \cdots. \tag{7.1.12}$$

Third, we must sum this series. It is easy to show that when N is large, the Nth term in this series is bounded in absolute value by $\varepsilon^N x^{2N} K^N (1 + |x|)/(2N)!$, where K is an upper bound for $|f(t)|$ in the interval $0 \leq |t| \leq |x|$. Thus, the series (7.1.12) is convergent for all x. We also conclude that if $x^2 K$ is small, then the perturbation series is rapidly convergent for $\varepsilon = 1$ and an accurate solution to the original problem may be achieved by taking only a few terms.

How do these perturbation methods for differential equations compare with the series methods that were introduced in Chap. 3? Suppose $f(x)$ in (7.1.7) has a convergent Taylor expansion about $x = 0$ of the form

$$f(x) = \sum_{n=0}^{\infty} f_n x^n. \tag{7.1.13}$$

Then another way to solve for $y(x)$ is to perform a local analysis of the differential equation near $x = 0$ by substituting the series solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad a_0 = a_1 = 1, \tag{7.1.14}$$

and computing the coefficients a_n. As shown in Chap. 3, the series in (7.1.14) is guaranteed to have a radius of convergence at least as large as that in (7.1.13).

By contrast, the perturbation series (7.1.9) converges for all finite values of x, and not just those inside the radius of convergence of $f(x)$. Moreover, the perturbation series converges even if $f(x)$ has no Taylor series expansion at all.

Example 3 *Comparison of Taylor and perturbation series.* The differential equation

$$y'' = -e^{-x} y, \qquad y(0) = 1,\ y'(0) = 1, \tag{7.1.15}$$

may be solved in terms of Bessel functions as

$$y(x) = \frac{[Y_0(2) + Y_0'(2)] J_0(2e^{-x/2}) - [J_0(2) + J_0'(2)] Y_0(2e^{-x/2})}{J_0(2) Y_0'(2) - J_0'(2) Y_0(2)}.$$

The local expansion (7.1.14) converges everywhere because e^{-x} has no finite singularities. Nevertheless, a fixed number of terms of the perturbation series (7.1.9) (see Prob. 7.11) gives a much better approximation than the same number of terms of the Taylor series (7.1.14) if x is large and positive (see Fig. 7.1).

In addition, the perturbation methods of Example 2 are immediately applicable to problems where local analysis cannot be used. For example, an approximate solution of the formidable-looking nonlinear two-point boundary-value problem

$$y'' + y = \frac{\cos x}{3 + y^2}, \qquad y(0) = y\left(\frac{\pi}{2}\right) = 2, \tag{7.1.16}$$

may be readily obtained using perturbation theory (see Prob. 7.14).

Thus, the ideas of perturbation theory apply equally well to problems requiring local or global analysis.

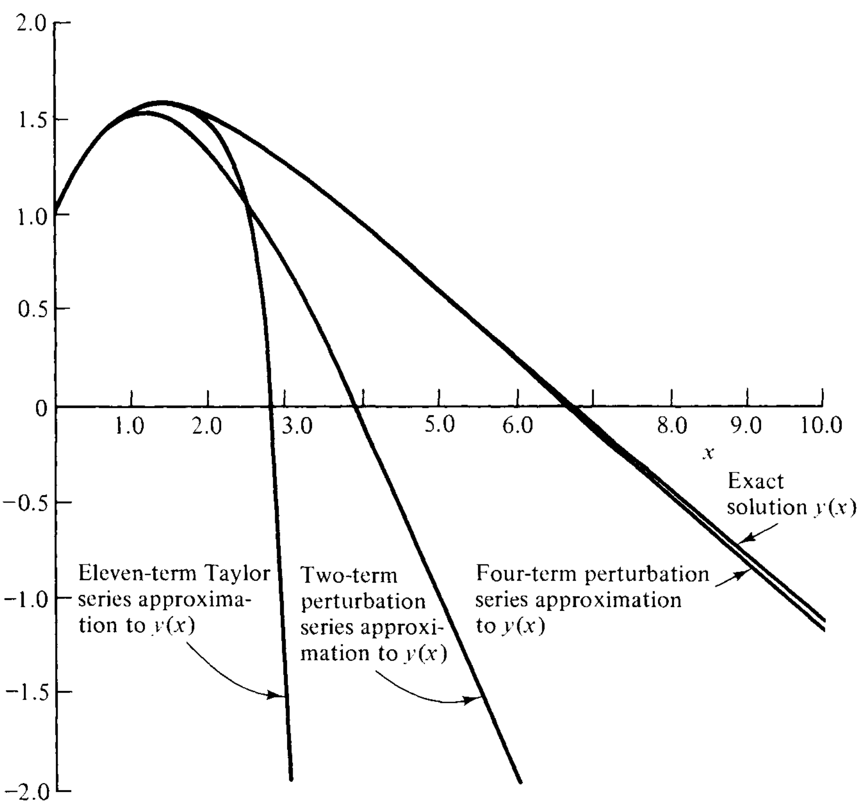

Figure 7.1 A comparison of Taylor series and perturbation series approximations to the solution of the initial-value problem $y'' = -e^{-x}y$ [$y(0) = 1$, $y'(0) = 1$] in (7.1.15). The exact solution to the problem is plotted. Also plotted are an 11-term Taylor series approximation of the form in (7.1.14) and 2- and 4-term perturbation series approximations of the form in (7.1.3) with $\varepsilon = 1$. The global perturbative approximation is clearly far superior to the local Taylor series.

(E) 7.2 REGULAR AND SINGULAR PERTURBATION THEORY

The formal techniques of perturbation theory are a natural generalization of the ideas of local analysis of differential equations in Chap. 3. Local analysis involves approximating the solution to a differential equation near the point $x = a$ by developing a series solution about a in powers of a small parameter, either $x - a$ for finite a or $1/x$ for $a = \infty$. Once the leading behavior of the solution near $x = a$ (which we would now refer to as the zeroth-order solution!) is known, the remaining coefficients in the series can be computed recursively.

The strong analogy between local analysis of differential equations and formal perturbation theory may be used to classify perturbation problems. Recall that there are two different types of series solutions to differential equations. A series solution about an *ordinary* point of a differential equation is always a Taylor series having a nonvanishing radius of convergence. A series solution about a *singular* point does not have this form (except in rare cases). Instead, it may either be a convergent series not in Taylor series form (such as a Frobenius series) or it may be a divergent series. Series solutions about singular points often have the remarkable property of being meaningful near a singular point yet not existing at the singular point. [The Frobenius series for $K_0(x)$ does not exist at $x = 0$ and the asymptotic series for Bi (x) does not exist at $x = \infty$.]

Perturbation series also occur in two varieties. We define a *regular* perturbation problem as one whose perturbation series is a power series in ε having a nonvanishing radius of convergence. A basic feature of all regular perturbation problems (which we will use to identify such problems) is that the exact solution for small but nonzero $|\varepsilon|$ smoothly approaches the unperturbed or zeroth-order solution as $\varepsilon \to 0$.

We define a *singular* perturbation problem as one whose perturbation series either does not take the form of a power series or, if it does, the power series has a vanishing radius of convergence. In singular perturbation theory there is sometimes no solution to the unperturbed problem (the exact solution as a function of ε may cease to exist when $\varepsilon = 0$); when a solution to the unperturbed problem does exist, its qualitative features are distinctly different from those of the exact solution for arbitrarily small but nonzero ε. In either case, the exact solution for $\varepsilon = 0$ is *fundamentally different in character* from the "neighboring" solutions obtained in the limit $\varepsilon \to 0$. If there is no such abrupt change in character, then we would have to classify the problem as a regular perturbation problem.

When dealing with a singular perturbation problem, one must take care to distinguish between the *zeroth-order* solution (the leading term in the perturbation series) and the solution of the unperturbed problem, since the latter may not even exist. There is no difference between these two in a regular perturbation theory, but in a singular perturbation theory the zeroth-order solution may depend on ε and may exist only for nonzero ε.

The examples of the previous section are all regular perturbation problems. Here are some examples of singular perturbation problems:

Example 1 *Roots of a polynomial.* How does one determine the approximate roots of

$$\varepsilon^2 x^6 - \varepsilon x^4 - x^3 + 8 = 0? \tag{7.2.1}$$

We may begin by setting $\varepsilon = 0$ to obtain the unperturbed problem $-x^3 + 8 = 0$, which is easily solved:

$$x = 2,\ 2\omega,\ 2\omega^2, \tag{7.2.2}$$

where $\omega = e^{2\pi i/3}$ is a complex root of unity. Note that the unperturbed equation has only three roots while the original equation has six roots. This abrupt change in the character of the solution, namely the disappearance of three roots when $\varepsilon = 0$, implies that (7.2.1) is a singular perturbation problem. Part of the exact solution ceases to exist when $\varepsilon = 0$.

The explanation for this behavior is that the three missing roots tend to ∞ as $\varepsilon \to 0$. Thus, for those roots it is no longer valid to neglect $\varepsilon^2 x^6 - \varepsilon x^4$ compared with $-x^3 + 8$ in the limit $\varepsilon \to 0$. Of course, for the three roots near 2, 2ω, and $2\omega^2$, the terms $\varepsilon^2 x^6$ and εx^4 are indeed small as $\varepsilon \to 0$ and we may assume a regular perturbation expansion for these roots of the form

$$x_k(\varepsilon) = 2e^{2\pi i k/3} + \sum_{n=1}^{\infty} a_{n,k}\varepsilon^n, \qquad k = 1, 2, 3. \tag{7.2.3}$$

Substituting (7.2.3) into (7.2.1) and comparing powers of ε, as in Example 1 of Sec. 7.1, gives a sequence of equations which determine the coefficients $a_{n,k}$.

To track down the three missing roots we first estimate their orders of magnitude as $\varepsilon \to 0$. We do this by considering all possible dominant balances between pairs of terms in (7.2.1). There are four terms in (7.2.1) so there are six pairs to consider:

(*a*) Suppose $\varepsilon^2 x^6 \sim \varepsilon x^4$ ($\varepsilon \to 0$) is the dominant balance. Then $x = O(\varepsilon^{-1/2})$ ($\varepsilon \to 0$). It follows that the terms $\varepsilon^2 x^6$ and εx^4 are both $O(\varepsilon^{-1})$. But $\varepsilon x^4 \ll x^3 = O(\varepsilon^{-3/2})$ as $\varepsilon \to 0$, so x^3 is the biggest term in the equation and is not balanced by any other term. Thus, the assumption that $\varepsilon^2 x^6$ and εx^4 are the dominant terms as $\varepsilon \to 0$ is inconsistent.
(*b*) Suppose $\varepsilon x^4 \sim x^3$ as $\varepsilon \to 0$. Then $x = O(\varepsilon^{-1})$. It follows that $\varepsilon x^4 \sim x^3 = O(\varepsilon^{-3})$. But $x^3 \ll \varepsilon^2 x^6 = O(\varepsilon^{-4})$ as $\varepsilon \to 0$. Thus, $\varepsilon^2 x^6$ is the largest term in the equation. Hence, the original assumption is again inconsistent.
(*c*) Suppose $\varepsilon^2 x^6 \sim 8$ so that $x = O(\varepsilon^{-1/3})$ ($\varepsilon \to 0$). Hence, $x^3 = O(\varepsilon^{-1})$ is the largest term, which is again inconsistent.
(*d*) Suppose $\varepsilon x^4 \sim 8$ so that $x = O(\varepsilon^{-1/4})$ ($\varepsilon \to 0$). Then $x^3 = O(\varepsilon^{-3/4})$ is the biggest term, which is also inconsistent.
(*e*) Suppose $x^3 \sim 8$. Then $x = O(1)$. This is a consistent assumption because the other two terms in the equation, $\varepsilon^2 x^6$ and εx^4, are negligible compared with x^3 and 8, and we recover the three roots of the unperturbed equation $x = 2$, 2ω, and $2\omega^2$.
(*f*) Suppose $\varepsilon^2 x^6 \sim x^3$ ($\varepsilon \to 0$). Then $x = O(\varepsilon^{-2/3})$. This is consistent because $\varepsilon^2 x^6 \sim x^3 = O(\varepsilon^{-2})$ is bigger than $\varepsilon x^4 = O(\varepsilon^{-5/3})$ and $8 = O(1)$ as $\varepsilon \to 0$.

Thus, the magnitudes of the three missing roots are $O(\varepsilon^{-2/3})$ as $\varepsilon \to 0$. This result is a clue to the structure of the perturbation series for the missing roots. In particular, it suggests a *scale transformation* for the variable x:

$$x = \varepsilon^{-2/3} y. \tag{7.2.4}$$

Substituting (7.2.4) into (7.2.1) gives

$$y^6 - y^3 + 8\varepsilon^2 - \varepsilon^{1/3} y^4 = 0. \tag{7.2.5}$$

This is now a *regular* perturbation problem for y in the parameter $\varepsilon^{1/3}$ because the unperturbed problem $y^6 - y^3 = 0$ has six roots $y = 1, \omega, \omega^2, 0, 0, 0$. Now, no roots disappear in the limit $\varepsilon^{1/3} \to 0$.

The perturbative corrections to these roots may be found by assuming a regular perturbation expansion in powers of $\varepsilon^{1/3}$ (it would not be possible to match powers in an expansion having only integral powers of ε):

$$y = \sum_{n=0}^{\infty} y_n(\varepsilon^{1/3})^n. \tag{7.2.6}$$

Having established that we are dealing with a singular perturbation problem, it is no surprise that the perturbation series for the roots x is not a series in integral powers of ε.

Nevertheless, when $y_0 = 0$ we find that $y_1 = 0$ and $y_2 = 2, 2\omega$, and $2\omega^2$. Thus, since the first two terms in this series vanish, $x = \varepsilon^{-2/3}y$ is not really $O(\varepsilon^{-2/3})$ but rather $O(1)$ and we have reproduced the three finite roots near $x = 2, 2\omega, 2\omega^2$. Moreover, only every third coefficient in (7.2.6), $y_2, y_5, y_8, \ldots$, is nonvanishing, so we have also reproduced the regular perturbation series in (7.2.3)!

Example 2 *Appearance of a boundary layer.* The boundary-value problem

$$\varepsilon y'' - y' = 0, \qquad y(0) = 0,\ y(1) = 1, \tag{7.2.7}$$

is a singular perturbation problem because the associated unperturbed problem

$$-y' = 0, \qquad y(0) = 0,\ y(1) = 1, \tag{7.2.8}$$

has no solution. (The solution to this first-order differential equation, y = constant, cannot satisfy both boundary conditions.) The solution to (7.2.7) cannot have a regular perturbation expansion of the form $y = \sum_{n=0}^{\infty} y_n(x)\varepsilon^n$ because y_0 does not exist.

There is a close parallel between this example and the previous one. Here, the highest derivative is multiplied by ε and in the limit $\varepsilon \to 0$ the unperturbed solution loses its ability to satisfy the boundary conditions because a solution is lost. In the previous example the highest power of x is multiplied by ε and in the limit $\varepsilon \to 0$ some roots are lost.

The exact solution to (7.2.7) is easy to find:

$$y(x) = \frac{e^{x/\varepsilon} - 1}{e^{1/\varepsilon} - 1}. \tag{7.2.9}$$

This function is plotted in Fig. 7.2 for several small positive values of ε. For very small but nonzero ε it is clear from Fig. 7.2 that y is almost constant except in a very narrow interval of thickness $O(\varepsilon)$ at $x = 1$, which is called a *boundary layer.* Thus, outside the boundary layer the exact solution satisfies the left boundary condition $y(0) = 0$ and almost but not quite satisfies the unperturbed equation $y' = 0$.

It is not obvious how to construct a perturbative approximation to a differential equation whose highest derivative is multiplied by ε until it is known how to construct an analytical expression for the zeroth-order approximation. A new technique called asymptotic matching must be introduced (see Sec. 7.4 and Chap. 9) to solve this problem.

Example 3 *Appearance of rapid variation on a global scale.* In the previous example we saw that the exact solution varies rapidly in the neighborhood of $x = 1$ for small ε and develops a discontinuity there in the limit $\varepsilon \to 0+$. A solution to a boundary-value problem may also develop discontinuities throughout a large region as well as in the neighborhood of a point.

The boundary-value problem $\varepsilon y'' + y = 0$ $[y(0) = 0,\ y(1) = 1]$ is a singular perturbation problem because when $\varepsilon = 0$, the solution to the unperturbed problem, $y = 0$, does not satisfy the boundary condition $y(1) = 1$. The exact solution, when ε is not of the form $(n\pi)^{-2}$ ($n = 0, 1, 2, \ldots$), is $y(x) = \sin(x/\sqrt{\varepsilon})/\sin(1/\sqrt{\varepsilon})$. Observe that $y(x)$ becomes discontinuous throughout the inter-

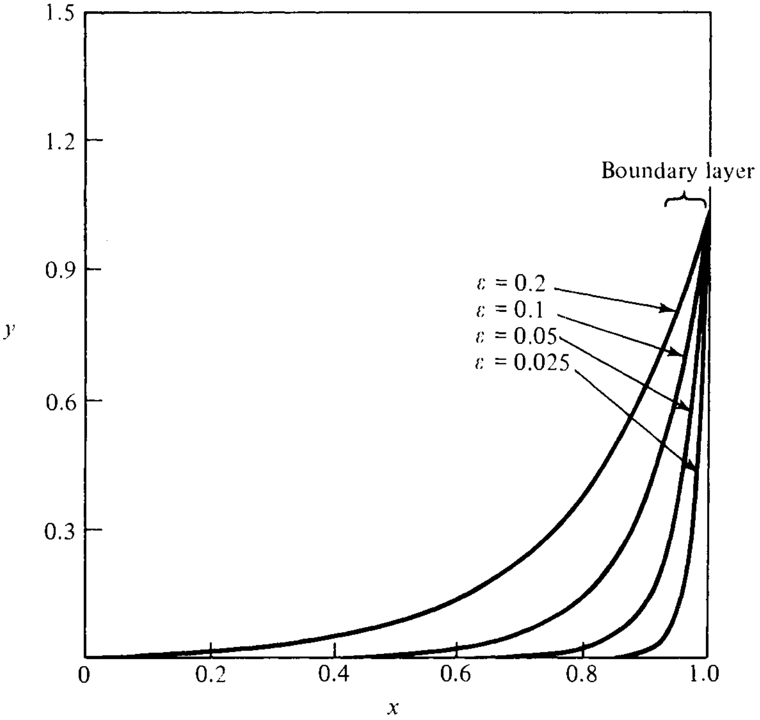

Figure 7.2 A plot of $y(x) = (e^{x/\varepsilon} - 1)/(e^{1/\varepsilon} - 1)$ $(0 \le x \le 1)$ for $\varepsilon = 0.2, 0.1, 0.05, 0.025$. When ε is small $y(x)$ varies rapidly near $x = 1$; this localized region of rapid variation is called a boundary layer. When ε is negative the boundary layer is at $x = 0$ instead of $x = 1$. This abrupt jump in the location of the boundary layer as ε changes sign reflects the singular nature of the perturbation problem.

val $0 \le x \le 1$ in the limit $\varepsilon \to 0+$ (see Fig. 7.3). When $\varepsilon = (n\pi)^{-2}$, there is no solution to the boundary-value problem.

When the solution to a differential-equation perturbation problem varies rapidly on a global scale for small ε, it is not obvious how to construct a leading-order perturbative approximation to the exact solution. The best procedure that has evolved is called WKB theory (see Chap. 10).

Example 4 *Perturbation theory on an infinite interval.* The initial-value problem

$$y'' + (1 - \varepsilon x)y = 0, \qquad y(0) = 1,\ y'(0) = 0, \tag{7.2.10}$$

is a regular perturbation problem in ε over the finite interval $0 \le x \le L$. In fact, the perturbation solution is just

$$\begin{aligned} y(x) = \cos x &+ \varepsilon(\tfrac{1}{4}x^2 \sin x + \tfrac{1}{4}x \cos x - \tfrac{1}{4} \sin x) \\ &+ \varepsilon^2(-\tfrac{1}{32}x^4 \cos x + \tfrac{5}{48}x^3 \sin x + \tfrac{7}{16}x^2 \cos x - \tfrac{7}{16}x \sin x) + \cdots, \end{aligned} \tag{7.2.11}$$

which converges for all x and ε, with increasing rapidity as $\varepsilon \to 0+$ for fixed x.

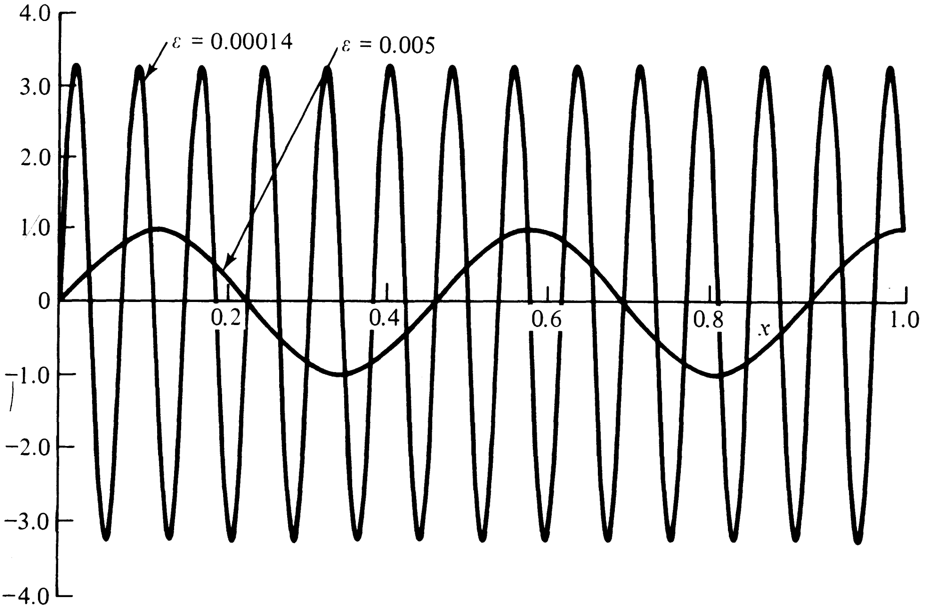

Figure 7.3 A plot of $y(x) = [\sin(x\varepsilon^{-1/2})]/[\sin(\varepsilon^{-1/2})]$ $(0 \le x \le 1)$ for $\varepsilon = 0.005$ and 0.00014. As ε gets smaller the oscillations become more violent; as $\varepsilon \to 0+$, $y(x)$ becomes discontinuous over the entire interval. The WKB approximation is a perturbative method commonly used to describe functions like $y(x)$ which exhibit rapid variation on a global scale.

However, this same initial-value problem must be reclassified as a singular perturbation problem over the semi-infinite interval $0 \le x < \infty$. While the exact solution does approach the solution to the unperturbed problem as $\varepsilon \to 0+$ for fixed x, it does not do so uniformly for all x (see Fig. 7.4). The zeroth-order solution is bounded and oscillatory for all x. But when $\varepsilon > 0$, local analysis of the exact solution for large x shows that it is a linear combination of exponentially increasing and decreasing functions (Prob. 7.20). This change in character of the solution occurs because it is certainly wrong to neglect εx compared with 1 when x is bigger than $1/\varepsilon$. In fact, a more careful argument shows that the term εx is not a small perturbation unless $x \ll \varepsilon^{-1/2}$ (Prob. 7.20).

Example 4 shows that the interval itself can determine whether a perturbation problem is regular or singular. We examine more examples having this property in the next section on eigenvalue problems. The feature that is common to all such examples is that an nth-order perturbative approximation bears less and less resemblance to the exact solution as x increases.

For these sorts of problems Chap. 11 introduces new perturbative procedures called multiple-scale methods which substantially improve the rather poor predictions of ordinary perturbation theory. The particular problem in Example 4 is reconsidered in Prob. 11.13.

Example 5 *Roots of a high-degree polynomial.* When a perturbation problem is regular, the perturbation series is convergent and the exact solution is a smooth analytic function of ε for

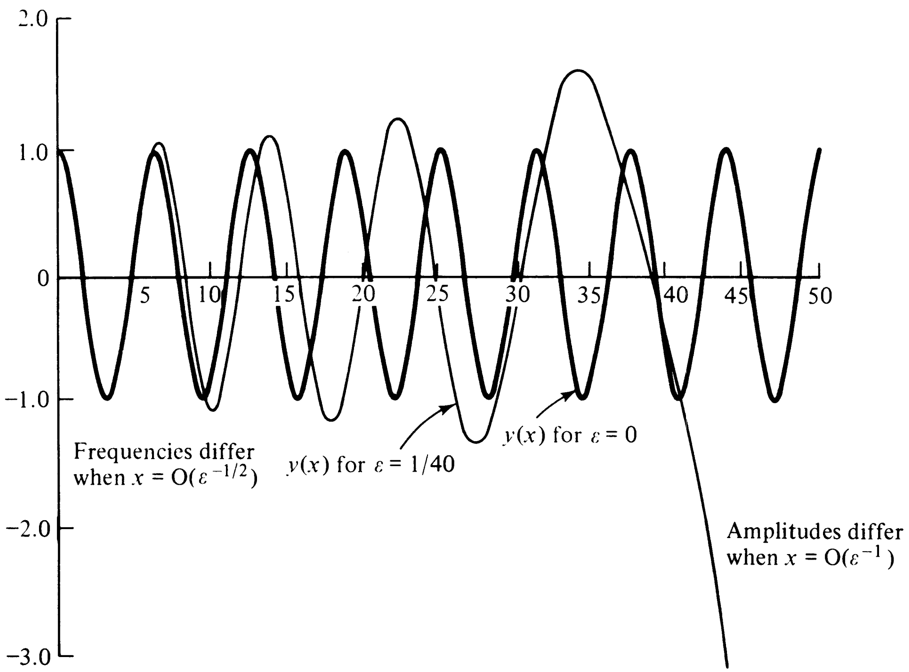

Figure 7.4 Exact solutions to the initial-value problem $y'' + (1 - \varepsilon x)y = 0$ $[y(0) = 1,\ y'(0) = 0]$ in (7.2.10) for $\varepsilon = 0$ and $\varepsilon = \frac{1}{40}$. Although this is a regular perturbation problem on the finite interval $0 \le x \le L$, it is a singular perturbation problem on the infinite interval $0 \le x \le \infty$ because the perturbed solution ($\varepsilon > 0$) is not close to the unperturbed solution ($\varepsilon = 0$), no matter how small ε is. When $x = O(\varepsilon^{-1/2})$ the frequencies begin to differ (the curves become phase shifted) and when $x = O(\varepsilon^{-1})$ the amplitudes differ (one curve remains finite while the other grows exponentially).

sufficiently small ε. However, just what is "sufficiently small" may vary enormously from problem to problem. A striking example by Wilkinson concerns the roots of the polynomial

$$\prod_{k=1}^{20} (x - k) + \varepsilon x^{19} = x^{20} - (210 - \varepsilon)x^{19} + \cdots + 20! \tag{7.2.12}$$

The perturbation εx^{19} is regular, since no roots are lost in the limit $\varepsilon \to 0$; the roots of the unperturbed polynomial lie at 1, 2, 3, ..., 20.

Let us now take $\varepsilon = 10^{-9}$ so that the perturbation in the coefficient of x^{19} is of relative magnitude $10^{-9}/210$, or roughly 10^{-11}. For such a small regular perturbation one might expect the 20 roots to be only very slightly displaced from their $\varepsilon = 0$ values. The actual displaced roots are given in Table 7.1. One is surprised to find that while some roots are relatively unchanged by the perturbation, others have paired into complex conjugates. The qualitative effect on the roots of varying ε is shown in Figs. 7.5 and 7.6. In these plots the paths of the roots are traced as a function of ε. As $|\varepsilon|$ increases, the roots coalesce into pairs of complex conjugate roots. Evidently, a "small" perturbation is one for which $|\varepsilon| < 10^{-11}$, while $|\varepsilon| \gtrsim 10^{-10}$ is a "large" perturbation for at least some of the roots. Low-order regular perturbation theory may be used to understand this behavior (Probs. 7.22 and 7.23).

Table 7.1 Roots of the Wilkinson polynomial (7.2.12) with $\varepsilon = 10^{-9}$

The first column lists the unperturbed ($\varepsilon = 0$) roots $1, 2, \ldots, 20$; the second column gives the results of first-order perturbation theory (see Prob. 7.22); the third column gives the exact roots. The unperturbed roots at 13 and 14, 15 and 16, and 17 and 18 are perturbed into complex-conjugate pairs. Observe that while first-order perturbation theory is moderately accurate for the real perturbed roots near $1, 2, \ldots, 12, 19, 20$, it cannot predict the locations of the complex roots (but see Prob. 7.23)

Unperturbed root	First-order perturbation theory	Exact root
1	1.000 000 000 0	1.000 000 000 0
2	2.000 000 000 0	2.000 000 000 0
3	3.000 000 000 0	3.000 000 000 0
4	4.000 000 000 0	4.000 000 000 0
5	5.000 000 000 0	5.000 000 000 0
6	5.999 999 941 8	5.999 999 941 8
7	7.000 002 542 4	7.000 002 542 4
8	7.999 994 030 4	7.999 994 031 5
9	9.000 839 327 5	9.000 841 033 5
10	9.992 405 941 6	9.992 518 124 0
11	11.046 444 571	11.050 622 592
12	11.801 496 835	11.832 935 987
13	13.605 558 629	$13.349\,018\,036 \pm 0.532\,765\,750\,0i$
14	12.667 031 557	
15	17.119 065 220	$15.457\,790\,724 \pm 0.899\,341\,526\,2i$
16	13.592 486 027	
17	18.904 402 150	$17.662\,434\,477 \pm 0.704\,285\,236\,9i$
18	17.004 413 300	
19	19.309 013 459	19.233 703 334
20	19.956 900 195	19.950 949 654

This example shows that the roots of high-degree polynomials may be extraordinarily sensitive to changes in the coefficients of the polynomial, even though the perturbation problem so obtained is regular. It should serve as ample warning to a "number cruncher" not to trust computer output without sufficient understanding of the nature of the problem being solved.

(I) 7.3 PERTURBATION METHODS FOR LINEAR EIGENVALUE PROBLEMS

In this section we show how perturbation theory can be used to approximate the eigenvalues and eigenfunctions of the Schrödinger equation

$$\left[-\frac{d^2}{dx^2} + V(x) + W(x) - E\right] y(x) = 0, \tag{7.3.1}$$

subject to the boundary condition

$$\lim_{|x|\to\infty} y(x) = 0. \tag{7.3.2}$$

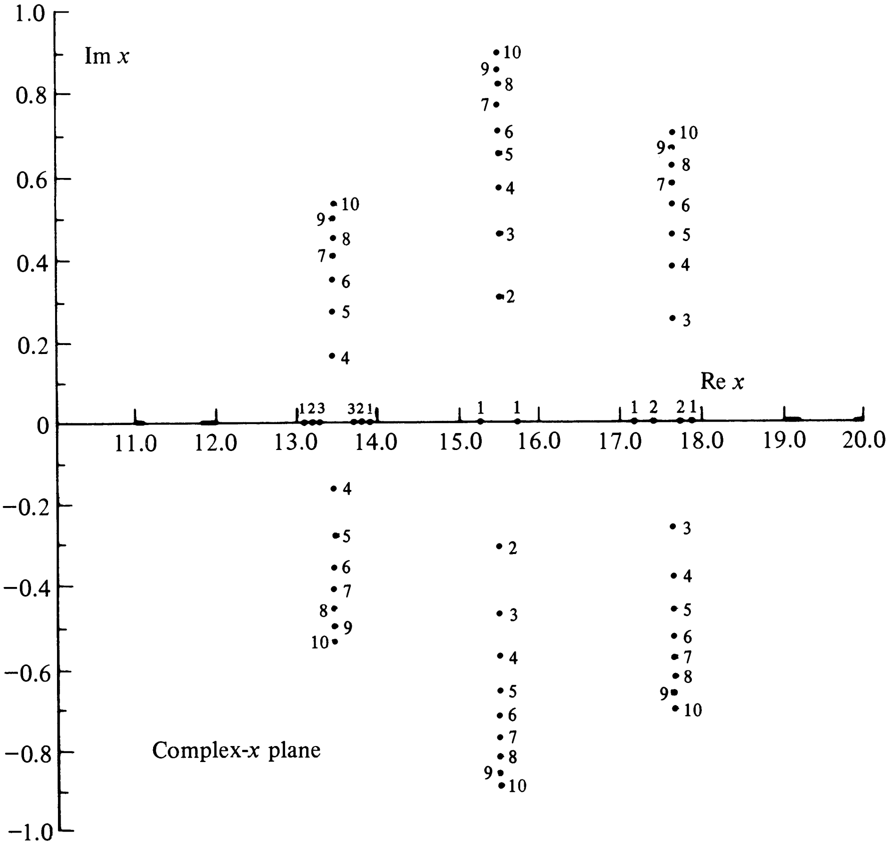

Figure 7.5 Roots of the Wilkinson polynomial $(x-1)(x-2)(x-3)\cdots(x-20)+\varepsilon x^{19}$ in (7.2.12) for 11 values of ε. When $\varepsilon = 0$ the roots shown are 10, 11, . . . , 20. As ε is allowed to increase very slowly, the roots move toward each other in pairs along the real-x axis and then veer off in opposite directions into the complex-x plane. We have plotted the roots for $\varepsilon = 0, 10^{-10}, 2 \times 10^{-10}, 3 \times 10^{-10}, \ldots, 10^{-9}$. Some of the roots are numbered to indicate the value of ε to which they correspond; that is, 6 means $\varepsilon = 6 \times 10^{-10}$, 3 means $\varepsilon = 3 \times 10^{-10}$, and so on. The roots starting at 11, 12, 19, and 20 move too slowly to be seen as individual dots. We conclude from this plot that very slight changes in the coefficients of a polynomial can cause drastic changes in the values of some of the roots; one must be cautious when performing numerical calculations.

In (7.3.1) E is called the energy eigenvalue and $V + W$ is called the potential. We assume that $V(x)$ and $W(x)$ are continuous functions and that both $V(x)$ and $V(x) + W(x)$ approach ∞ as $|x| \to \infty$.

We suppose that the function $V(x) + W(x)$ is so complicated that (7.3.1) is not soluble in closed form. One can still prove from the above assumptions that nontrivial solutions $[y(x) \not\equiv 0]$ satisfying (7.3.1) and (7.3.2) exist for special discrete values of E, the allowed eigenvalues of the equation (see Sec. 1.8). On the other hand, we assume that removing the term $W(x)$ from (7.3.1) makes the equation an

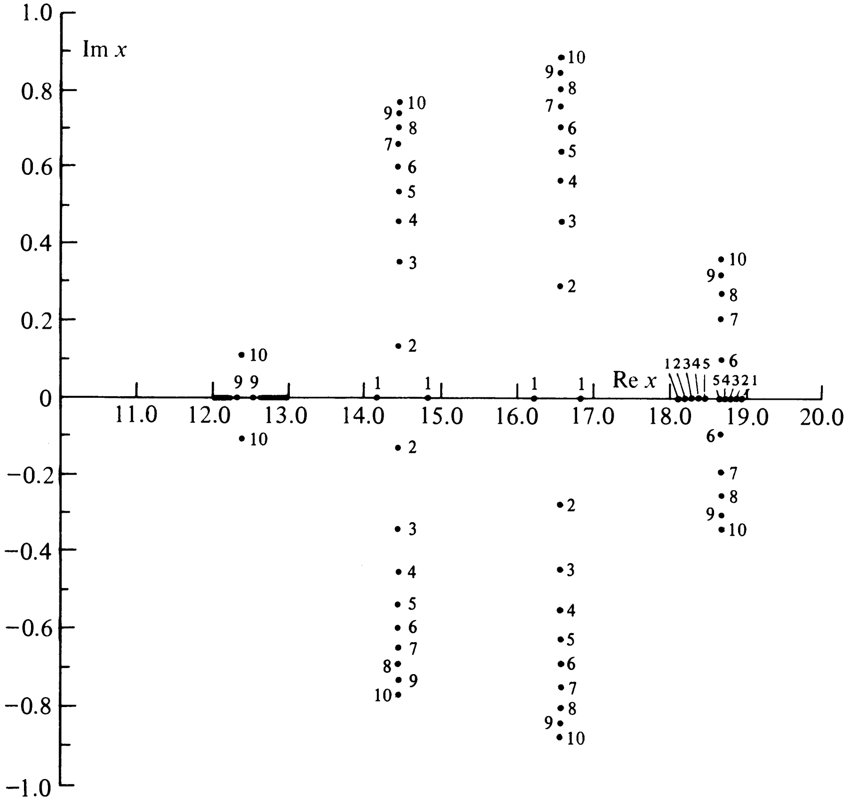

Figure 7.6 Same as in Fig. 7.5 except that the values of ε are 0, -10^{-10}, -2×10^{-10}, $-3 \times 10^{-10}, \ldots, -10^{-9}$. The roots pair up and veer off into the complex-x plane, but the pairs are not the same as in Fig. 7.5.

exactly soluble eigenvalue problem. This suggests using perturbation theory to solve the family of eigenvalue problems in which $W(x)$ is replaced by $\varepsilon W(x)$:

$$\left[-\frac{d^2}{dx^2} + V(x) + \varepsilon W(x) - E\right] y(x) = 0. \tag{7.3.3}$$

Our assumptions on the nature of $V(x)$ and $W(x)$ leave no choice about where to introduce the parameter ε if the unperturbed problem is to be exactly soluble.

Example 1 *An exactly soluble eigenvalue problem.* Several exactly soluble eigenvalue problems are given in Sec. 1.8. One such example, which is used extensively in this section, is obtained if we take $V(x) = x^2/4$. The unperturbed problem is the Schrödinger equation for the quantum-mechanical harmonic oscillator, which is just the parabolic cylinder equation

$$-y'' + \frac{x^2}{4} y - Ey = 0. \tag{7.3.4}$$

We have already shown that solutions to this equation behave like $e^{\pm x^2/4}$ as $|x| \to \infty$.

There is a discrete set of values of E for which a solution that behaves like $e^{-x^2/4}$ as $x \to \infty$ also behaves like $e^{-x^2/4}$ as $x \to -\infty$ (see Example 4 of Sec. 3.5 and Example 9 of Sec. 3.8). These values of E are

$$E = n + \tfrac{1}{2}, \qquad n = 0, 1, 2, \ldots, \tag{7.3.5}$$

and the associated eigenfunctions are parabolic cylinder functions

$$y_n(x) = D_n(x) = e^{-x^2/4}\,\mathrm{He}_n\,(x), \tag{7.3.6}$$

where $\mathrm{He}_n\,(x)$ is the Hermite polynomial of degree n: $\mathrm{He}_0\,(x) = 1$, $\mathrm{He}_1\,(x) = x$, $\mathrm{He}_2\,(x) = x^2 - 1, \ldots.$

In general, once an eigenvalue E_0 and an eigenfunction $y_0(x)$ of the unperturbed problem

$$\left[-\frac{d^2}{dx^2} + V(x) - E_0\right] y_0(x) = 0 \tag{7.3.7}$$

have been found, we may seek a perturbative solution to (7.3.3) of the form

$$E = \sum_{n=0}^{\infty} E_n \varepsilon^n, \tag{7.3.8}$$

$$y(x) = \sum_{n=0}^{\infty} y_n(x)\varepsilon^n. \tag{7.3.9}$$

Substituting (7.3.8) and (7.3.9) into (7.3.3) and comparing powers of ε gives the following sequence of equations:

$$\left[-\frac{d^2}{dx^2} + V(x) - E_0\right] y_n(x) = -Wy_{n-1}(x) + \sum_{j=1}^{n} E_j y_{n-j}(x),$$
$$n = 1, 2, 3, \ldots, \tag{7.3.10}$$

whose solutions must satisfy the boundary conditions

$$\lim_{|x| \to \infty} y_n(x) = 0, \qquad n = 1, 2, 3, \ldots. \tag{7.3.11}$$

Equation (7.3.10) is linear and inhomogeneous. The associated homogeneous equation is just the unperturbed problem and thus is soluble by assumption. However, technically speaking, only *one* of the two linearly independent solutions of the unperturbed problem (the one that satisfies the boundary conditions) is assumed known. Therefore, we proceed by the method of reduction of order (see Sec. 1.4); to wit, we substitute

$$y_n(x) = y_0(x)F_n(x), \tag{7.3.12}$$

where $F_0(x) = 1$, into (7.3.10). Simplifying the result using (7.3.7) and multiplying by the integrating factor $y_0(x)$ gives

$$\frac{d}{dx}\left[y_0^2(x)F_n'(x)\right] = y_0^2(x)\left[W(x)F_{n-1}(x) - \sum_{j=1}^{n} E_j F_{n-j}(x)\right]. \tag{7.3.13}$$

If we integrate this equation from $-\infty$ to ∞ and use $y_0^2(x)F_n'(x) = y_0(x)y_n'(x) - y_0'(x)y_n(x) \to 0$ as $|x| \to \infty$, we obtain the formula for the coefficient E_n:

$$E_n = \frac{\displaystyle\int_{-\infty}^{\infty} y_0(x)\left[W(x)y_{n-1}(x) - \sum_{j=1}^{n-1} E_j y_{n-j}(x)\right] dx}{\displaystyle\int_{-\infty}^{\infty} y_0^2(x)\, dx},$$

$$n = 1, 2, 3, \ldots, \qquad (7.3.14)$$

from which we have eliminated all reference to $F_n(x)$. [The sum on the right side of (7.3.14) is defined to be 0 when $n = 1$.]

Integrating (7.3.13) twice gives the formula for $y_n(x)$:

$$y_n(x) = y_0(x) \int_a^x \frac{dt}{y_0^2(t)} \int_{-\infty}^t ds\, y_0(s)\left[W(s)y_{n-1}(s) - \sum_{j=1}^{n} E_j y_{n-j}(s)\right],$$

$$n = 1, 2, 3, \ldots. \qquad (7.3.15)$$

Observe that in (7.3.15) a is an arbitrary number at which we choose to impose $y_n(a) = 0$. This means we have fixed the overall normalization of $y(x)$ so that $y(a) = y_0(a)$ [assuming that $y_0(a) \neq 0$]. If $y_0(t)$ vanishes between a and x, the integral in (7.3.15) seems formally divergent; however, $y_n(x)$ satisfies a differential equation (7.3.10) which has no finite singular points. Thus, it is possible to define $y_n(x)$ everywhere as a finite expression (see Prob. 7.24).

Equations (7.3.14) and (7.3.15) together constitute an iterative procedure for calculating the coefficients in the perturbation series for E and $y(x)$. Once the coefficients $E_0, E_1, \ldots, E_{n-1}, y_0, y_1, \ldots, y_{n-1}$ are known, (7.3.14) gives E_n, and once E_n has been calculated (7.3.15) gives y_n. The remaining question is whether or not these perturbation series are convergent.

Example 2 *A regular perturbative eigenvalue problem.* Let $V(x) = x^2/4$ and $W(x) = x$. It may be shown (Prob. 7.25) that the perturbation series for $y(x)$ is convergent for all ε and that the series for E has vanishing terms of order ε^n for $n \geq 3$. This is a regular perturbation problem.

Example 3 *A singular perturbative eigenvalue problem.* It may be shown (Prob. 7.26) that if $V(x) = x^2/4$ and $W(x) = x^4/4$, then the perturbation series for the smallest eigenvalue for positive ε is

$$E(\varepsilon) \sim \tfrac{1}{2} + \tfrac{3}{4}\varepsilon - \tfrac{21}{8}\varepsilon^2 + \tfrac{333}{16}\varepsilon^3 + \cdots, \qquad \varepsilon \to 0+. \qquad (7.3.16)$$

The terms in this series appear to be getting larger and suggest that this series may be divergent for all $\varepsilon \neq 0$. Indeed, (7.3.16) diverges for all ε because the nth term satisfies $E_n \sim -(-3)^n\Gamma(n + \frac{1}{2})\sqrt{6}/\pi^{3/2}$ $(n \to \infty)$. (This is a nontrivial result that we do not explain here.)

The divergence of the perturbation series in Example 3 indicates that the perturbation problem is singular. A simple way to observe the singular behavior is to compare $e^{-x^2/4}$, the controlling factor of the large-x behavior of the unperturbed ($\varepsilon = 0$) solution, with $e^{-x^3\sqrt{\varepsilon}/6}$, the controlling factor of the large-x behavior for $\varepsilon \neq 0$. There is an abrupt change in the nature of the solution when we pass to the limit ($\varepsilon \to 0+$). This phenomenon occurs because the perturbing term $\varepsilon x^4/4$ is not small compared with $x^2/4$ when x is large.

If the functions $V(x)$ and $W(x)$ in Example 3 were interchanged, then the resulting eigenvalue problem would be a regular perturbation problem because εx^2 is a small perturbation of x^4 for all $|x| < \infty$. However, the unperturbed problem, $(-d^2/dx^2 + x^4/4 - E_0)y_0(x) = 0$, is not soluble in closed form. Thus, it would not be possible to use (7.3.14) and (7.3.15) to compute the coefficients in the perturbation series analytically.

Also note that if the boundary conditions in Example 3 were given at $x = \pm A$, $A < \infty$, then the perturbation theory would be regular. This is because here εx^4 is a small perturbation of x^2. However, it is much more difficult to solve the unperturbed problem on a finite interval.

Thus, one is forced to accept a solution to Example 3 in the form of a divergent series. Fortunately, this series is one of many that may be summed by Padé theory to give a finite and unique result (see Sec. 8.3).

Example 4 *Another regular perturbation problem.* When $V = x^2/4$ and $W = |x|$ the perturbation problem is regular. But unlike the problem in Example 2, this perturbation series is not convergent for all ε; the series in (7.3.8) and (7.3.9) have finite radii of convergence. The significance of the finite radius of convergence is discussed in Sec. 7.5.

(D) 7.4 ASYMPTOTIC MATCHING

The purpose of this section is to introduce the notion of matched asymptotic expansions. Asymptotic matching is an important perturbative method which is used often in both boundary-layer theory (Chap. 9) and WKB theory (Chap. 10) to determine analytically the approximate global properties of the solution to a differential equation. Asymptotic matching is usually used to determine a uniform approximation to the solution of a differential equation and to find other global properties of differential equations such as eigenvalues. Asymptotic matching may also be used to develop approximations to integrals.

The principle of asymptotic matching is simple. The interval on which a boundary-value problem is posed is broken into a sequence of two or more *overlapping* subintervals. Then, on each subinterval perturbation theory is used to obtain an asymptotic approximation to the solution of the differential equation valid on that interval. Finally, the matching is done by requiring that the asymptotic approximations have the same functional form on the overlap of every pair of intervals. This gives a sequence of asymptotic approximations to the solution of the differential equation; by construction, each approximation satisfies all the boundary conditions given at various points on the interval. Thus, the end result is an approximate solution to a boundary-value problem valid over the entire interval.

Asymptotic matching bears a slight resemblance to an elementary technique for solving boundary-value problems called *patching*. Patching is helpful when the differential equation can be solved in closed form. Here is a simple example:

Example 1 *Patching.* The method of patching may be used to solve the boundary-value problem $y'' - y = e^{-|x|}$ $[y(\pm\infty) = 0]$. There are two regions to consider. When $x \le 0$, the most general solution which satisfies the boundary condition $y(-\infty) = 0$ is

$$y(x) = ae^x + \tfrac{1}{2}xe^x, \tag{7.4.1}$$

where a is a constant to be determined. When $x \geq 0$, the most general solution which satisfies $y(\infty) = 0$ is

$$y(x) = be^{-x} - \tfrac{1}{2}xe^{-x}, \tag{7.4.2}$$

where b is another constant to be determined. Solutions (7.4.1) and (7.4.2) are now patched together at the one point common to both regions, namely, $x = 0$. That is, we require that $y(x)$ and $y'(x)$ be continuous at $x = 0$. This gives a pair of equations for a and b whose solution is $a = b = -\frac{1}{2}$. Substituting these values of a and b into (7.4.1) and (7.4.2) gives the exact solution to the boundary-value problem: $y(x) = -\frac{1}{2}e^{-|x|} - \frac{1}{2}|x|e^{-|x|}$.

Matching is different from patching because an asymptotic approximation to the solution of the differential equation rather than the exact solution gets matched. Moreover, matching is done by comparing functions over an *interval* while patching is done by comparing functions and their derivatives at a *point*. In general, the length of the matching interval approaches ∞ as ε, the perturbing parameter, approaches 0. Here are several examples to introduce the techniques of asymptotic matching.

Example 2 *Asymptotic matching for a first-order differential equation.* The first-order differential equation

$$y' + (\varepsilon x^2 + 1 + 1/x^2)y = 0, \qquad y(1) = 1,$$

is exactly soluble on the interval $1 \leq x < \infty$. Nevertheless, we will use the principles of asymptotic matching to study the approximate behavior of the solution as $\varepsilon \to 0+$. When x is not too large, the term εx^2 is negligible so an approximate equation for y is

$$y_L' + (1 + 1/x^2)y_L = 0,$$

where the subscript L refers to the *left* region. The solution to this equation which satisfies $y_L(1) = 1$ is

$$y_L = e^{-x+1/x}. \tag{7.4.3}$$

When x is large, εx^2 is no longer negligible but $1/x^2$ is. Therefore, an approximate equation valid as $x \to +\infty$ is

$$y_R' + (\varepsilon x^2 + 1)y_R = 0,$$

where the subscript R refers to the *right* region. The solution to this equation is

$$y_R = ae^{-\varepsilon x^3/3 - x}, \tag{7.4.4}$$

where a is a constant.

There is a common region of validity of the two solutions (7.4.3) and (7.4.4) which enables us to determine the approximate value of a. For those values of x lying in the range

$$\varepsilon^{-1/5} \ll x \ll \varepsilon^{-1/4}, \qquad \varepsilon \to 0+, \tag{7.4.5}$$

x is so large that (7.4.3) may be approximated by $y_L(x) = e^{-x+1/x} \sim e^{-x}$ ($\varepsilon \to 0+$), but x is still small enough that (7.4.4) may be approximated by $y_R(x) = ae^{-\varepsilon x^3/3 - x} \sim ae^{-x}$ ($\varepsilon \to 0+$). In the overlap region both solutions have the *same functional dependence* on x. If both asymptotic expansions are to agree in the overlap region (7.4.5), then we must choose $a \sim 1$ ($\varepsilon \to 0+$). This is asymptotic matching; we have obtained a global approximation to the original differential equation. The condition $y(1) = 1$ translates into the condition that $a \sim 1$ ($\varepsilon \to 0+$), which completely determines the approximation to $y(x)$ as $x \to \infty$. Note that the extent of the matching region in (7.4.5) becomes infinite as $\varepsilon \to 0+$.

The matching region (7.4.5) that we have chosen is not the only possible choice. We could have matched on the interval $\varepsilon^{-1/6} \ll x \ll \varepsilon^{-1/5}$ ($\varepsilon \to 0+$) or even the interval

$\varepsilon^{-1/99} \ll x \ll \varepsilon^{-32/99}$ ($\varepsilon \to 0+$)! These regions all work because they satisfy the general matching criterion that x lie in the asymptotic interval $1 \ll x \ll \varepsilon^{-1/3}$ as $\varepsilon \to 0+$ (see Prob. 7.30).

Example 3 *Asymptotic matching in higher order.* Let us return to the differential equation in Example 2 and carry out the asymptotic matching to first order in ε. First we consider the left region. In Example 2 we merely discarded the term εx^2. Now let us seek an orderly perturbative expansion of the solution in powers of ε: $y_L = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots$, where $y_0(1) = 1$, $y_1(1) = 0$, $y_2(1) = 0, \ldots$. As before, $y_0 = e^{-x+1/x}$. y_1 satisfies the inhomogenous equation

$$y_1'(x) + \left(1 + \frac{1}{x^2}\right) y_1(x) = -x^2 y_0(x),$$

whose solution is

$$y_1(x) = \left(\frac{1}{3} - \frac{x^3}{3}\right) e^{-x+1/x}.$$

Hence a first-order approximation to y_L is

$$y_L = e^{-x+1/x}[1 + \varepsilon(1 - x^3)/3 + O(\varepsilon^2 x^6)], \qquad \varepsilon \to 0+, \tag{7.4.6}$$

where the form of the error term is suggested by an examination of $y_2(x)$ (see Prob. 7.30).

Next we consider the right region. A more accurate estimate of the behavior of y for large x is found by the usual technique of substituting $y = e^{S(x)}$, as explained in Chap. 3. This method normally generates only the asymptotic expansion of $y(x)$ valid as $x \to \infty$, but in this example it gives the exact solution because the differential equation for $y(x)$ is first order: $y_R(x) = ae^{-\varepsilon x^3/3 - x + 1/x}$. In the overlap region this expression is approximately

$$y_R(x) = ae^{-x+1/x}[1 - \varepsilon x^3/3 + O(\varepsilon^2 x^6)], \qquad \varepsilon \to 0+. \tag{7.4.7}$$

Comparing (7.4.6) and (7.4.7) determines a to first order in ε: $a = 1 + \varepsilon/3 + O(\varepsilon^2)$ ($\varepsilon \to 0+$). Note that the exact value of a is $e^{\varepsilon/3}$. The method of matched asymptotic expansions has given the first two terms in the expansion of a for small ε.

Example 4 *Asymptotic matching for a second-order differential equation.* Unlike the first-order differential equation of Examples 2 and 3, the equation

$$y'' + \left(\nu + \frac{1}{2} - \frac{x^2}{4} - \varepsilon x^4\right) y = 0, \tag{7.4.8}$$

where ν is a parameter and $\varepsilon \to 0+$, does not have a closed-form solution. Nevertheless, the method of matched asymptotic expansions may be used to obtain an approximate solution to the boundary-value problem $y(0) = 1$, $y(+\infty) = 0$.

When x is so small that εx^4 is negligible compared with $x^2/4$, the original differential equation (7.4.8) may be approximated by the simpler equation $y'' + (\nu + \frac{1}{2} - x^2/4)y = 0$. This is a parabolic cylinder equation; a solution which decays exponentially for large x and satisfies $y(0) = 1$ is

$$y(x) = D_\nu(x)/D_\nu(0). \tag{7.4.9}$$

[We could also include in (7.4.9) a linearly independent solution like $D_\nu(-x)$ (when ν is nonintegral) which grows exponentially for large x, but such a solution would immediately be rejected in the course of asymptotic matching, as we will shortly see.]

When x is large and positive, we can perform a local analysis of (7.4.8). The usual procedure is to substitute $y = e^{S(x)}$ and to look for the exponentially decaying solution. The leading term in the asymptotic expansion of $y(x)$ for large x has the rather complicated form (see Prob. 7.31)

$$y(x) \sim a\left(\frac{x^2}{4} + \varepsilon x^4\right)^{-1/4} \left(\frac{\sqrt{\frac{1}{4} + \varepsilon x^2} - \frac{1}{2}}{\sqrt{\frac{1}{4} + \varepsilon x^2} + \frac{1}{2}}\right)^{\nu/2 + 1/4} \exp\left[-\frac{1}{3\varepsilon}\left(\varepsilon x^2 + \frac{1}{4}\right)^{3/2}\right], \qquad x \to +\infty. \tag{7.4.10}$$

The constant a cannot be determined by a local analysis of (7.4.8). However, asymptotic matching supplies the connection between the behavior at ∞ and the boundary condition $y(0) = 1$, and can thus be used to determine a approximately.

The asymptotic matching procedure relies on the existence of an overlap region. We will seek an overlap region where x is so large that it is valid to use (7.4.10) but where the term εx^4 is still negligible compared with $x^2/4$ so that it is valid to use (7.4.9). The interval

$$\varepsilon^{-1/4} \le x \le \varepsilon^{-1/3} \tag{7.4.11}$$

satisfies these constraints. Note that this interval becomes infinitely long as $\varepsilon \to 0+$ (see Prob. 7.32).

When x lies in this overlap region, the asymptotic approximation in (7.4.10) may be greatly simplified because εx^2 is small compared with $\frac{1}{4}$:

$$-\frac{1}{3\varepsilon}\left(\frac{1}{4} + \varepsilon x^2\right)^{3/2} \sim -\frac{1}{24\varepsilon} - \frac{1}{4}x^2,$$

$$\left(\frac{\sqrt{\frac{1}{4} + \varepsilon x^2} - \frac{1}{2}}{\sqrt{\frac{1}{4} + \varepsilon x^2} + \frac{1}{2}}\right) \sim \varepsilon x^2,$$

$$\left(\frac{x^2}{4} + \varepsilon x^4\right)^{-1/4} \sim \left(\frac{2}{x}\right)^{1/2}, \qquad \varepsilon \to 0+.$$

Thus,

$$y(x) \sim a 2^{1/2} \varepsilon^{\nu/2 + 1/4} e^{-1/24\varepsilon} x^{\nu} e^{-x^2/4}, \qquad \varepsilon \to 0+. \tag{7.4.12}$$

On the other hand, the expression in (7.4.9) may be replaced by its asymptotic behavior for large x when x lies in the overlap region (7.4.11) (see Chap. 3):

$$y(x) \sim \frac{1}{D_\nu(0)} x^\nu e^{-x^2/4}, \qquad \varepsilon \to 0+. \tag{7.4.13}$$

Throughout the overlap region the two asymptotic expansions (7.4.12) and (7.4.13) match; they exhibit the same dependence on x! We thus conclude that to lowest order in ε, an expression for a is

$$a = \frac{1}{D_\nu(0)} 2^{-1/2} \varepsilon^{-\nu/2 - 1/4} e^{1/24\varepsilon}.$$

Our treatment of asymptotic matching has no doubt raised several questions of principle. How do we know that an overlap region necessarily exists? If it does exist, how can we predict its size? What do we do if there is no overlap region? We postpone a discussion of these serious questions until Part IV. Our goal here is merely to introduce the mechanical aspects of asymptotic matching. With this in mind, we give three more examples which, although they are somewhat involved, clearly illustrate the depth of analytical power which the method of asymptotic matching can provide.

Example 5 *Nonlinear eigenvalue problem.* Asymptotic matching may be used to find an asymptotic approximation to the large positive eigenvalues E of the boundary-value problem

$$(x^2 - 1)y''(x) + xy'(x) + (E^2 - 2Ex)y(x) = 0, \qquad y(1) = 0, \; y(\infty) = 0. \tag{7.4.14}$$

This eigenvalue problem is different from those considered in Sec. 7.3 because the eigenvalue E appears nonlinearly.

The differential equation (7.2.3) may be recast into a somewhat more familiar form by substituting $x = \cosh s$. In terms of the new variable s (7.4.14) becomes

$$y''(s) + (E^2 - 2E \cosh s)y = 0,$$

which is called the associated Mathieu equation (Mathieu equation of imaginary argument). It becomes the ordinary Mathieu equation

$$y''(t) + (\alpha + \beta \cos t)y = 0$$

when we replace s by it.

This is an interesting but terribly impractical result! Recognizing that we are solving the Mathieu equation is by itself no progress; at best it assures us that it will be fruitless to search for a simple analytical expression for E. We thus return to (7.4.14) and attempt a perturbative solution. Let us take E to be large and positive and choose the perturbing parameter for this problem to be $1/E$. We will attempt to find an approximate formula for those eigenvalues E which are large.

When E is large, we can decompose the full interval $1 \le x < \infty$ into two smaller and overlapping regions: region I for which $1 \le x \ll E$ $(E \to +\infty)$ and region II for which $1 \ll x$ $(E \to +\infty)$. Note that the overlap region, which includes values of x lying between but far from 1 and E, becomes infinitely long as $E \to +\infty$. We will show that it is fairly easy to solve (7.4.14) in regions I and II separately. We will then require that both approximations agree in the overlap region. This requirement translates into a condition which determines the eigenvalues.

Throughout region I the term $2Ex$ is small compared with E^2. When x is of order 1, it is valid to neglect $2Ex$ compared with E^2. When we do so, the resulting differential equation

$$(x^2 - 1)y_1''(x) + xy_1'(x) + E^2 y_1(x) = 0$$

is soluble. To solve this equation we again let $x = \cosh s$ and obtain

$$y_1''(s) + E^2 y_1(s) = 0,$$

whose solutions are $y_1 = \cos Es$ and $y_1 = \sin Es$. But since

$$s = \operatorname{arc\,cosh} x = \ln [x + (x^2 - 1)^{1/2}]$$

and since $y(x)$ must satisfy the boundary condition $y(1) = 0$, we have

$$y_{\mathrm{I}}(x) = A \sin [E \ln (x + \sqrt{x^2 - 1})], \tag{7.4.15}$$

where A is an undetermined constant.

One should note that (7.4.15) is not quite valid throughout region I, but only when $x = 0(1)$. To understand why, one can substitute (7.4.15) into (7.4.14) and observe that terms containing E do not all cancel. A solution which uniformly satisfies (7.4.14) up to terms of order 1 for all x in region I is (see Prob. 7.35)

$$y_{\mathrm{I}}(x) = A\left(1 + \frac{x}{2E}\right) \sin [E \ln (x + \sqrt{x^2 - 1}) - \sqrt{x^2 - 1}]. \tag{7.4.16}$$

Equation (7.4.16) is a higher-order perturbative approximation to the solution of (7.4.14) than is (7.4.15). In general, one can solve (7.4.14) to all orders in powers of $1/E$ as an expression of the form

$$y_{\mathrm{I}}(x) = A\left[1 + \sum_{n=1}^{\infty} f_n(x)E^{-n}\right] \sin \left[E \ln (x + \sqrt{x^2 - 1}) + \sum_{n=0}^{\infty} g_n(x)E^{-n}\right] \tag{7.4.17}$$

(see Prob. 7.35), where the two series in brackets are asymptotic series in powers of $1/E$ valid as $E \to \infty$.

The matching region is the interval $1 \ll x \ll E$ as $E \to +\infty$. Since x is large on this interval, we may approximate (7.4.16) by its asymptotic expansion

$$y_{\text{overlap}}(x) = Af(x) \sin g(x),$$

$$f(x) \sim 1 + \frac{x}{2E}, \tag{7.4.18}$$

$$g(x) \sim E \ln (2x) - x, \qquad E \to \infty,\ 1 \ll x \ll E.$$

This completes the analysis of region I.

Throughout region II, the term -1 is small compared with x^2 so we may replace (7.4.14) with the approximate differential equation

$$x^2 y_{\text{II}}''(x) + xy_{\text{II}}' + (E^2 - 2Ex)y_{\text{II}} = 0. \tag{7.4.19}$$

Although the difference between (7.4.14) and (7.4.19) may seem slight, (7.4.19) is now soluble. The substitution $t = \sqrt{8Ex}$ converts (7.4.19) into

$$t^2 y''(t) + ty'(t) + (4E^2 - t^2)y(t) = 0,$$

which is a modified Bessel equation. Two linearly independent solutions are $y_{\text{II}} = I_{2iE}(t),\ K_{2iE}(t)$.

In Example 2 of Sec. 3.5 we showed that for large positive argument $I_\nu(x)$ grows exponentially like $e^x/\sqrt{x}$ and that $K_\nu(x)$ decays exponentially like $e^{-x}/\sqrt{x}$. Thus, the most general solution of (7.4.19) which satisfies the boundary condition $y(\infty) = 0$ is

$$y_{\text{II}} = BK_{2iE}(\sqrt{8Ex}), \tag{7.4.20}$$

where B is an arbitrary multiplicative constant.

Finally, recall from Prob. 6.71 that when p and z are real and $p \gg z$, the leading behavior of $K_{ip}(z)$ is

$$\sqrt{2\pi}\,(p^2 - z^2)^{-1/4} e^{-p\pi/2} \sin [p \cosh^{-1} (p/z) - \sqrt{p^2 - z^2} + \pi/4].$$

In our case $p = 2E$ and $z = \sqrt{8Ex}$, so in the overlap region the condition $p \gg z$ actually holds. Hence, the leading behavior of $y_{\text{II}}(x)$ in the overlap region is

$$-B\left(1 + \frac{x}{2E}\right)\sqrt{\frac{\pi}{E}}\, e^{-E\pi} \sin\left(E \ln 2x - x + 2E - E \ln 4E - \frac{\pi}{4}\right),$$

$$E \to \infty,\ 1 \ll x \ll E. \tag{7.4.21}$$

This completes the analysis of region II.

We now have two asymptotic approximations (7.4.18) and (7.4.21) to $y(x)$ in the overlap region. Since they both approximate the same function, they must agree over the entire overlap region. Thus, $2E - E \ln 4E - \pi/4$ must be an integral multiple of π. This condition gives a simple approximate formula for the eigenvalues E which becomes increasingly accurate as $E \to \infty$. It states that as $n \to \infty$ the nth eigenvalue E_n satisfies the equation

$$E_n \ln 4E_n - 2E_n = (n + \tfrac{3}{4})\pi, \qquad n = 0, 1, 2, \ldots. \tag{7.4.22}$$

How accurate is this result? In Table 7.2 we compare the exact eigenvalues (obtained numerically on a computer) with the solutions of (7.4.22) for values of n ranging from 0 to 8. Observe that the percentage error does indeed decrease as n increases. But what is most remarkable is that the error is never more than 3.38 percent. This entire calculation rests on the assumption that E is large. However, it is never clear just how large is large. One might imagine that E must be a million or so before one can believe the result in (7.4.22). However, the computer calculation shows that $E = 3$ is large enough to give 3 percent errors. It is a common experience that asymptotic calculations tend to give errors which are far smaller than what one might reasonably have expected.

The asymptotic match gives the eigenfunctions as well as the eigenvalues. Demanding that (7.4.18) and (7.4.21) agree in the overlap region imposes a relation between the constants A and B: $A = B\sqrt{\pi/E}\, e^{-E\pi}(-1)^n$. Thus, the eigenfunctions are now known approximately for $1 \le x < \infty$ up to an overall multiplicative constant. Of course, without further information [such as $y(6) = 19$] the normalization of $y(x)$ cannot be determined because the boundary-value problem is homogeneous.

Approximate Evaluation of an Integral

Asymptotic matching may be used to determine asymptotically the behavior of some integrals. Consider, for example, the integral

$$F(\varepsilon) = \int_0^\infty e^{-t-\varepsilon/t}\, dt \tag{7.4.23}$$

as $\varepsilon \to 0+$.

The leading behavior of $F(\varepsilon)$ as $\varepsilon \to 0+$ is easy to find. We simply set $\varepsilon = 0$ in (7.4.23): $F(0) = \int_0^\infty e^{-t}\, dt = 1$. Even though the integrand does not approach e^{-t} uniformly near $t = 0$ as $\varepsilon \to 0+$, we will verify shortly that the leading behavior of $F(\varepsilon)$ is correctly given by

$$F(\varepsilon) \sim 1, \qquad \varepsilon \to 0+. \tag{7.4.24}$$

It is more difficult to find the corrections to this leading behavior. Differentiating $F(\varepsilon)$ gives $F'(\varepsilon) = -\int_0^\infty e^{-t-\varepsilon/t}\, dt/t$, so $\lim_{\varepsilon\to 0+} F'(\varepsilon)$ does not exist. Apparently, the perturbative expansion of $F(\varepsilon)$ for small ε has the curious property that while its first term in (7.4.24) is almost trivial to find, successive terms require some real ingenuity. It is much more common for the first term (the

Table 7.2 A comparison of the exact eigenvalues E_n of (7.4.14) obtained from computer calculations with the approximations to E_n in (7.4.22) obtained from asymptotic matching

The percentage relative error [percentage relative error = (approximate − exact)/exact] decreases as E_n increases and the error is never more than 3.38 percent. Observe that the relative error falls off roughly as $1/E$, the small parameter in the asymptotic analysis which led to the above results. The form of (7.4.22) suggests that there are logarithmic corrections to the $1/E$ behavior of the relative error

n	E_n (from computer)	E_n (from asymptotic matching)	Percentage relative error
0	3.6975	3.5724	−3.38
1	5.3723	5.2569	−2.15
2	6.8195	6.7031	−1.71
3	8.1423	8.0223	−1.47
4	9.3826	9.2583	−1.33
5	10.5625	10.4337	−1.22
6	11.6955	11.5623	−1.14
7	12.7904	12.6532	−1.07
8	13.8535	13.7128	−1.02

unperturbed problem) to require the ingenuity and the remaining terms to require nothing more than paper and patience.

We will obtain the corrections to (7.4.24) in two ways. First, we express $F(\varepsilon)$ in terms of a modified Bessel function and use this representation to quickly write down the appropriate expansion of $F(\varepsilon)$ about $\varepsilon = 0$. This method has nothing to do with matched asymptotic expansions but it rapidly establishes the answer. Second, and here is the point of this discussion, we use matched asymptotic expansions to rederive the expansion of $F(\varepsilon)$ without using any special properties of modified Bessel functions.

First, we represent $F(\varepsilon)$ as a modified Bessel function. By differentiating $F(\varepsilon)$ in (7.4.23) twice we see that $F(\varepsilon)$ satisfies the differential equation $\varepsilon F''(\varepsilon) = F(\varepsilon)$. The substitution $F(\varepsilon) = ty(t)$, where $t = 2\sqrt{\varepsilon}$ converts this equation into the modified Bessel equation of order 1:

$$t^2y''(t) + ty'(t) - (1 + t^2)y(t) = 0.$$

Therefore,

$$F(\varepsilon) = 2A\sqrt{\varepsilon}\, I_1(2\sqrt{\varepsilon}) + 2B\sqrt{\varepsilon}\, K_1(2\sqrt{\varepsilon}),$$

where A and B are constants to be determined from boundary conditions at 0 and ∞.

To calculate A and B we use the asymptotic relations $I_1(x) \sim e^x(2\pi x)^{-1/2}$, $K_1(x) \sim e^{-x}(2x/\pi)^{-1/2}$ $(x \to +\infty)$ and $I_1(x) \sim x/2$, $K_1(x) \sim 1/x$ $(x \to 0+)$. Since the integral in (7.4.23) vanishes as $\varepsilon \to +\infty$, the first set of asymptotic relations implies that $A = 0$. Also, comparing the leading asymptotic behavior in (7.4.24) with the second set of asymptotic relations gives $B = 1$. Thus,

$$F(\varepsilon) = 2\sqrt{\varepsilon}\, K_1(2\sqrt{\varepsilon}). \tag{7.4.25}$$

Finally, we look up the behavior of $K_1(x)$ as $x \to 0+$ in the Appendix and learn that

$$F(\varepsilon) \sim 1 + \varepsilon \ln \varepsilon + \varepsilon(2\gamma - 1) + \tfrac{1}{2}\varepsilon^2 \ln \varepsilon + \varepsilon^2(\gamma - \tfrac{5}{4}) + \tfrac{1}{12}\varepsilon^3 \ln \varepsilon + (\tfrac{1}{6}\gamma - \tfrac{5}{18})\varepsilon^3 + \cdots, \qquad \varepsilon \to 0+, \tag{7.4.26}$$

where $\gamma \doteq 0.5772$ is Euler's constant.

Having established the answer, we proceed to the second and main point of this discussion: namely, an independent derivation of (7.4.26) directly from asymptotic matching. We observe that asymptotic matching may be useful here because the character of the integrand $e^{-t-\varepsilon/t}$ is very different in the two regions $t \ll 1$ $(\varepsilon \to 0+)$ and $\varepsilon \ll t$ $(\varepsilon \to 0+)$. In the *inner* region $(t \ll 1)$ it is valid to expand the integrand as

$$e^{-t-\varepsilon/t} \sim e^{-\varepsilon/t}(1 - t + \tfrac{1}{2}t^2 - \tfrac{1}{6}t^3 + \cdots), \qquad t \to 0+; \tag{7.4.27}$$

in the *outer* region $(t \gg \varepsilon)$ we have

$$e^{-t-\varepsilon/t} \sim e^{-t}\left(1 - \frac{\varepsilon}{t} + \frac{1}{2}\frac{\varepsilon^2}{t^2} - \frac{1}{6}\frac{\varepsilon^3}{t^3} + \cdots\right), \qquad \varepsilon/t \to 0+. \tag{7.4.28}$$

The terms "inner" and "outer" are borrowed from the terminology of boundary-layer theory (Chap. 9).

Since these two asymptotic regions overlap for $\varepsilon \ll t \ll 1$ ($\varepsilon \to 0+$), it is natural to write

$$F(\varepsilon) = \int_0^\delta e^{-t-\varepsilon/t}\,dt + \int_\delta^\infty e^{-t-\varepsilon/t}\,dt, \tag{7.4.29}$$

where $\delta(\varepsilon)$ is arbitrary subject to the asymptotic constraint $\varepsilon \ll \delta \ll 1$ ($\varepsilon \to 0+$).

Our plan is to calculate each of the two integrals on the right side of (7.4.29) asymptotically as $\varepsilon \to 0+$ and then to show that the two results match asymptotically; i.e., their sum depends only on ε and not on the arbitrary matching parameter δ. We will perform this match to several orders in the inner matching variable δ and the outer matching variable ε/δ. Note that in terms of the variable δ/ε, the extent of the overlap region becomes infinite as $\varepsilon \to 0+$.

Leading-Order (Zeroth-Order) Match

An approximation to the integral over the inner region $0 \le t \le \delta$ that is correct to zeroth order in δ is $\int_0^\delta e^{-t-\varepsilon/t}\,dt = O(\delta)$ ($\delta \to 0+$) because the integrand is bounded by 1 for all $t > 0$. On the other hand, a zeroth-order approximation to the integral over the outer region $\delta \le t < \infty$ is

$$\begin{aligned}\int_\delta^\infty e^{-t-\varepsilon/t}\,dt &= \int_\delta^\infty e^{-t}\,dt + \int_\delta^\infty e^{-t}(e^{-\varepsilon/t} - 1)\,dt\\ &= e^{-\delta} + O(\varepsilon/\delta), \qquad \varepsilon/\delta \to 0+,\end{aligned}$$

because

$$\left|\int_\delta^\infty e^{-t}(e^{-\varepsilon/t} - 1)\,dt\right| \le \varepsilon\int_\delta^\infty \frac{e^{-t}}{t}\,dt \le \frac{\varepsilon}{\delta}\int_\delta^\infty e^{-t}\,dt \le \frac{\varepsilon}{\delta},$$

where we have used the inequality $|e^{-x} - 1| \le x$ for all $x \ge 0$.

Combining the contributions from the inner and outer regions gives

$$\begin{aligned}F(\varepsilon) = \int_0^\infty e^{-t-\varepsilon/t}\,dt &= e^{-\delta} + O(\delta) + O(\varepsilon/\delta)\\ &= 1 + O(\delta) + O(\varepsilon/\delta), \qquad \delta \to 0+, \varepsilon/\delta \to 0+,\end{aligned} \tag{7.4.30}$$

because $e^{-\delta} = 1 + O(\delta)$ ($\delta \to 0+$). Note that the dependence on δ has dropped out to zeroth order (albeit in a trivial way) in both the inner expansion parameter δ and the outer expansion parameter ε/δ. We have rederived the leading behavior of $F(\varepsilon)$ in (7.4.24).

To find higher-order corrections to the leading behavior of $F(\varepsilon)$ we must match to higher order in powers of δ and ε/δ, which we now do.

First-Order Match

To find an approximation to the integral over the inner region $0 \le t \le \delta$ which is valid to first order in δ, we use the inequality $|e^{-x} - 1| \le x$ $(x \ge 0)$ to write

$$\int_0^\delta e^{-t-\varepsilon/t}\,dt = \int_0^\delta e^{-\varepsilon/t}\,dt + \int_0^\delta e^{-\varepsilon/t}(e^{-t} - 1)\,dt$$

$$= \int_0^\delta e^{-\varepsilon/t}\,dt + O(\delta^2), \qquad \delta \to 0+.$$

Setting $s = \varepsilon/t$, we obtain

$$\int_0^\delta e^{-t-\varepsilon/t}\,dt = \varepsilon \int_{\varepsilon/\delta}^\infty e^{-s}s^{-2}\,ds + O(\delta^2), \qquad \delta \to 0+.$$

The integral on the right is an incomplete gamma function. It was shown in Example 4 of Sec. 6.2 that when $N = 0, 1, 2, 3, 4, \ldots,$

$$\int_x^\infty \frac{e^{-s}}{s^{N+1}}\,ds = C_N + \frac{(-1)^{N+1}}{N!}\ln x - \sum_{\substack{n=0\\ n\ne N}}^\infty (-1)^n \frac{x^{n-N}}{n!\,(n-N)},$$

$$x \to 0+, \qquad (7.4.31)$$

where
$$C_0 = -\gamma,$$

$$C_N = \frac{(-1)^{N+1}}{N!}\left(\gamma - \sum_{n=1}^N \frac{1}{n}\right), \qquad N = 1, 2, 3, \ldots,$$

and $\gamma \doteqdot 0.5772$ is Euler's constant. Therefore, in the limit $\varepsilon/\delta \to 0+$, the first-order contribution to $F(\varepsilon)$ from the inner region is

$$\int_0^\delta e^{-t-\varepsilon/t}\,dt = \varepsilon[C_1 + \ln(\varepsilon/\delta) + \delta/\varepsilon + O(\varepsilon/\delta)] + O(\delta^2),$$

$$\delta \to 0+,\ \varepsilon/\delta \to 0+. \qquad (7.4.32)$$

We have retained two error terms in this expansion but we cannot yet conclude anything about the relative sizes of $O(\varepsilon^2/\delta)$ and $O(\delta^2)$.

Next, we compute a first-order approximation to the integral in the outer region $\delta \le t < \infty$. Using $|e^{-x} - 1 + x| \le x^2$ $(x \ge 0)$ and (7.4.31) with $N = 1$, we have

$$\int_\delta^\infty e^{-t-\varepsilon/t}\,dt = \int_\delta^\infty e^{-t}(1 - \varepsilon/t)\,dt + \int_\delta^\infty e^{-t}(e^{-\varepsilon/t} - 1 + \varepsilon/t)\,dt$$

$$= \int_\delta^\infty e^{-t}(1 - \varepsilon/t)\,dt + O(\varepsilon^2/\delta), \qquad \varepsilon/\delta \to 0+ \qquad (7.4.33)$$

(see Prob. 7.38). Therefore, using (7.4.31) with $N = 0$,

$$\int_\delta^\infty e^{-t-\varepsilon/t}\,dt = e^{-\delta} - \varepsilon[C_0 - \ln\delta + O(\delta)] + O(\varepsilon^2/\delta)$$

$$= 1 - \delta - \varepsilon C_0 + \varepsilon\ln\delta + O(\delta^2) + O(\varepsilon^2/\delta),$$

$$\delta \to 0+,\ \varepsilon/\delta \to 0+, \qquad (7.4.34)$$

where we neglect the error term $\varepsilon\,O(\delta)$ because $\varepsilon\delta \ll \delta^2$ $(\varepsilon/\delta \to 0+)$.

Now we combine the contributions in (7.4.32) and (7.4.34) from the inner and outer regions. Even though the parameter δ appears explicitly in these two formulas, it cancels to second order in δ and ε/δ when the formulas are added together:

$$\int_0^\infty e^{-t-\varepsilon/t}\,dt = 1 + \varepsilon\ln\varepsilon + \varepsilon(2\gamma - 1) + O(\delta^2) + O(\varepsilon^2/\delta),$$

$$\delta \to 0+,\ \varepsilon/\delta \to 0+, \qquad (7.4.35)$$

where we have used $C_0 = -\gamma$ and $C_1 = \gamma - 1$. We have now reproduced the first three terms in (7.4.26).

It is interesting to note that the original condition on δ, $\varepsilon \ll \delta \ll 1$ $(\varepsilon \to 0+)$, is not adequate to ensure that the error terms in (7.4.35) are smaller than the retained terms. The constraint on δ must be sharpened to read $\varepsilon \ll \delta \ll \varepsilon^{1/2}$ $(\varepsilon \to 0+)$. However, even though this new relation restricts δ more than $\varepsilon \ll \delta \ll 1$ $(\varepsilon \to 0+)$, the matching of the inner and outer integrals still occurs over an infinite range in terms of the matching variable δ/ε: $1 \ll \delta/\varepsilon \ll \varepsilon^{-1/2}$ $(\varepsilon \to 0+)$.

Third-Order Match

Now we use asymptotic matching to calculate the first seven terms in the series (7.4.26). We will see that the number of expansion terms in the inner and outer expansions proliferate rapidly. To reproduce (7.4.26) we have to calculate the inner integral accurate to $O(\delta^6)$ $(\delta \to 0+)$ and the outer integral to $O(\varepsilon^6/\delta^5)$! We must retain terms to this order if we are to achieve a proper match to order $\varepsilon^3 \ln\varepsilon$.

To fifth order in δ, the inner integral is

$$\int_0^\delta e^{-t-\varepsilon/t}\,dt = \int_0^\delta e^{-\varepsilon/t}[1 - t + \tfrac{1}{2}t^2 - \tfrac{1}{6}t^3 + \tfrac{1}{24}t^4]\,dt + O(\delta^6), \qquad \delta \to 0+.$$

Setting $s = \varepsilon/t$, we obtain

$$\int_0^\delta e^{-t-\varepsilon/t}\,dt = \varepsilon\int_{\varepsilon/\delta}^\infty e^{-s}\left(\frac{1}{s^2} - \frac{\varepsilon}{s^3} + \frac{\varepsilon^2}{2s^4} - \frac{\varepsilon^3}{6s^5} + \frac{\varepsilon^4}{24s^6}\right)ds + O(\delta^6), \qquad \delta \to 0+.$$

Using (7.4.31) with $N = 1, \ldots, 5$, we get

$$\int_0^\delta e^{-t-\varepsilon/t}\,dt = \varepsilon\left[C_1 + \ln(\varepsilon/\delta) + \frac{\delta}{\varepsilon} - \frac{\varepsilon}{2\delta} + \frac{\varepsilon^2}{12\delta^2} - \frac{\varepsilon^3}{72\delta^3} + \frac{\varepsilon^4}{480\delta^4} + \mathrm{O}\left(\frac{\varepsilon^5}{\delta^5}\right)\right]$$
$$- \varepsilon^2\left[C_2 - \frac{1}{2}\ln(\varepsilon/\delta) + \frac{\delta^2}{2\varepsilon^2} - \frac{\delta}{\varepsilon} + \frac{\varepsilon}{6\delta} - \frac{\varepsilon^2}{48\delta^2} + \mathrm{O}\left(\frac{\varepsilon^3}{\delta^3}\right)\right]$$
$$+ \frac{1}{2}\varepsilon^3\left[C_3 + \frac{1}{6}\ln(\varepsilon/\delta) + \frac{\delta^3}{3\varepsilon^3} - \frac{\delta^2}{2\varepsilon^2} + \frac{\delta}{2\varepsilon} + \mathrm{O}\left(\frac{\varepsilon}{\delta}\right)\right]$$
$$- \frac{1}{6}\varepsilon^4\left[\frac{\delta^4}{4\varepsilon^4} - \frac{\delta^3}{3\varepsilon^3} + \mathrm{O}\left(\frac{\delta^2}{\varepsilon^2}\right)\right]$$
$$+ \frac{1}{24}\varepsilon^5\left[\frac{\delta^5}{5\varepsilon^5} + \mathrm{O}\left(\frac{\delta^4}{\varepsilon^4}\right)\right] + \mathrm{O}(\delta^6), \qquad \delta \to 0+, \varepsilon/\delta \to 0+.$$

The outer integral is expanded similarly:

$$\int_\delta^\infty e^{-t-\varepsilon/t}\,dt = \int_\delta^\infty e^{-t}\left(1 - \frac{\varepsilon}{t} + \frac{\varepsilon^2}{2t^2} - \frac{\varepsilon^3}{6t^3} + \frac{\varepsilon^4}{24t^4} - \frac{\varepsilon^5}{120t^5}\right)dt + \mathrm{O}\left(\frac{\varepsilon^6}{\delta^5}\right),$$
$$\frac{\varepsilon}{\delta} \to 0+ \qquad (7.4.36)$$

(see Prob. 7.38). Using (7.4.31) with $N = 0, \ldots, 4$ gives

$$\int_\delta^\infty e^{-t-\varepsilon/t}\,dt = e^{-\delta} - \varepsilon\left[C_0 - \ln\delta + \delta - \frac{1}{4}\delta^2 + \frac{1}{18}\delta^3 + \mathrm{O}(\delta^4)\right]$$
$$+ \frac{1}{2}\varepsilon^2\left[C_1 + \ln\delta + \frac{1}{\delta} - \frac{1}{2}\delta + \mathrm{O}(\delta^2)\right]$$
$$- \frac{1}{6}\varepsilon^3\left[C_2 - \frac{1}{2}\ln\delta + \frac{1}{2\delta^2} - \frac{1}{\delta} + \mathrm{O}(\delta)\right]$$
$$+ \frac{1}{24}\varepsilon^4\left[\frac{1}{3\delta^3} - \frac{1}{2\delta^2} + \mathrm{O}\left(\frac{1}{\delta}\right)\right]$$
$$- \frac{1}{120}\varepsilon^5\left[\frac{1}{4\delta^4} + \mathrm{O}\left(\frac{1}{\delta^3}\right)\right] + \mathrm{O}\left(\frac{\varepsilon^6}{\delta^5}\right),$$
$$\delta \to 0+, \varepsilon/\delta \to 0+.$$

The order of accuracy to which we compute the inner and outer expansions is not arbitrary; the error terms are chosen so that these expansions match through terms of order ε^3. Indeed, if we add together the inner and outer expansions, we obtain

$$F(\varepsilon) = \int_0^\infty e^{-t-\varepsilon/t}\,dt$$

$$= 1 + \varepsilon \ln \varepsilon + \varepsilon(2\gamma - 1) + \frac{1}{2}\varepsilon^2 \ln \varepsilon$$

$$+ \varepsilon^2\left(\gamma - \frac{5}{4}\right) + \frac{1}{12}\varepsilon^3 \ln \varepsilon + \varepsilon^3\left(\frac{1}{6}\gamma - \frac{5}{18}\right) + O(\delta^6)$$

$$+ O\left(\frac{\varepsilon^6}{\delta^5}\right) + O\left(\frac{\varepsilon^5}{\delta^3}\right) + O\left(\frac{\varepsilon^4}{\delta}\right) + O(\varepsilon^2\delta^2) + O(\varepsilon\delta^4) + O(\varepsilon^3\delta),$$

$$\delta \to 0+,\ \varepsilon/\delta \to 0+, \qquad (7.4.37)$$

where we have substituted the values $C_0 = -\gamma$, $C_1 = \gamma - 1$, $C_2 = -\frac{1}{2}\gamma + \frac{3}{4}$, $C_3 = \frac{1}{6}\gamma - \frac{11}{36}$. We have thus reproduced the series (7.4.26).

Note that all the error terms in (7.4.37) are negligible with respect to ε^3 if the constraint on δ is sharpened to read $\varepsilon^{3/5} \ll \delta \ll \varepsilon^{1/2}$ $(\varepsilon \to 0+)$. In successively higher orders the constraint on δ becomes increasingly tight. However, in terms of the matching variable δ/ε the extent of the matching interval is always infinite.

In Fig. 7.7 we compare the series for $F(\varepsilon)$ in (7.4.30), (7.4.35), and (7.4.37) with a numerical evaluation of the integral in (7.4.23).

In the next example we use the method of matched asymptotic expansions to obtain higher-order terms in the expansion of a generalized Fourier integral.

Example 6 *Use of asymptotic matching to improve the predictions of stationary-phase analysis.* In this example we use asymptotic matching to find the large-x behavior of the integral

$$I(x) = \int_0^{\pi/2} e^{ix\cos t}\,dt. \qquad (7.4.38)$$

The method of stationary phase (see Example 3 of Sec. 6.5) quickly gives the leading behavior of $I(x)$:

$$\int_0^{\pi/2} e^{ix\cos t}\,dt \sim \sqrt{\frac{\pi}{2x}}\,e^{i(x-\pi/4)}, \qquad x \to +\infty. \qquad (7.4.39)$$

However, we did not explain in Sec. 6.5 how to obtain the higher-order corrections to this leading behavior.

In Sec. 6.5 we showed that the leading-order behavior is completely determined by a *local* analysis of the integrand in the neighborhood of the stationary point, which for the integral (7.4.38) lies at $t = 0$. On the other hand, higher-order corrections to the leading behavior may arise from regions in the domain of integration away from the stationary point. Therefore, some form of *global* analysis is required to obtain higher-order corrections; this example shows how asymptotic matching can be used.

As usual, the procedure consists of dividing the domain of integration into two regions: the first is a narrow region $0 \le t \le \delta$ containing the stationary point at $t = 0$; the second is the remainder of the integration interval $\delta < t \le \pi/2$. For now we say only that $\delta = \delta(x)$ is a small parameter satisfying $\delta \ll 1$ $(x \to +\infty)$. Later we will impose more restrictive conditions on

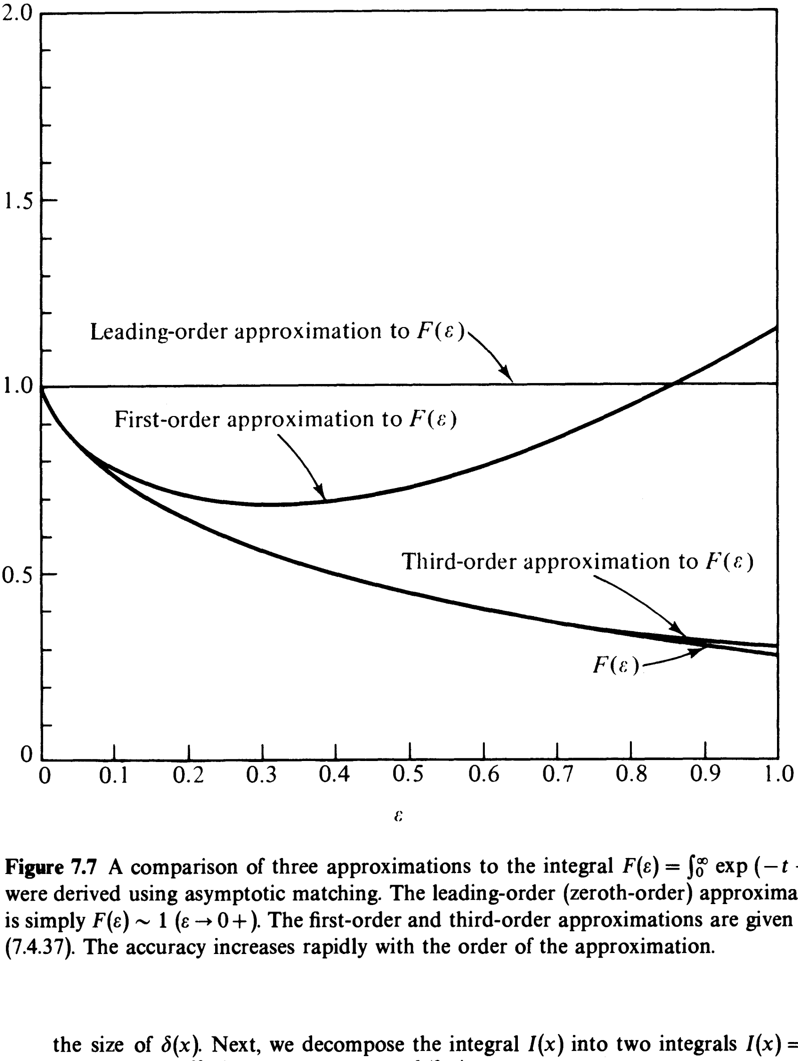

Figure 7.7 A comparison of three approximations to the integral $F(\varepsilon) = \int_0^\infty \exp(-t - \varepsilon/t)\,dt$ which were derived using asymptotic matching. The leading-order (zeroth-order) approximation in (7.4.30) is simply $F(\varepsilon) \sim 1$ $(\varepsilon \to 0+)$. The first-order and third-order approximations are given in (7.4.35) and (7.4.37). The accuracy increases rapidly with the order of the approximation.

the size of $\delta(x)$. Next, we decompose the integral $I(x)$ into two integrals $I(x) = I_1(x) + I_2(x)$, where $I_1(x) = \int_0^\delta e^{ix\cos t}\,dt$ and $I_2(x) = \int_\delta^{\pi/2} e^{ix\cos t}\,dt$. We will find asymptotic approximations to $I_1(x)$ and $I_2(x)$ as $x \to +\infty$ and $\delta(x) \to 0+$. These approximations will each depend on δ but, as we will see, their sum $I(x)$ will not depend on δ.

First, we approximate $I_1(x)$ as $x \to +\infty$. Since $\delta \ll 1$ as $x \to +\infty$, we have $\cos t = 1 - \frac{1}{2}t^2 + O(\delta^4)$ $(0 \le t \le \delta,\ \delta \to 0+)$. Therefore, $I_1(x) = e^{ix}\int_0^\delta e^{-ixt^2/2}\,dt + O(x\delta^5)$ $(x \to \infty,\ x^{1/4}\delta \to 0+)$. This result is valid if we impose the condition on δ that $x^{1/4}\delta \to 0+$ as $x \to +\infty$; we are free to impose this condition which specifies just how rapidly $\delta \to 0$ as $x \to +\infty$. To approximate the above integral further, we write $\int_0^\delta e^{-ixt^2/2}\,dt = \int_0^\infty e^{-ixt^2/2}\,dt - \int_\delta^\infty e^{-ixt^2/2}\,dt$. In both of these integrals we rotate the contour of integration by 45° in the complex-t plane. This enables us to do the first integral exactly and to approximate the second using integration by parts twice. The result for $I_1(x)$ is

$$I_1(x) = \sqrt{\frac{\pi}{2x}}\,e^{i(x-\pi/4)} + \frac{i}{x\delta}\,e^{ix(1-\delta^2/2)} - \frac{1}{x^2\delta^3}\,e^{ix(1-\delta^2/2)} - \frac{3ie^{ix(1-\delta^2/2)}}{x^3\delta^5} + O(x\delta^5) + O\left(\frac{1}{x^4\delta^7}\right),$$
$$x \to +\infty,\ x^{2/5}\delta \to 0+,\ x^{1/2}\delta \to +\infty. \tag{7.4.40}$$

To make the error incurred upon integrating by parts smaller than the smallest retained term, we have imposed two new conditions on the magnitude of δ: $x^{2/5}\delta \to 0+$ (which supplants $x^{1/4}\delta \to 0+$ because it is more stringent) and $x^{1/2}\delta \to +\infty$ as $x \to +\infty$. Observe that the conditions $x \to +\infty$, $\delta \to 0+$, $x^{2/5}\delta \to 0+$, and $x^{1/2}\delta \to +\infty$ are all satisfied if we take

$$x^{-1/2} \ll \delta \ll x^{-2/5}, \qquad x \to +\infty. \tag{7.4.41}$$

Next, we approximate $I_2(x)$ as $x \to +\infty$. Since there are no stationary points of the integrand for $\delta < t \le 1$, it is valid to integrate by parts. Three integrations by parts give

$$\begin{aligned} I_2(x) &= \frac{ie^{ix\cos t}}{x\sin t}\bigg|_\delta^{\pi/2} - \frac{e^{ix\cos t}}{x^2\sin^3 t}\cos t\bigg|_\delta^{\pi/2} - \frac{i(1+2\cos^2 t)e^{ix\cos t}}{x^3\sin^5 t}\bigg|_\delta^{\pi/2} + \mathrm{O}\left(\frac{1}{x^4\delta^7}\right) \\ &= \frac{i}{x} - \frac{ie^{ix\cos\delta}}{x\sin\delta} + \frac{e^{ix\cos\delta}\cos\delta}{x^2\sin^3\delta} + \frac{ie^{ix\cos\delta}(2\cos^2\delta+1)}{x^3\sin^5\delta} + \mathrm{O}\left(\frac{1}{x^3}\right) + \mathrm{O}\left(\frac{1}{x^4\delta^7}\right), \\ &\qquad x \to +\infty,\ x^{3/7}\delta \to +\infty. \end{aligned} \tag{7.4.42}$$

We have imposed the additional condition $x^{-3/7} \ll \delta (x \to +\infty)$ to ensure that the error term is smaller than the smallest retained term. Notice that $x^{1/2}\delta = (x^{3/7}\delta)x^{1/14} \to \infty$ when $x^{3/7}\delta \to +\infty$ and $x \to +\infty$.

Finally, we add together the asymptotic approximations to $I_1(x)$ and $I_2(x)$ in (7.4.40) and (7.4.42) and obtain

$$I(x) = \sqrt{\frac{\pi}{2x}}e^{i(x-\pi/4)} + \frac{i}{x} + \mathrm{O}\left(\frac{1}{x^4\delta^7}\right) + \mathrm{O}(x\delta^5) + \mathrm{O}\left(\frac{1}{x^3}\right),$$
$$x \to +\infty,\ x^{2/5}\delta \to 0+,\ x^{3/7}\delta \to +\infty. \tag{7.4.43}$$

This is the answer; we have found the first *two* terms (one term beyond the leading behavior) in the expansion of $I(x)$ as $x \to +\infty$. Observe that the parameter δ cancels out of the asymptotic expansion of $I_1(x)$ and $I_2(x)$ and appears only in error terms in the final answer for $I(x)$.

The consistency of the asymptotic match depends on the error terms being smaller than the smallest retained term, i/x. There are two error terms that must be checked: $x^{-4}\delta^{-7} \ll x^{-1}$ and $x\delta^5 \ll x^{-1}$ $(x \to +\infty)$. These conditions are satisfied because it is possible to impose the asymptotic conditions $x^{-3/7} \ll \delta \ll x^{-2/5}$ $(x \to +\infty)$, which is a further refinement of the condition in (7.4.41). To stress the delicacy of this condition we rewrite it as

$$x^{-15/35} \ll \delta \ll x^{-14/35}, \qquad x \to +\infty. \tag{7.4.44}$$

The consistency of the asymptotic match depends on the existence of a parameter δ which satisfies (7.4.44). If it were not true that $x^{-15/35} \ll x^{-14/35}$ $(x \to +\infty)$, then there would have been no such δ!

We can now explain why it was necessary to integrate by parts three times even though (*a*) the final answer in (7.4.43) was determined after just one integration by parts and (*b*) further integration by parts generated terms depending on δ which cancelled when we added I_1 and I_2. Three integrations by parts were necessary to establish the consistency of asymptotic matching. If we had done just one or two integrations by parts, the final asymptotic condition on δ, instead of being consistent like that in (7.4.44), would have been inconsistent. For example, after one or two integrations by parts we would have had

$$x^{-1/3} \ll \delta \ll x^{-2/5} \qquad \text{or} \qquad x^{-2/5} \ll \delta \ll x^{-2/5}, \qquad x \to +\infty, \tag{7.4.45}$$

respectively, which are impossible conditions (see Prob. 7.41). In general, the result of an asymptotic match cannot be trusted until the matching scheme has been shown to be consistent. The condition in (7.4.44) is crucial because it shows that the parameter δ, which determines the location of the matching region, exists. This is the subtlety of asymptotic matching; the rest is straightforward calculation.

(TD) 7.5 MATHEMATICAL STRUCTURE OF PERTURBATIVE EIGENVALUE PROBLEMS

The series $\sum_{n=0}^{\infty} \varepsilon^n$ diverges for $|\varepsilon| \geq 1$. Naturally, this divergence reflects the singularity structure of the function $f(\varepsilon)$ that the series approximates; here, $f(\varepsilon) = 1/(1-\varepsilon)$ has a pole at $\varepsilon = 1$. Several examples of perturbative eigenvalue problems having perturbation series in the form of power series in ε were given in Sec. 7.3. In those problems, also, when the perturbation series for the eigenvalue had a finite or vanishing radius of convergence, the exact eigenvalue considered as a function of ε also had singularities in the complex-ε plane.

In this section we discuss the origin and meaning of such singularities. We will argue that the presence of singularities as well as their type is not a chance event, but is a predictable phenomenon characteristic of a broad class of perturbative eigenvalue problems.

Example 1 *Eigenvalues of a* 2×2 *matrix.* We begin by considering the simplest eigenvalue problem of all. Let A and B be real symmetric 2×2 matrices of the form

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad B = \begin{pmatrix} x & z \\ z & y \end{pmatrix},$$

and consider the problem of finding the eigenvalues of $A + B$ by perturbation theory. To do this we replace B with εB and express each eigenvalue as a power series in ε. Each power series begins with the numbers a or b and is expected to be convergent for sufficiently small ε because the perturbation problem is regular.

To find the radii of convergence we solve the problem exactly. Setting the determinant of $A + \varepsilon B - I\lambda$ equal to zero gives the following formula for the two eigenvalues $\lambda_\pm$:

$$\lambda_\pm = \tfrac{1}{2}\{a + b + \varepsilon x + \varepsilon y \pm [(a - b + \varepsilon x - \varepsilon y)^2 + 4\varepsilon^2 z^2]^{1/2}\}. \tag{7.5.1}$$

$\lambda_\pm$ are analytic functions of ε except at the zeros of the square-root term. Thus, $\lambda_\pm(\varepsilon)$ have a pair of square-root branch points, symmetrically placed about the real axis, at

$$\varepsilon = \frac{a - b}{y - x \pm 2iz}. \tag{7.5.2}$$

The radius of convergence of the perturbation series is $|a - b|/[(x - y)^2 + 4z^2]^{1/2}$.

Observe that the radius of convergence vanishes when $a = b$. But a and b are the unperturbed eigenvalues, so that, if $a = b$, the unperturbed problem is degenerate. Thus, when $a = b$, the exact solution of the perturbed problem (which is nondegenerate if $\varepsilon \neq 0$) undergoes an abrupt change (the appearance of degeneracy) in the limit $\varepsilon \to 0$. The perturbation problem must therefore be singular when $a = b$ and this conclusion is consistent with the vanishing of the radius of convergence.

The noteworthy feature of this example is that the two eigenvalues in (7.5.1) are analytic continuations of each other and together they form a single two-valued function $\lambda(\varepsilon)$. $\lambda(\varepsilon)$ is defined on a two-sheeted (Riemann) surface; on the lower sheet $\lambda(\varepsilon) = \lambda_-(\varepsilon)$ and on the upper sheet $\lambda(\varepsilon) = \lambda_+(\varepsilon)$. Analytic continuation around either of the two branch points (7.5.2) exchanges the identities of the two eigenvalues because the sign of the square root in (7.5.1) changes; this phenomenon is called *level crossing*.

The existence of square-root branch-point singularities, the appearance of level crossing, and the unification of the eigenvalues into a single many-valued analytic function of ε are not just special properties of the simple problem in Example 1. Rather, these seem to be very general features of perturbative eigenvalue problems having perturbation series that diverge for sufficiently large $|\varepsilon|$. Of course, one could argue that all this analytical structure is artificial because the original problem did not involve ε. However, we are often forced to introduce a perturbation parameter ε when there is no other analytical way to make progress in computing the eigenvalues. And, when the perturbation series is divergent, the recovery of the eigenvalues depends upon a clear understanding of the analytical structure of the perturbed model, as will be shown in Chap. 8.

Next we consider a more general eigenvalue problem and show that an approximate solution displays the same basic analytic structure as the above example.

Let us reconsider (7.3.3):

$$\left[-\frac{d^2}{dx^2} + V(x) + \varepsilon W(x) - E(\varepsilon)\right] y(x) = 0 \tag{7.5.3}$$

with the boundary conditions that $y(x) \to 0$ as $|x| \to \infty$. However, instead of immmediately expanding the solution in power series in ε, let us instead represent the solution as an infinite linear combination of eigenfunctions of the unperturbed problem, which we assume are all known. We will then use the differential equation to determine the coefficients in this expansion. If we label the nth unperturbed eigenfunction and eigenvalue by the superscript (n), then

$$-\frac{d^2}{dx^2} y_0^{(n)} + V(x)y_0^{(n)} - E_0^{(n)}y_0^{(n)} = 0, \tag{7.5.4}$$

where $y_0^{(n)}(x) \to 0$ as $|x| \to \infty$.

The Schrödinger equation (7.5.4) is a Sturm-Liouville eigenvalue problem (see Sec. 1.8). Therefore, the eigenfunctions $y_0^{(n)}$ are *complete*. For these Schrödinger eigenfunctions completeness means that an arbitrary square-integrable function, such as an exact eigenfunction solution $y(x)$ of (7.5.3), can be expanded as the infinite linear combination

$$y(x) = \sum_{n=0}^{\infty} a_n y_0^{(n)}(x). \tag{7.5.5}$$

Eq. (7.5.5) is *not* a perturbation expansion because there is no perturbing parameter.

We assume that $W(x)y_0^{(n)}(x)$ may also be expanded in terms of the same eigenfunctions $y_0^{(m)}$:

$$W(x)y_0^{(n)}(x) = \sum_{m=0}^{\infty} A_m^n y_0^{(m)}(x). \tag{7.5.6}$$

Applying the differential equation (7.5.3) to (7.5.5), using (7.5.6), and equating coefficients of $y_0^{(n)}$ for each n gives an infinite matrix equation satisfied by the coefficients a_n:

$$M\mathbf{a} \equiv \begin{bmatrix} E_0^{(0)} - E + \varepsilon A_0^0 & \varepsilon A_0^1 & \varepsilon A_0^2 & \cdots \\ \varepsilon A_1^0 & E_0^{(1)} - E + \varepsilon A_1^1 & \varepsilon A_1^2 & \cdots \\ \varepsilon A_2^0 & \varepsilon A_2^1 & E_0^{(2)} - E + \varepsilon A_2^2 & \\ \vdots & \vdots & & \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} = 0. \tag{7.5.7}$$

If the matrix M were finite dimensional, the condition that there be nontrivial solutions $\mathbf{a}$ satisfying (7.5.7) is that $\det M = 0$ (Cramer's rule). Unfortunately, when M is infinite dimensional, as it is here, $\det M$ (called a Hill determinant) may not exist. Therefore we devise a sequence of approximations in which we truncate the matrix M. Let M_N be an $N \times N$ matrix whose entries are the same as the first N rows and columns of M. We thus approximate (7.5.7) by

$$M_N \mathbf{a}_N = 0, \tag{7.5.8}$$

where $\mathbf{a}_N = (a_0, a_1, \ldots, a_{N-1})$. We have replaced the complicated equation in (7.5.3) by a comparatively trivial sequence of matrix equations.

Note The matrix M_N can be obtained directly using the orthogonality properties of the eigenfunctions $y_0^{(n)}$. This process, called the Galerkin method, involves three steps. First, we seek an approximation to $y(x)$ in the form

$$y_{\text{approx}}(x) = \sum_{n=0}^{N-1} a_n y_0^{(n)}. \tag{7.5.9}$$

Second, we substitute $y_{\text{approx}}(x)$ into the perturbed differential equation and reexpand the result as a series of the $y_0^{(n)}$. Third, equations for the expansion coefficients a_n ($n = 0, 1, \ldots, N-1$) are obtained by equating coefficients of $y_0^{(n)}$($n = 0, 1, \ldots, N-1$) to 0. The result is precisely the matrix equation (7.5.8). This so-called Galerkin procedure is very useful in numerical analysis.

The approximation (7.5.8) can also be derived in a somewhat different way using a Rayleigh-Ritz variational procedure. There, one also begins with the series (7.5.9). The coefficients a_n which give the "best fit" to $y(x)$ are determined by applying a variational principle to minimize the difference between $y(x)$ and $y_{\text{approx}}(x)$. The "best fit" is achieved when (7.5.8) is satisfied.

Next, we define $D_N(E, \varepsilon) = \det M_N$. The limit of $D_N(E, \varepsilon)$ as $N \to \infty$ may or may not exist, but we are not really interested in this limit. We are actually concerned with the behavior of the *roots* of the equation

$$D_N(E, \varepsilon) = 0. \tag{7.5.10}$$

Do the roots of this equation approach the exact eigenvalues of the differential equation (7.5.3) as $N \to \infty$? A glance at (7.5.7) shows that $D_N(E, \varepsilon)$ is an Nth-order polynomial in E and ε. Thus, given any ε we can obtain N values for E. Leaving aside all questions of rigor we will simply assume that for every value of ε these values of E do approach the correct eigenvalues of the exact problem in (7.5.3) as $N \to \infty$. We have thus replaced the complicated differential-equation eigenvalue problem (7.5.3) by a much simpler matrix eigenvalue problem very similar in structure to the one considered in Example 1.

Example 2 *Singular perturbation of the parabolic cylinder equation.* When $V = x^2/4$ and $W = x^4/4$, $D_N(E, \varepsilon)$ satisfies a five-term recursion relation (see Prob. 7.43). The results of a numerical computation of the zeros of $D_N(E, \varepsilon)$ for $\varepsilon = 1$ are given in Table 7.3. As N increases, the eigenvalues rapidly converge to the eigenvalues of the differential equation. The eigenvalues approach their limits in order of their size, the smaller ones converging more rapidly than the larger ones. This sequential (nonuniform) convergence of the eigenvalues typically occurs when infinite matrices are approximated by truncated finite matrices.

Table 7.3 Numerical calculation of the first five eigenvalues of (7.3.3) with $V(x) = x^2/4$, $W(x) = x^4/4$, and $\varepsilon = 1$

The eigenvalues are the limits as $N \to \infty$ of the zeros of $D_N(E, \varepsilon = 1)$. This table shows that as N increases zeros rapidly converge to the exact eigenvalues listed on the bottom line [obtained by Padé summation (see Chap. 8)]. The entries in the table form a checkerboard pattern with every other entry absent because the values of the zeros only change when N increases by 2. This effect is connected with the fact that the perturbed and unperturbed eigenfunctions are either even or odd functions of x

N	$E^{(0)}$	$E^{(1)}$	$E^{(2)}$	$E^{(3)}$	$E^{(4)}$
1	1.250 000				
2		5.250 000			
3	0.855 087		12.644 91		
4		3.273 837		24.226 16	
5	0.808 229		7.382 825		40.558 95
6		2.843 872		13.867 49	
7	0.805 870		5.860 713		23.373 00
8		2.752 576		10.308 12	
9	0.805 614		5.361 362		16.794 95
10		2.740 927		8.842 45	
11	0.804 698		5.215 487		13.702 25
12		2.740 828		8.240 62	
13	0.804 076		5.185 265		12.170 52
14		2.740 060		8.020 67	
15	0.803 838		5.182 772		11.435 98
16		2.738 944		7.957 72	
17	0.803 781		5.182 628		11.117 65
18		2.738 253		7.946 76	
19	0.803 774		5.181 493		11.002 51
20		2.737 979		7.946 43	
21	0.803 774		5.180 331		10.972 07
22		2.737 907		7.945 87	
23	0.803 773		5.179 657		10.967 97
24		2.737 897		7.944 56	
25	0.803 772		5.179 385		10.967 97
26		2.737 897		7.943 41	
27	0.803 771		5.179 310		10.967 08
28		2.737 896		7.942 77	
29	0.803 771		5.179 298		10.965 69
30		2.737 894		7.942 50	
∞	0.803 771	2.737 893	5.179 292	7.942 40	10.963 58

Let us examine the structure of the roots $E_N(\varepsilon)$ of $D_N(E, \varepsilon)$. Equation (7.5.10) is an implicit algebraic relation between E and ε. Therefore, the solution $E_N(\varepsilon)$ is one, or possibly several, multivalued functions (having altogether N values) and the only singularities that $E_N(\varepsilon)$ may exhibit are poles or branch points. However, from the specific form of the $D_N(E, \varepsilon)$ one may show that $E_N(\varepsilon)$ may not have poles or branch points at which $E_N(\varepsilon) = \infty$ (see Prob. 7.44). The only singularities that $E_N(\varepsilon)$ may have are branch points at which $E_N(\varepsilon)$ remains *finite*. Level crossing of the approximate eigenvalues occurs as the solutions of (7.5.10) are analytically continued around these branch points.

At a branch point of $E_N(\varepsilon)$ we expect at least two eigenvalues to become degenerate [see (7.5.1)]. Thus, at a branch point (7.5.10) must have at least a double root. The condition for a double root is

$$\frac{\partial}{\partial E} D_N(E, \varepsilon) = 0. \tag{7.5.11}$$

Since $\partial D_N/\partial E$ is a polynomial of degree $N - 1$ in both E and ε, the simultaneous solutions of (7.5.10) and (7.5.11) may yield at most $N(N - 1)$ branch points (see Prob. 7.45). These branch points typically occur as $\frac{1}{2}N(N - 1)$ complex conjugate pairs because (7.5.10) and (7.5.11) are real [see (7.5.7)].

A double root of (7.5.10) implies that $E_N(\varepsilon)$ has a square-root branch point in the ε plane. A more complicated branch point of $E_N(\varepsilon)$ would require $D_N(E, \varepsilon)$ to have a multiple root; e.g., a cube-root branch point would occur if $(\partial^2/\partial E^2)D_N(E, \varepsilon) = 0$ holds simultaneously with (7.5.10) and (7.5.11). Of course, it is not impossible for three or more simultaneous equations in two unknowns to have a solution, but it is very unlikely. The existence of a level-crossing point which is not a square-root singularity must be viewed as purely fortuitous; even if such a branch point could exist for some fixed N, it would probably disappear as soon as N is increased by 1. We conclude that in a typical $N \times N$ matrix eigenvalue problem with parameter ε there are $N(N - 1)$ square-root branch points in the ε plane.

Now let us consider what may happen to the solution of the finite matrix problem (7.5.8) as $N \to \infty$. There are four possibilities and we consider each in turn.

Possibility 1 As $N \to \infty$, the locations of the branch points stabilize, remain well separated from each other and the origin, and maintain their identities as square-root branch points. If this occurs, then the radius of convergence of the perturbation series for each eigenvalue is then nonzero and exactly equal to the distance to the nearest singularity in the complex plane at which this eigenvalue crosses with (the analytic continuation of) another eigenvalue. If possibility 1 occurs, the perturbation theory is regular.

Example 3 *Regular perturbation of the parabolic cylinder equation.* An eigenvalue equation which displays the behavior described above is

$$(d^2/dx^2 + x^2/4 + \varepsilon|x| - E)y(x) = 0, \qquad \lim_{|x|\to\infty} y(x) = 0.$$

For this differential equation the exact eigenfunctions $y(x)$ are always either even or odd under the reflection $x \to -x$. The determinant $D_N(E, \varepsilon)$ in (7.5.10) factors into a product of two determinants, $D_N(E, \varepsilon) = D_N^{(\text{even})}(E, \varepsilon) D_N^{(\text{odd})}(E, \varepsilon)$, where $D_N^{(\text{even})}(E, \varepsilon)$ contains the entries A_{2m}^{2n} and $D_N^{(\text{odd})}(E, \varepsilon)$ contains the entries A_{2m+1}^{2n+1} [see (7.5.6)]; A_{2m}^{2n+1} and A_{2m+1}^{2n} both vanish. The eigenvalues associated with even eigenfunctions and the eigenvalues associated with odd eigenfunctions are qualitatively similar, so we restrict our attention to the even eigenfunctions and their eigenvalues. The numbers A_{2m}^{2n} in equation (7.5.6) are given by (see Prob. 7.46)

$$A_{2m}^{2n} = (2/\pi)^{1/2} \frac{(-1)^{n+m+1}[2(n+m)+1](2n)!}{2^{m+n}[4(n-m)^2-1]m!\,n!}.$$

A simultaneous numerical solution of (7.5.10) and (7.5.11) gives branch points for various values of N. The locations of these branch points stabilize as N gets large (see Table 7.4). In Fig. 7.8 we plot a portion of the upper-half complex-ε plane showing the limiting values of some branch points for large N. The branch points occur in complex-conjugate pairs: each branch point in the upper-half ε plane is associated with another (not shown) in the lower-half plane. Each pair of branch points is joined by a branch cut (not shown).

What happens when the eigenvalues for this problem are analytically continued around a branch point? Contours which emerge from the origin in the ε plane, encircle a branch point, and return to the origin are indicated in Fig. 7.8. These contours all consist of sequences of line segments. The simplest contour has its corners numbered sequentially 0, 1, 2, 3, 4, 1, 0. In Fig. 7.9 a portion of the complex-E plane is plotted showing the images of this contour in the ε plane. Note that when $\varepsilon = 0$ the eigenvalues assume their unperturbed values $\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \frac{13}{2}$. [Only the eigenvalues for even eigenfunctions $y(x) = y(-x)$ are shown here; the unperturbed eigenvalues $\frac{3}{2}$, $\frac{7}{2}, \frac{11}{2}, \ldots$ are associated with odd functions $y(x) = -y(x)$ and behave similarly as functions of complex ε.] As the argument of each eigenvalue follows the contour in the ε plane, the eigenvalues simultaneously trace out curves in the E plane. The first two eigenvalues undergo level crossing (they exchange identities), while the other eigenvalues return to their original positions.

Figures 7.10 to 7.12 show how other pairs of levels cross when the eigenvalues are analytically continued around other branch points in Fig. 7.8. These figures are not schematic representations; they demonstrate the actual numerical behavior of the eigenvalues. The numerical error is approximately equal to the thickness of the curves.

The radius of convergence of the perturbation series for an eigenvalue increases as the size of the unperturbed eigenvalue increases because, as can be seen from Fig. 7.8, the distance to the nearest branch point at which this eigenvalue crosses with another also increases.

Table 7.4 Stabilization of the branch point "A" as $N \to \infty$

This branch point is one of five which are plotted in Fig. 7.9. The Nth approximation to a branch point is a value of ε which simultaneously solves $D_N(E, \varepsilon) = 0$ and $(\partial/\partial E) D_N(E, \varepsilon) = 0$ [see (7.5.10) and (7.5.11)]

N	Nth approximation to A
3	$-1.136 + 0.5552i$
4	$-1.209 + 0.5623i$
5	$-1.206 + 0.5741i$
6	$-1.205 + 0.5731i$
7	$-1.205 + 0.5730i$
8	$-1.205 + 0.5730i$

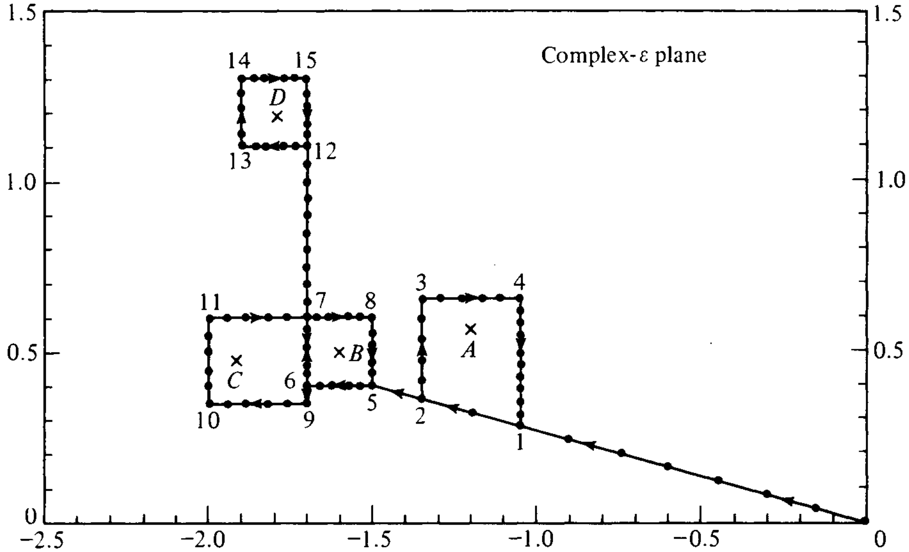

Figure 7.8 A portion of the complex-ε plane for the perturbation problem in Example 3 of Sec. 7.5. Shown are four level-crossing points A, B, C, and D. Paths made up of short line segments which start at the origin and go around the points A, B, C, and D are indicated. Particular points along these paths are labeled by the numbers 1 through 15. The images of these paths in the complex-E plane are shown in Figs. 7.9 to 7.12. Figures 7.8 through 7.12 were drawn with the help of B. Svetitsky and H. Happ.

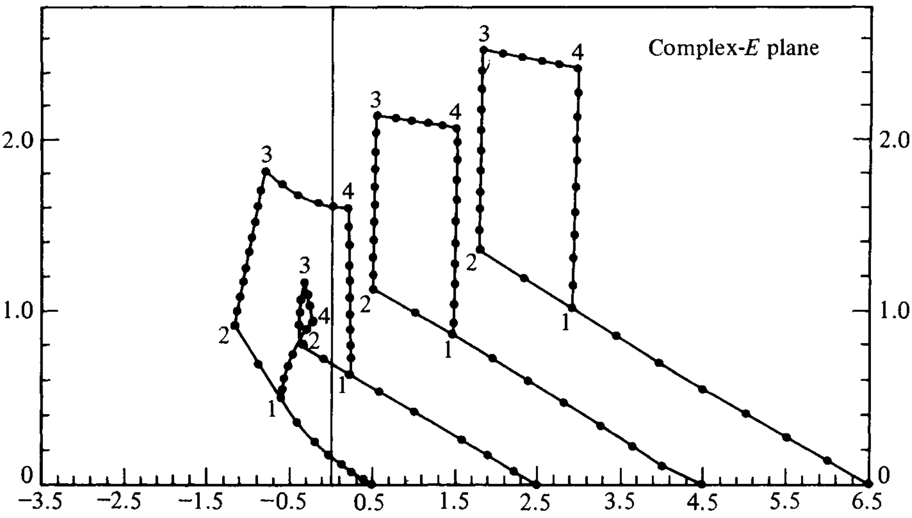

Figure 7.9 Level crossing of the first two even-parity eigenvalues in the complex-E plane. This figure shows the images of the path in the complex-ε plane that encircles the branch point A (see Fig. 7.8). For example, the images of the line segment from the origin to "1" in the ε plane are paths from 0.5, 2.5, 4.5, and 6.5 in the E plane to the four points marked "1". As we go from the origin in the ε plane to "1", "2", "3", "4", "1", and back to the origin, all of the eigenvalues in the E plane return to their original positions except for two, which exchange their identities.

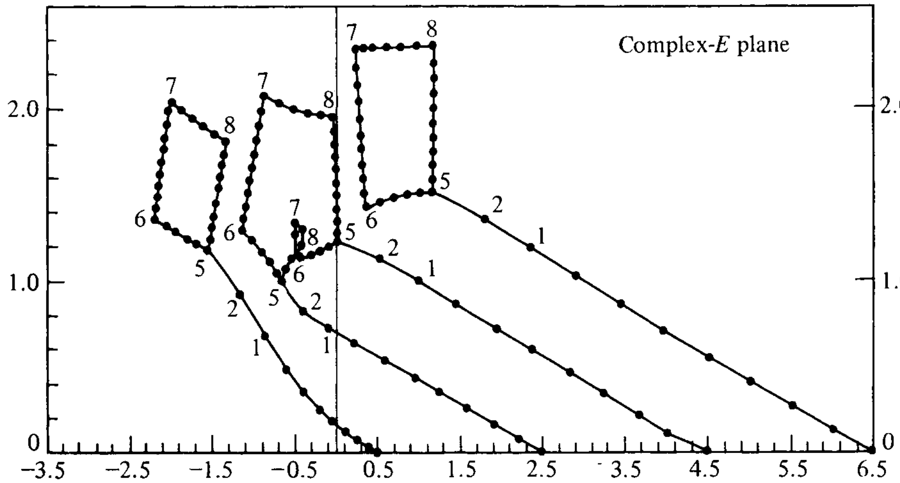

Figure 7.10 Level crossing of the second and third even-parity eigenvalues in the complex-E plane. This figure shows the images of the path in the complex-ε plane that encircles the branch point labeled "B" (see Fig. 7.8).

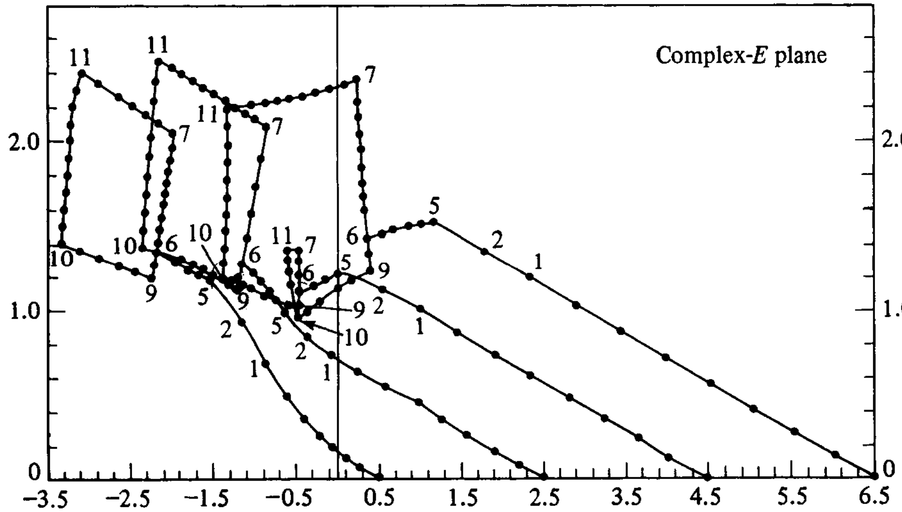

Figure 7.11 Level crossing of the third and fourth even-parity eigenvalues in the complex-E plane. This rather complicated figure shows the images of the path in the complex-ε plane that encircles the branch point labeled "C" (see Fig. 7.8).

Possibility 2 As $N \to \infty$, the branch points all move out to ∞. If this were to happen, it would mean that the eigenvalues of the original differential equation would all be entire functions of ε. Thus, the perturbation problem in question would be regular. We will give an interesting argument, based on the properties of

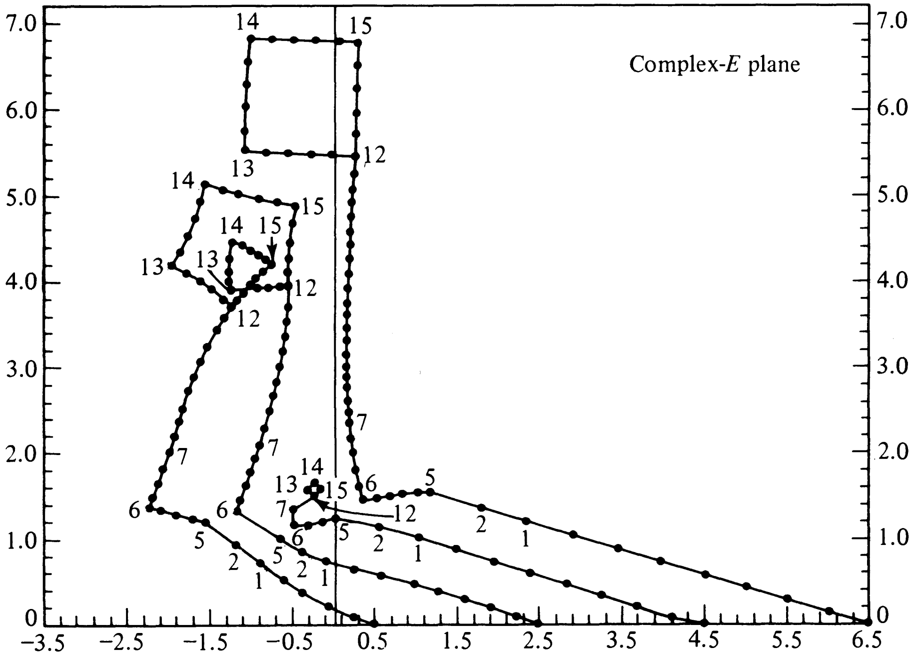

Figure 7.12 The images of the path in the complex-ε plane that encircles the branch point labeled "D" (see Fig. 7.8). The effect of traversing this path is to exchange the first two even-parity eigenvalues in the complex-E plane.

Herglotz functions, which rules out this possibility whenever W is one-signed for all x.

An analytic function is said to be *Herglotz* if $\operatorname{Im} f > 0$ when $\operatorname{Im} z > 0$, $\operatorname{Im} f = 0$ when $\operatorname{Im} z = 0$, and $\operatorname{Im} f < 0$ when $\operatorname{Im} z < 0$. For example, $f(z) = 16 + 7z$ is a Herglotz function. It is a rigorous result of complex variable theory that an entire function [$f(z)$ is *entire* if it is analytic for all $|z| < \infty$] which is Herglotz is linear [$f(z) = a + bz$]. (For the proof see Prob. 7.47.)

Next, we argue that whenever (7.5.3) is a regular perturbation problem, $E(\varepsilon)$ or $-E(\varepsilon)$ is Herglotz. If (7.5.3) is a regular perturbation problem, then for all complex ε, $\varepsilon W(x)$ becomes insignificant compared with $V(x)$ as $|x| \to \infty$. Thus, the asymptotic behavior of $y(x)$ for large $|x|$ is independent of ε. Having established this we simply multiply (7.5.3) by y^* (the complex conjugate of y) and integrate from $-\infty$ to ∞. After one integration by parts (in which the boundary term vanishes) we have

$$\int_{-\infty}^{\infty} y'(x)^* y'(x)\,dx + \int_{-\infty}^{\infty} V(x)y(x)y^*(x)\,dx$$
$$+ \varepsilon \int_{-\infty}^{\infty} W(x)y^*(x)y(x)\,dx = E\int_{-\infty}^{\infty} y^*(x)y(x)\,dx.$$

Taking the imaginary part of this equation gives

$$\operatorname{Im} E = \frac{(\operatorname{Im} \varepsilon) \int_{-\infty}^{\infty} W(x) y^*(x) y(x)\, dx}{\int_{-\infty}^{\infty} y^*(x) y(x)\, dx}.$$

Thus, assuming that W is one-signed, $\operatorname{Im} E$ has the same sign as $\operatorname{Im} \varepsilon$ or $-\operatorname{Im} \varepsilon$.

The Herglotz property of $E(\varepsilon)$ explains why the branch points, which always occur in complex-conjugate pairs in the approximate theory using truncated matrices, must remain as complex-conjugate pairs in the exact theory. More importantly, this theorem almost excludes the possibility that as $N \to \infty$, the branch points all move out to ∞ leaving no singularities in the complex-ε plane. If $E(\varepsilon)$ were both Herglotz and entire, its perturbation series would have the unlikely form $E(\varepsilon) = a + b\varepsilon$. We would conclude that if the first three terms in the perturbation series for $E(\varepsilon)$ are computed and the coefficient of ε^2 is nonzero, then $E(\varepsilon)$ must almost certainly have singularities (branch points) somewhere in the finite-ε plane. (In Prob. 7.27 it is shown that when $V = x^2/4$ and $W = |x|$ the coefficient of ε^2 is nonzero. This is consistent with the existence of singularities in the ε plane in Fig. 7.8.)

This conclusion does not contradict the results of Example 2 of Sec. 7.3, where there was a nonvanishing ε^2 correction to the eigenvalues but the eigenvalues were still entire functions of ε. (Why?)

Possibility 3 Some of the singularities coalesce in the limit $N \to \infty$ to form more complicated kinds of singularities than square-root branch points. This can happen, but the kinds of singularities one might expect to find in the limit are restricted by the condition that $E(\varepsilon)$ be Herglotz. For example, poles or essential singularities cannot occur because in any neighborhood of an essential singularity or a pole it is possible to find both signs of $\operatorname{Im} E$. However, complicated kinds of branch points including logarithmic branch points might well occur.

Possibility 4 As $N \to \infty$, the square-root branch points form a sequence having a limit point at $\varepsilon = 0$. Thus, in any neighborhood of the origin, however small, one can always find a branch point at which a given eigenvalue crosses with some other eigenvalue. As a result, the perturbation series for any eigenvalue has a zero radius of convergence and we have a singular perturbation theory.

Example 4 *Singular perturbation of a parabolic cylinder equation.* Possibility 4 occurs in the singular perturbation theory for which $V = x^2/4$ and $W = x^4/4$. However, it is easiest to understand how such a remarkable configuration as a converging sequence of branch points comes about by studying the *regular* perturbation problem in which the roles of V and W are reversed:

$$-y''(x) + \frac{1}{4} x^4 y(x) + \frac{\varepsilon}{4} x^2 y(x) - E(\varepsilon) y(x) = 0. \tag{7.5.12}$$

To make the connection between these two theories we perform a simple scaling transformation of the independent variable: $x = \varepsilon^{-1/4} t$. The result is the *singular* perturbation problem in which we are interested:

$$-y''(t) + \tfrac{1}{4} t^2 y(t) + \tfrac{1}{4} \delta t^4 y(t) - F(\delta) y(t) = 0, \tag{7.5.13}$$

where $\delta = \varepsilon^{-3/2}$ and $F(\delta) = \delta^{1/3}E(\delta^{-2/3})$. This scaling transformation is reminiscent of the one used in (7.2.4) to convert a singular perturbation problem to a regular one.

We can now point out four interesting features of the singular problem:

(*a*) Observe first that small values of $|\delta|$ correspond with large values of $|\varepsilon|$. Thus, since there are no level-crossing points of $E(\varepsilon)$ in the regular perturbation theory for values of $|\varepsilon|$ less than some nonzero number, it follows that $F(\delta)$, an eigenvalue of the singular perturbation theory, has no crossing points for δ lying outside some circle in the δ plane.

(*b*) The eigenvalue $F(\delta)$ has a sequence of branch points approaching the origin in the δ plane if and only if $E(\varepsilon)$ has a sequence of branch points approaching ∞ in the ε plane. One can show rigorously that, apart from a possible isolated singularity at ∞, $E(\varepsilon)$ *does* have singularities in the ε plane outside any circle $|\varepsilon| = R$. The proof (by Simon) is based on the Herglotz property of $E(\varepsilon)$ (see Prob. 7.48). Numerical calculations show that these singularities are square-root branch points.

(*c*) If we allow δ to become large, $E(\delta^{-2/3})$ approaches the small-ε (unperturbed) value of $E(\varepsilon)$. Thus, the large-δ behavior of $F(\delta)$ is

$$F(\delta) \sim E(0)\delta^{1/3}. \tag{7.5.14}$$

(*d*) Apart from any singularities of $E(\delta^{-2/3})$, $F(\delta)$ clearly has a *cube-root* singularity at the origin in the δ plane. [This is consistent with (7.5.14).] Thus, we visualize the sequence of square-root branch points in the δ plane as converging on a three-sheeted surface. The approximate locations of the branch points near the origin may be determined using WKB theory. The results are that there are actually four separate sequences of branch points which approach the origin on either side of and asymptotic to the lines arg $\delta = \pm 3\pi/2$. This is represented schematically in Fig. 7.13.

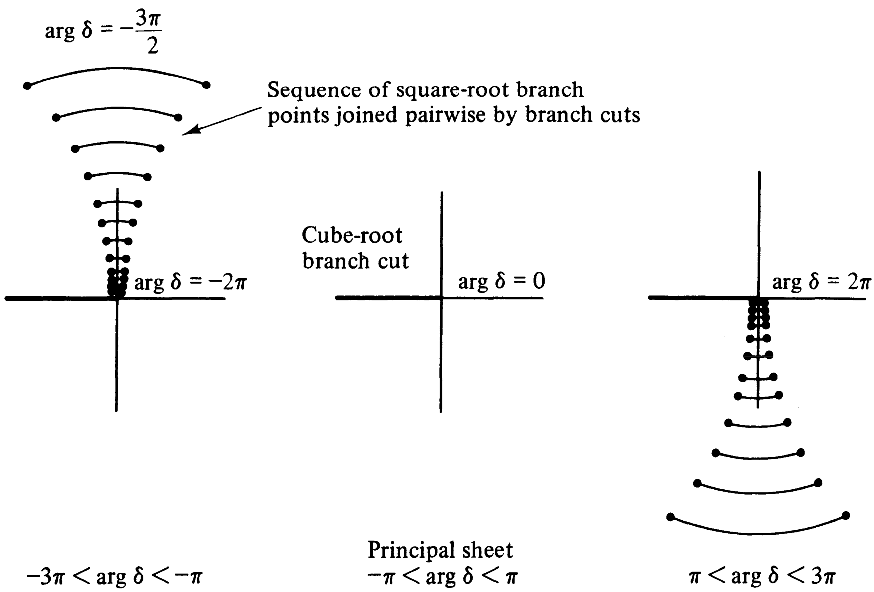

Figure 7.13 Schematic representation of a slice of the Riemann surface on which the function $F(\delta)$ in (7.5.13) is defined. The complete Riemann surface consists of an infinite number of triplets of planes; one such triplet is shown. Each triplet of planes has two pairs of sequences of square-root branch points which approach the origin in the directions arg $\delta = -3\pi/2$ and arg $\delta = 3\pi/2$. Each sequence of pairs of branch points is joined pairwise by branch cuts. Level crossing occurs at every branch point.

We conclude this qualitative summary of the analytic properties of perturbative eigenvalue problems with one general observation. We have seen that when we introduce a perturbing parameter ε into an eigenvalue problem having discrete and apparently unrelated eigenvalues, the eigenvalues suddenly become unified in the sense that they are now recognizable as analytic continuations of each other and therefore identifiable as branches of a single (or possibly several) analytic function(s) $E(\varepsilon)$. The discreteness of the eigenvalues for each value of ε is of course maintained, but a new and remarkable picture of what is meant by a discrete eigenvalue spectrum emerges—there is one eigenvalue for each sheet of the Riemann surface. Thus, although the introduction of the parameter ε initially may have seemed to complicate the original eigenvalue problem, it has ended up organizing and clarifying it.

PROBLEMS FOR CHAPTER 7

Sections 7.1 and 7.2

(E) **7.1** Use second-order perturbation theory to find approximations to the roots of the following equations:

(*a*) $x^2 + x + 6\varepsilon = 0$;
(*b*) $x^3 - \varepsilon x - 1 = 0$;
(*c*) $x^3 + \varepsilon x^2 - x = 0$.

(E) **7.2** Obtain a perturbative solution to the roots of $x^2 - 2.0004x + 0.9998 = 0$ and compare with the exact roots (found by solving the quadratic equation). Why does the most straightforward application of perturbation series fail?

Clue: To find the correct perturbation expansion, study the exact solution of the perturbed quadratic.

(I) **7.3** Formulate a perturbation procedure to solve the equation $(x + 1)^n = \varepsilon x$. How rapidly do the roots vary as a function of ε? Why?

(E) **7.4** Why is $x^3 - x^2 + \varepsilon = 0$ a singular perturbation problem? That is, in what sense does the exact solution undergo an abrupt change in character in the limit $\varepsilon \to 0$? Use perturbation theory to approximate the roots for small ε.

(I) **7.5** Analyze in the limit $\varepsilon \to 0$ the roots of the polynomials:

(*a*) $\varepsilon x^3 + x^2 - 2x + 1 = 0$;
(*b*) $\varepsilon x^8 - \varepsilon^2 x^6 + x - 2 = 0$;
(*c*) $\varepsilon^2 x^8 - \varepsilon x^6 + x - 2 = 0$.

(I) **7.6** Prove that the perturbation expansions (7.1.3) of the roots of the polynomial (7.1.2) in Example 1 of Sec. 7.1 converge for $|\varepsilon| < 1$.

(I) **7.7** (*a*) Show that there is one real root of $1 + (x^2 + \varepsilon)^{1/2} = e^x$ when ε is small and positive. Find a leading-order perturbative approximation to this root as $\varepsilon \to 0+$. Also find a second-order approximation.

(*b*) Find leading-order approximations to the roots of $1 + (x^n + \varepsilon)^{1/n} = e^x$ as $\varepsilon \to 0+$, where $n = 1, 2, 3, \ldots$.

(I) **7.8** Obtain a perturbative solution to $\tan\theta = 1/\theta$ when θ is large.

Clue: When θ is large, the graphs of $\tan\theta$ and θ^{-1} intersect near $\theta = N\pi$, where N is a positive integer. Thus, letting $\theta = N\pi + x$, where x is small, we have $\tan x = 1/(N\pi + x)$. Now let $x = \sum_{n=1}^{\infty} a_n \varepsilon^n$, where $\varepsilon = 1/N\pi$, and solve for the first few a_n.

(I) **7.9** If one expresses the solution y of the equation $y = x - \varepsilon \sin(2y)$ as a power series in ε, what are the first three coefficients?

(I) **7.10** Compute all of the coefficients in the perturbation series solution to the initial-value problem $y' = y + \varepsilon xy$ [$y(0) = 1$]. Show that the series converges for all values of ε. Also, compute the perturbation series indirectly by expanding the explicit exact solution in powers of ε.

(I) **7.11** Find the perturbative solution to (7.1.15).

Clue: The solution of $y_n'' = -e^{-x}y_{n-1}$ ($n = 0, \ldots$) has the form $y_n(x) = \sum_{m=0}^{n} (a_{nm} + b_{nm}x)e^{-mx}$ Find recurrence relations for a_{nm}, b_{nm}. Does the perturbation series give a uniform approximation to the solution for $0 \le x \le \infty$?

(I) **7.12** Solve perturbatively $y'' = (\sin x)y$ [$y(0) = 1$, $y'(0) = 1$].
Is the resulting perturbation series uniformly valid for $0 \le x \le \infty$? Why?

(I) **7.13** Compare the perturbation methods of Example 2 of Sec. 7.1 with the techniques of local analysis. Suppose $f(x)$ in (7.1.7) is x^3 or $\sin x$. How does the second-order approximation to $y(x)$ in (7.1.12) compare with the result of computing the first few coefficients in (7.1.14)?

(I) **7.14** Convert (7.1.16) into a perturbation problem by introducing ε in the right side. Then obtain a first-order approximation to the answer. How accurate is this approximation when $\varepsilon = 1$? You may estimate the accuracy by computing the next term in the perturbation series or by solving (7.1.16) numerically on a computer.

(I) **7.15** Compare local analysis with perturbation theory for the problem $y'' = \varepsilon(a + bx)y$ [$y(0) = A$, $y'(0) = B$] with general nonzero a, b, A, B and $\varepsilon = 1$. Which contains more information, N terms of the local approximations to $y(x)$ (the Taylor series) or N terms of the perturbation series? How many terms of the Taylor series are equivalent to N terms of the perturbation series?

(D) **7.16** Solve perturbatively the boundary-value problem $y'' + y + \varepsilon f(x)y = 0$ [$y(0) = 1$, $y(\pi) = 0$] with $f(x) = O(1)$ for $0 \le x \le \pi$.

Clue: Solve exactly the model problem with $f(x) \equiv 1$ to infer the form of the perturbation series that should be used. Should this problem be regarded as regular or singular?

(I) **7.17** (*a*) Explain the following paradox. We can use perturbation theory as in Example 2 of Sec. 7.1 to solve the initial-value problem $d^ny/dx^n = \varepsilon y$ [$y(0) = y_0$, $y'(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1}$] as a power series in ε. On the other hand, solutions to $y^{(n)} = \varepsilon y$ have the form $e^{\omega\varepsilon^{1/n}x}$, where ω is an nth root of unity. Such solutions may be expanded in powers of $\varepsilon^{1/n}$. Which expansion is correct?

(*b*) Carefully contrast this perturbation problem, which is regular, with the polynomial perturbation problem $x^n = \varepsilon f(x)$, where $f(x)$ is a polynomial of degree at most $n - 1$ and $f(0) = 1$. The latter problem is singular.

(I) **7.18** Analyze the limiting behavior as $\varepsilon \to 0$ of the solution to the equation $y' = -y - \varepsilon y^2$ [$y(0) = 1$] by solving the differential equation exactly. Should this problem be classified as regular or singular for x positive? x negative? Why? Repeat the above analysis for the initial condition $y(0) = -1$.

(E) **7.19** Show that the solution to the initial-value problem $y'' + y + \varepsilon y = 0$ [$y(0) = 1$, $y'(0) = 0$] remains bounded for all real x. Obtain a first-order perturbative approximation to $y(x)$ and show that it is unbounded as $x \to \infty$. Conclude that the first-order approximation is valid only for $|x| < O(\varepsilon^{-1})$. To approximate $y(x)$ for $|x| > O(\varepsilon^{-1})$ it is best to use the multiple-scale methods discussed in Chap. 11.

(I) **7.20** (*a*) Find the asymptotic behavior of the solutions $y(x)$ to (7.2.10) as $x \to +\infty$.

(*b*) Solve (7.2.10) exactly in terms of well-known special functions.

(*c*) On the basis of the solution to part (*b*), analyze the behavior of $y(x)$ for $x = O(1)$ as $\varepsilon \to 0+$.

(*d*) How large must x be for the latter asymptotic estimate of $y(x)$ to break down?

(*e*) Show that εx is not a small perturbation in (7.2.10) unless $\varepsilon^{1/2}x \ll 1$ ($\varepsilon \to 0+$). See Prob. 11.13.

(I) **7.21** Use the regular perturbation methods explained in Example 2 of Sec. 7.1 to all orders in powers of ε. Show that when x is of order $\varepsilon^{-1/2}$, the solution to (7.2.10) satisfies $y(x) = \cos(x - \varepsilon x^2/4) + O(\varepsilon^{1/2})$ ($\varepsilon \to 0+$).

Clue: Show that when $x = O(\varepsilon^{-1/2})$

$$y(x) = \cos x \left[1 - \frac{1}{2!}\left(\frac{\varepsilon x^2}{4}\right)^2 + \frac{1}{4!}\left(\frac{\varepsilon x^2}{4}\right)^4 - \cdots\right]$$
$$+ \sin x \left[\left(\frac{\varepsilon x^2}{4}\right) - \frac{1}{3!}\left(\frac{\varepsilon x^2}{4}\right)^3 + \cdots\right] + O(\varepsilon^{1/2}), \qquad \varepsilon \to 0+.$$

(See Sec. 11.1.)

(I) **7.22** (*a*) Apply regular perturbation theory to first order in ε to estimate the effect of the εx^{19} perturbation upon the roots of (7.2.12). Show that the root at $x = k$ changes by the amount

$$(-1)^{k+1}\varepsilon \frac{k^{19}}{(k-1)!\,(20-k)!} + O(\varepsilon^2), \qquad \varepsilon \to 0.$$

(*b*) Show that the unperturbed root at $x = 16$ is most sensitive to ε. Estimate the magnitude of ε necessary to perturb each of the roots by 1 percent.

(*c*) For what value of ε does first-order perturbation theory predict "crossing" of the unperturbed roots $x = 16, 17$?

(*d*) Apply second-order perturbation theory to the roots at $x = 16, 17$. For what values of ε do the roots cross? Compare with the result of part (*c*).

(*e*) What are the radii of convergence of these perturbation series for the roots?

(D) **7.23** (*a*) Develop a perturbation theory for the roots of (7.2.12) that accounts for the fact that the roots may become complex-conjugate pairs when ε is large enough.

Clue: Seek real quadratic factors of $P(x)$ in the form $x^2 + a(\varepsilon)x + b(\varepsilon)$ where a and b are real.

Carry out this calculation to first order in ε for the roots whose unperturbed values are 16 and 17. What are your predictions for these roots when $\varepsilon = -10^{-10}$, -2×10^{-10}, and -10^{-9}?

(*b*) Repeat the above calculations to second order in ε.

(*c*) What is the radius of convergence of the Taylor series expansions of $a(\varepsilon)$ and $b(\varepsilon)$ about $\varepsilon = 0$ for the unperturbed roots at 16 and 17 [$a(0) = -33$, $b(0) = 272$]?

Section 7.3

(I) **7.24** Suppose that $y_0(x)$ in (7.3.15) vanishes when $x = x_0$, making this integral expression for $y_n(x)$ formally divergent when $x > x_0 > a$. Show that it is possible to continue this integral representation for $y_n(x)$ through x_0 to give a finite expression for $y_n(x)$ for all x. Demonstrate this effect explicitly for the eigenvalue problem (7.3.3) with $V(x) = x^2/4$, $W(x) = x^4/4$, $a = 0$, $y_0(x) = (x^2 - 1)e^{-x^2/4}$, $E_0 = \frac{5}{2}$, and $n = 1$.

(I) **7.25** (*a*) Show by explicit computation that if $V(x) = x^2/4$ and $W(x) = x$ in (7.3.3), then the perturbation series for $y(x)$ is convergent for all ε and the series for $E(\varepsilon)$ has vanishing terms of order ε^n for $n \geq 3$.

(*b*) Rederive these results by showing that the effect of the perturbation is equivalent to a translation of x by 2ε and the energy level by $-\varepsilon^2$ in the unperturbed problem $V(x) = x^2/4$, $W(x) = 0$.

(I) **7.26** Show that if $V(x) = x^2/4$ and $W(x) = x^4/4$ in (7.3.3), then the perturbation series for the smallest eigenvalue is given by (7.3.16).

(D) **7.27** (*a*) Show that if $V(x) = x^2/4$ and $W(x) = |x|$ in (7.3.3), then the perturbation series for the smallest eigenvalue for positive ε begins $E(\varepsilon) = \frac{1}{2} + (2/\pi)^{1/2}\varepsilon - (\pi - 4 + 2\ln 2)\varepsilon^2/\pi + O(\varepsilon^3)$ $(\varepsilon \to 0+)$.

Clue: The expression for the coefficient of ε^2 is a tricky integral which can be evaluated in closed form by judicious use of polar coordinates.

(*b*) Show that the eigenvalues $E(\varepsilon)$ satisfy the transcendental equation

$$\frac{d}{dx} D_{\varepsilon^2 + E - 1/2}(x + 2\varepsilon)\bigg|_{x=0} = 0$$

when the eigenfunction is even in x, and

$$D_{\varepsilon^2+E-1/2}(2\varepsilon)=0$$

when the eigenfunction is odd.

Section 7.4

(E) **7.28** Use the method of patching to solve $y''(x)-y(x)=\delta(x)$ $[y(\pm\infty)=0]$.

(E) **7.29** Use patching to solve $y'=y$ $(y>x)$, $y'=x$ $(y\le x)$, with $y(0)=0$.

(I) **7.30** (*a*) One would normally think that the upper edge of the left region in Examples 2 and 3 of Sec. 7.4 would be the largest value of x for which εx^2 is still small compared with 1 (see the differential equation). This would suggest that the left region consists of those x for which $x \ll \varepsilon^{-1/2}$ $(\varepsilon\to 0+)$. Actually, the region of validity of the left solution is $x\ll\varepsilon^{-1/3}$ $(\varepsilon\to 0+)$. Show that this is true and explain why.

(*b*) Explain why the largest possible matching region for the differential equation in Example 2 of Sec. 7.4 is $1\ll x\ll\varepsilon^{-1/3}$ $(\varepsilon\to 0+)$.

(D) **7.31** Derive (7.4.10).

(I) **7.32** Show that the largest possible matching region for Example 4 of Sec. 7.4 is $1\ll x\ll\varepsilon^{-1/2}$ $(\varepsilon\to 0+)$.

(D) **7.33** Carry out the analysis of Example 4 of Sec. 7.4 to first order in ε for $v=0$. Determine the size of the largest matching region.

(I) **7.34** Use asymptotic matching to solve the initial-value problem $y''+(v+\frac{1}{2}-\frac{1}{4}x^2-\varepsilon x^4)y=0$ $[y(0)=y'(0)=1]$. Determine the leading behavior of $y(x)$ as $x\to+\infty$.

(D) **7.35** (*a*) Show that $y_1(x)$ in (7.4.16) uniformly satisfies (7.4.14) up to terms of order 1 for all x in region I.

(*b*) More generally, show that (7.4.14) has a solution of the form (7.4.17) which is accurate throughout region I to any order in powers of $1/E$. Calculate $f_2(x)$ and $g_1(x)$.

(D) **7.36** Use matched asymptotic expansions to derive two terms in the asymptotic expansions of the following integrals as $x\to+\infty$:

(*a*) $\displaystyle\int_0^\pi e^{ixt^2}\cos t\,dt;$

(*b*) $\displaystyle\int_0^\pi e^{ixt^4}\cos t\,dt;$

(*c*) $\displaystyle\int_0^1 \exp\,(ix\sqrt{1-t^2})t^{-1/4}(1-t^2)^{-1/2}\,dt;$

(*d*) $\displaystyle\int_0^1 e^{ix\ln(1+t^2)}\cos t\,dt;$

(*e*) $\displaystyle\int_0^1 e^{ix(\sinh t-t)}\sqrt{1-t^2}\,dt.$

For each of these integrals give the order of the error term after two terms in the asymptotic expansion.

(D) **7.37** Use matched asymptotic expansions to show that

$$\int_0^{\pi/4}\frac{\mathrm{Ai}\,(-x\sin t)}{\sqrt{\sin t}}\,dt\sim\sqrt{\frac{2\pi}{x}}\frac{3^{-1/3}}{\Gamma(\frac{5}{6})}-\frac{2^{11/8}\cos\,(2^{1/4}x/3+\pi/4)}{\sqrt{\pi}\,x^{7/4}}+O(x^{-5/2}),\qquad x\to+\infty.$$

(I) **7.38** Verify the error estimates given in (7.4.33) and (7.4.36).

Clue: Show that

$$\left|e^{-x}-\sum_{n=0}^{N}\frac{(-x)^n}{n!}\right|\le\frac{x^{N+1}}{(N+1)!}$$

for all $x\ge 0$.

(I) **7.39** Examine the $\varepsilon \to 0+$ behavior of the integral $G(\varepsilon) = \int_0^\infty \exp(-t - \varepsilon/\sqrt{t})\,dt$.

(*a*) Show that if we expand the integrand in powers of ε and keep only the first two terms we obtain $G(\varepsilon) \sim 1 - \varepsilon\sqrt{\pi}$ $(\varepsilon \to 0+)$.

(*b*) The method of matched asymptotic expansions serves to verify this behavior and to find the higher-order corrections to it. Divide the region of integration into two overlapping regions, the *inner* region $(t \ll 1)$ and the *outer* region $(t \gg \varepsilon^2)$ $(\varepsilon \to 0+)$, and decompose $G(\varepsilon)$ into inner and outer integrals: $G(\varepsilon) = \int_0^\delta \exp(-t - \varepsilon/\sqrt{t})\,dt + \int_\delta^\infty \exp(-t - \varepsilon/\sqrt{t})\,dt$, where $\delta(\varepsilon)$ satisfies $\varepsilon^2 \ll \delta \ll 1$ $(\varepsilon \to 0+)$. Using (7.4.31) show that

$$\int_0^\delta \exp(-t - \varepsilon/\sqrt{t})\,dt = \int_0^\delta \exp(-\varepsilon/\sqrt{t})\,dt + O(\delta^2), \qquad \delta \to 0+,$$

$$= 2\varepsilon^2\left[\frac{\delta}{2\varepsilon^2} - \frac{\sqrt{\delta}}{\varepsilon} + O\left(\ln\frac{\delta}{\varepsilon^2}\right)\right] + O(\delta^2), \qquad \delta \to 0+, \varepsilon^2/\delta \to 0+,$$

and using (6.2.5) show that

$$\int_\delta^\infty \exp(-t - \varepsilon/\sqrt{t})\,dt \doteq \int_\delta^\infty e^{-t}\left(1 - \frac{\varepsilon}{\sqrt{t}}\right)dt + O\left(\frac{\varepsilon^2}{\delta}\right), \qquad \varepsilon^2/\delta \to 0+,$$

$$= e^{-\delta} - \varepsilon[\sqrt{\pi} - 2\sqrt{\delta} + O(\delta^{3/2})] + O(\varepsilon^2/\delta), \qquad \delta \to 0+, \varepsilon^2/\delta \to 0+.$$

Combine the inner and outer expansions to show that $G(\varepsilon) = 1 - \varepsilon\sqrt{\pi} + O(\delta^2) + O(\varepsilon^2/\delta)$ $(\delta \to 0+, \varepsilon^2/\delta \to 0+)$. Show that the error terms are negligible compared to ε provided that $\varepsilon \ll \delta \ll \sqrt{\varepsilon}$ $(\varepsilon \to 0+)$.

(*c*) Perform a higher-order match to derive the first five terms in the expansion of $G(\varepsilon)$. In particular, show that

$$\int_0^\delta \exp(-t - \varepsilon/\sqrt{t})\,dt = \int_0^\delta \exp(-\varepsilon/\sqrt{t})(1 - t + t^2/2)\,dt + O(\delta^4)$$

$$= 2\varepsilon^2\left[C_2 - \frac{1}{2}\ln\frac{\varepsilon}{\sqrt{\delta}} + \frac{\delta}{2\varepsilon^2} - \frac{\sqrt{\delta}}{\varepsilon} + \frac{\varepsilon}{6\sqrt{\delta}} + O\left(\frac{\varepsilon^2}{\delta}\right)\right]$$

$$- 2\varepsilon^4\left[\frac{\delta^2}{4\varepsilon^4} - \frac{\delta^{3/2}}{3\varepsilon^3} + \frac{\delta}{4\varepsilon^2} + O\left(\frac{\sqrt{\delta}}{\varepsilon}\right)\right]$$

$$+ \varepsilon^6\left[\frac{\delta^3}{6\varepsilon^6} + O\left(\frac{\delta^{5/2}}{\varepsilon^5}\right)\right] + O(\delta^4), \qquad \delta \to 0+, \varepsilon^2/\delta \to 0+,$$

and that

$$\int_\delta^\infty \exp(-t - \varepsilon/\sqrt{t})\,dt = \int_\delta^\infty e^{-t}\left(1 - \frac{\varepsilon}{\sqrt{t}} + \frac{\varepsilon^2}{2t} - \frac{\varepsilon^3}{6t^{3/2}}\right)dt + O\left(\frac{\varepsilon^4}{\delta}\right)$$

$$= e^{-\delta} - \varepsilon\left[\sqrt{\pi} - 2\sqrt{\delta} + \frac{2}{3}\delta^{3/2} + O(\delta^{5/2})\right]$$

$$+ \frac{1}{2}\varepsilon^2[C_0 - \ln\delta + \delta + O(\delta^2)]$$

$$- \frac{1}{6}\varepsilon^3\left[-2\sqrt{\pi} + \frac{2}{\sqrt{\delta}} + O(\sqrt{\delta})\right] + O\left(\frac{\varepsilon^4}{\delta}\right), \qquad \delta \to 0+, \varepsilon^2/\delta \to 0+.$$

Combine these inner and outer expansions and substitute $C_0 = -\gamma$ and $C_2 = -\gamma/2 + \frac{3}{4}$ to show that

$$G(\varepsilon) = 1 - \varepsilon\sqrt{\pi} - \varepsilon^2\ln\varepsilon - \frac{3}{2}(\gamma - 1)\varepsilon^2 + \frac{1}{3}\sqrt{\pi}\,\varepsilon^3 + O\left(\frac{\varepsilon^4}{\delta}\right) + O(\varepsilon^3\sqrt{\delta})$$

$$+ O(\varepsilon^2\delta^2) + O(\varepsilon\delta^{5/2}) + O(\delta^4), \qquad \delta \to 0+, \varepsilon^2/\delta \to 0+.$$

Note that the error terms are negligible compared with ε^3 in the limit $(\varepsilon \to 0+)$ provided that $\varepsilon \ll \delta \ll \varepsilon^{4/5}$ $(\varepsilon \to 0+)$.

(D) **7.40** (*a*) Use matched asymptotic expansions to show that $\int_0^\infty e^{-t-\varepsilon/t^2}\,dt \sim 1 - \sqrt{\pi\varepsilon} - \frac{1}{2}\varepsilon \ln \varepsilon - \frac{3}{2}\varepsilon(\gamma - 1) + \cdots$ $(\varepsilon \to 0+)$.

(*b*) Investigate the asymptotic behavior of the integral $\int_0^\infty e^{-t-\varepsilon/t^\alpha}\,dt$ as $\varepsilon \to 0+$ for fixed α. For which values of α does Euler's constant γ appear in this expansion?

(I) **7.41** Show how the impossible conditions (7.4.45) in Example 6 of Sec. 7.4 arise after one or two integrations by parts.

(D) **7.42** Use asymptotic matching to determine the first two terms in the asymptotic expansion as $x \to +\infty$ of $\int_0^1 e^{ixt}t^{-1/2}(1-t)^{-1/4}\,dt$.

Section 7.5

(D) **7.43** (*a*) Use the recursion property of the Hermite polynomials $\mathrm{He}_n(x)$, $\mathrm{He}_{n+1}(x) = x\,\mathrm{He}_n(x) - n\,\mathrm{He}_{n-1}(x)$, to derive the coefficients of the matrix M in (7.5.7) for $V(x) = x^2/4$ and $W(x) = x^4/4$.

(*b*) Using the result of part (*a*) derive a five-term recursion relation for $D_N(E, \varepsilon) = \det M_N$, where M_N is the $N \times N$ truncation of M.

(TI) **7.44** Show from the specific form of $D_N(E, \varepsilon)$ that $E_N(\varepsilon)$, the solution of (7.5.10), may not have poles or branch points at which $E_N(\varepsilon) = \infty$.

(I) **7.45** (*a*) If $P(x, y)$ and $Q(x, y)$ are polynomials of degree M and N, respectively, in x and y together, show that there are at most MN roots of the simultaneous equations $P(x, y) = 0$, $Q(x, y) = 0$.

Clue: Suppose that $P(x, y) = \sum_{m=0}^{M} p_{M-m}(y)x^m$, $Q(x, y) = \sum_{n=0}^{N} q_{N-n}(y)x^n$ where $p_n(y)$, $q_n(y)$ are polynomials in y of degree at most n, and consider the $(M + N)$ equations for fixed y, $x^pP(x, y) = 0$ $(p = 0, \ldots, N-1)$, $x^qQ(x, y) = 0$ $(q = 0, \ldots, M-1)$, as $(M + N)$ *linear* equations in the $(M + N)$ "unknowns" $1, x, x^2, \ldots, x^{M+N-1}$. The condition for a solution to exist is that the determinant of the coefficients of the system be zero. Show that this determinant, called Sylvester's eliminant, is a polynomial of at most degree MN in y.

(*b*) Give an example of such a pair of polynomial equations having *fewer* than MN roots.

(D) **7.46** Show that if $V(x) = x^2/4$ and $W(x) = |x|$ in (7.3.3), then A_{2m}^{2n} in (7.5.6) is given by

$$A_{2m}^{2n} = \left(\frac{2}{\pi}\right)^{1/2} \frac{(-1)^{n+m+1}[2(n+m)+1](2n)!}{2^{m+n}[4(n-m)^2 - 1]m!\,n!}.$$

(TI) **7.47** Show that if $f(z)$ is entire and Herglotz it is linear.

Clue: An entire function has a Taylor series which converges for all $|z| < \infty$. Letting $z = re^{i\theta}$ in this series gives $\mathrm{Im}\, f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n \sin(n\theta)$ $(r < \infty)$. Multiply this series by $m \sin\theta \pm \sin(m\theta)$ $(m > 1)$ and integrate from $\theta = 0$ to $\theta = \pi$. Show that $m\sin\theta \pm \sin(m\theta)$ has the same sign as $\sin\theta$, which has the same sign as $\mathrm{Im}\, z = r\sin\theta$. Therefore, by the Herglotz property of f we have $ma_1 r \pm a_m r^m \geq 0$ $(m > 1)$. But we may take r arbitrarily large. Thus, for this inequality to remain true we must have $a_m = 0$ $(m \geq 0)$ and $f(z) = a_0 + a_1 z$.

(TI) **7.48** Show that in addition to a possible isolated singularity at ∞, $E(\varepsilon)$ in (7.5.12) must have other singularities in the ε plane outside any circle $|\varepsilon| = R$.

Clue: Assume the contrary, namely, that $E(\varepsilon)$ is analytic except for an isolated singularity at ∞. Then

$$E(\varepsilon) = \sum_{n=-\infty}^{\infty} a_n \varepsilon^n \qquad (*)$$

converges for all finite $|\varepsilon| > R$. But $E(\varepsilon)$ is Herglotz. By the same argument as in Prob. 7.47, show that $a_n = 0$ for $n \geq 2$. Using $F(\varepsilon) = \delta^{1/3}E(\delta^{-2/3})$ in (7.5.13) rewrite (*) as

$$F(\delta) = \sum_{n=-1}^{\infty} a_{-n}\delta^{(2n+1)/3} \qquad (**)$$

where this series *converges* for $|\delta|$ small enough but $\delta \neq 0$. However, we already know that the perturbation series for $F(\delta)$ has the form $F(\delta) \sim \sum_{n=0}^{\infty} F_N \delta^N$ $(\delta \to 0)$. Equating coefficients of powers of δ in both series yields many contradictions. For example, the absence of terms δ^0 and δ^2 in (**) implies that F_0 and F_2 vanish but $F_0 = \frac{1}{2}$, $F_1 = \frac{3}{4}$, $F_2 = -\frac{21}{8}$ for the smallest eigenvalue [see (7.3.16)].

(I) **7.49** Show that if (7.5.13) is generalized to $-y''(t) + \frac{1}{4}t^2 y(t) + \frac{1}{4}\delta t^{2N} y(t) - F(\delta)y(t) = 0$, then the cube-root branch cut on Fig. 7.13 is replaced by an $(N + 1)$th-root branch point.