The problem I have is about the derivation of the fact that the escape rate is the leading eigenvalue of the Perron-Frobenius-Operator. In order to calculate the escape rate, one has to examine the asymptotic behaviour of the quantity

\[ \Gamma_n = \frac{1}{|M|} \int_M dx \int_M dy \delta(y - f^n(x)). \]  

(1)

Of course the \( dx \)-integral is nothing but the Perron-Frobenius-Operator \( L^n \) acting on an uniform initial density \( i(x) = 1 \forall x \in \mathcal{M} \):

\[ \Gamma_n = \frac{1}{|M|} \int_M dy (L^n i)(y). \]  

(2)

If I understood it correctly, you argue in the following way: the initial density \( i(x) \) can be expanded in terms of eigenfunctions of \( L \),

\[ i(x) = \sum \alpha c_\alpha \varphi_\alpha(x), \]  

(3)

and therefore, for large \( n \), \( \Gamma_n \) is dominated by \( \lambda_0 \), the leading eigenvalue of \( L \): \( \Gamma_n \sim \lambda_0^n \) as \( n \to \infty \).

My first and most important question is the following: is the decomposition (3) really possible in an open system?

If trajectories can escape and the invariant set \( \Lambda \) is only a subset of \( \mathcal{M} \) of zero Lebesgue measure, I think the eigenfunctions \( \varphi_\alpha \) must be zero almost everywhere. Why? The eigenvalue condition

\[ (L^n \varphi_\alpha)(y) = \int_M dx \delta(y - f^n(x)) \varphi_\alpha(x) = \lambda_\alpha^n \varphi_\alpha(y) \]  

(4)

yields that \( \varphi_\alpha \) can have nonzero values only on the set \( \bigcap_{k=0}^n f^k(\mathcal{M}) \). This set becomes arbitrary small for large \( n \), and (4) holds for every \( n \), if \( f \) is invertible, it holds even for negative \( n \). Then, all the \( \varphi_\alpha \) must be concentrated on the invariant set \( \Lambda \), or at least on the set \( \Lambda^\infty_+ := \bigcap_{k=0}^\infty f^k(\mathcal{M}) \), and it is impossible to expand \( i(x) = 1 \forall x \in \mathcal{M} \) in terms of the eigenfunctions \( \varphi_\alpha \).

So how does it work? Do I have to think of the \( \varphi_\alpha \) as functions that are a little bit smoothed around \( \Lambda^\infty_+ \)? For large \( n \), only points close to \( \Lambda^\infty_+ \) contribute to the \( dy \)-integral in (1). Or am I dead wrong?

If this problem is solved, there are two questions remaining. Is \( \{ \varphi_\alpha \} \) a basis for a (properly chosen) function space? And can I be sure that the coefficient \( c_0 \) in (3) isn’t zero? Otherwise \( \lambda_0 \) would not be dominating.

Thank you very much for looking at this.