The problem I have is about the derivation of the fact that the escape rate is the leading eigenvalue of the Perron-Frobenius-Operator. In order to calculate the escape rate, one has to examine the asymptotic behaviour of the quantity

$$\Gamma_n = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} dx \int_{\mathcal{M}} dy \, \delta(y - f^n(x)). \tag{1}$$

Of course the dx-integral is nothing but the Perron-Frobenius-Operator $\mathcal{L}^n$ acting on an uniform initial density $i(x) = 1 \, \forall x \in \mathcal{M}$:

$$\Gamma_n = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} dy \, (\mathcal{L}^n i) \, (y). \tag{2}$$

If I understood it correctly, you argue in the following way: the initial density $i(x)$ can be expanded in terms of eigenfunctions of $\mathcal{L}$,

$$i(x) = \sum_{\alpha} c_\alpha \varphi_\alpha(x), \tag{3}$$

and therefore, for large n, Γ_n is dominated by λ_0, the leading eigenvalue of $\mathcal{L}$: $\Gamma_n \sim \lambda_0^n$ as $n \to \infty$.

My first and most important question is the following: is the decomposition (3) really possible in an open system?

If trajectories can escape and the invariant set Λ is only a subset of $\mathcal{M}$ of zero Lebesgue measure, I think the eigenfunctions φ_α must be zero almost everywhere. Why? The eigenvalue condition

$$\begin{aligned}(\mathcal{L}^n \varphi_\alpha)(y) &= \int_{\mathcal{M}} dx \, \delta(y - f^n(x)) \varphi_\alpha(x) \\ &= \lambda_\alpha^n \varphi_\alpha(y)\end{aligned} \tag{4}$$

yields that φ_α can have nonzero values only on the set $\cap_{k=0}^n f^k(\mathcal{M})$. This set becomes arbitrary small for large n, and (4) holds for every n, if f is invertible, it holds even for negative n. Then, all the φ_α must be concentrated on the invariant set Λ, or at least on the set $\Lambda_+^\infty := \cap_{k=0}^\infty f^k(\mathcal{M})$, and it is impossible to expand $i(x) = 1 \, \forall x \in \mathcal{M}$ in terms of the eigenfunctions φ_α.

So how does it work? Do I have to think of the φ_α as functions that are a little bit smoothed around Λ_+^∞? For large n, only points close to Λ_+^∞ contribute to the dy-integral in (1). Or am I dead wrong?

If this problem is solved, there are two questions remaining. Is $\{\varphi_\alpha\}$ a basis for a (properly chosen) function space? And can I be sure that the coefficient c_0 in (3) isn't zero? Otherwise λ_0 would not be dominating.

Thank you very much for looking at this.