Chapter 13

Slice & dice

Physicists like symmetry more than Nature
— Rich Kerswell

If the symmetry is continuous, the notion of fundamental domain is not applicable. Instead, the dynamical system should be reduced to a lower-dimensional, desymmetrized system, with ‘ignorable’ coordinates separated out (but not forgotten).

We shall describe here two ways of reducing a continuous symmetry. In the ‘method of slices’ of sect. 13.2 we slice the state space in such a way that an entire class of symmetry-equivalent points is represented by a single point. In the Hilbert polynomial basis approach of sect. 13.7 we replace the equivariant dynamics by the dynamics rewritten in terms of invariant coordinates. In either approach we retain the option of computing in the original coordinates, and then, when done, projecting the solution onto the symmetry reduced state space, or ‘post-processing’.

In the method of slices symmetry reduction is achieved by cutting the group orbits with a finite set of slice hyperplanes, one for each continuous group parameter, with each group orbit of symmetry-equivalent points represented by a single point, its intersection with the slice. The procedure is akin to (but distinct from) cutting across continuous-time parametrized trajectories by means of Poincaré sections. As is the case for Poincaré sections, choosing a ‘good’ slice is a dark art. We describe two strategies: (i) Foliation of state space by group orbits is a purely group-theoretic phenomenon that has nothing to do with dynamics, so we construct slices based on a decomposition of state space into irreducible linear representations of the symmetry group G. (ii) Nonlinear dynamics strongly couples such linear symmetry eigenmodes, so locally optimal slices should be constructed from physically important recurrent states, or ‘templates’. Our guiding principle is to chose a slice such that the distance between a ‘template’ state $\hat{x}'$ and nearby group orbits is *minimized*, i.e., we identify the point $\hat{x}$ on the group orbit (12.2) of a nearby state x which is the closest match to the template point $\hat{x}'$.

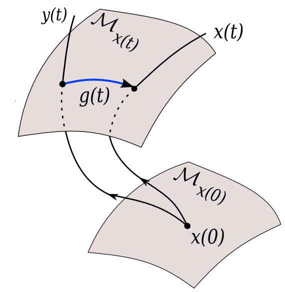


Figure 13.1: The freedom to pick a moving frame: A point x on the full state space trajectory $x(t)$ is equivalent up to a group rotation $g(t)$ to the point $\hat{x}$ on the curve $\hat{x}(t)$ if the two points belong to the same group orbit $\mathcal{M}_x$, see (12.2).

We start our discussion by explaining the freedom of redefining dynamics in a moving frame.

13.1 Only dead fish go with the flow: moving frames

The idea: As the symmetries commute with dynamics, we can evolve a solution $x(\tau)$ for as long as we like, and then rotate it to any equivalent point (see figure 13.1) on its group orbit. We can map each point along any solution $x(\tau)$ to the unique representative $\hat{x}(\tau)$ of the associated group orbit equivalence class, by a coordinate transformation

$$x(\tau) = g(\tau)\,\hat{x}(\tau)\,. \tag{13.1}$$

Equivariance guarantees that the two states are physically equivalent.

Definition: Moving frame. For a given $x \in \mathcal{M}$ and a given space of 'representative shapes' $\hat{\mathcal{M}}$ there exists a unique group element $g = g(x, \tau)$ that at instant τ rotates x into $gx = \hat{x} \in \hat{\mathcal{M}}$. The map that associates to a state space point x a Lie group action $g(x, \tau)$ is called a *moving frame*. exercise A2.1 exercise 13.1

Using decomposition (13.1) one can always write the full state space trajectory as $x(\tau) = g(\tau)\,\hat{x}(\tau)$, where the $(d-N)$-dimensional reduced state space trajectory $\hat{x}(\tau)$ is to be fixed by some condition, and $g(\tau)$ is then the corresponding curve on the N-dimensional group manifold of the group action that rotates $\hat{x}$ into x at time τ. The time derivative is then $\dot{x} = v(g\hat{x}) = \dot{g}\hat{x} + g\hat{v}$, with the reduced state space velocity field given by $\hat{v} = d\hat{x}/dt$. Rewriting this as $\hat{v} = g^{-1}v(g\,\hat{x}) - g^{-1}\dot{g}\,\hat{x}$ and using the equivariance condition (12.14) leads to

$$\hat{v} = v - g^{-1}\dot{g}\,\hat{x}\,. \tag{13.2}$$

The Lie group element (12.5) and its time derivative describe the group tangent flow

$$g^{-1}\dot{g} = g^{-1}\frac{d}{dt}e^{\phi\cdot\mathbf{T}} = \dot{\phi}\cdot\mathbf{T}\,.$$

This is the group tangent velocity $g^{-1}\dot{g}\,\hat{x} = \dot{\phi}\cdot t(\hat{x})$ evaluated at the point $\hat{x}$, i.e., with $g = 1$. The flow $\hat{v} = d\hat{x}/dt$ in the $(d-N)$ directions transverse to the group flow is now obtained by subtracting the flow along the group tangent direction,

$$\hat{v}(\hat{x}) = v(\hat{x}) - \dot{\phi}(\hat{x})\cdot t(\hat{x})\,. \tag{13.3}$$

We can pick any coordinate transformation (13.1) between the 'lab' and 'moving frame', any time, any way we like; equivariance guarantees that the states and the equations of motion (13.3) in the two frames are physically equivalent. This is a immense freedom, and with freedom comes responsibility, the responsibility of choosing a good frame.

13.2 Symmetry reduction

> Maybe when I'm done with grad school I'll be able to figure it all out . . .
> — Rebecca Wilczak, undergraduate

Given Lie group G acting smoothly on a C^∞ manifold $\mathcal{M}$, we can think of each group orbit as an equivalence class. *Symmetry reduction* is the identification of a unique point on a group orbit as the representative of its equivalence class. We call the set of all such group orbit representatives the *reduced state space* $\mathcal{M}/G$. This space has many names in the literature - it is alternatively called 'desymmetrized state space', 'symmetry-reduced space', 'orbit space' (because every group orbit in the original space is mapped to a single point in the orbit space), 'base manifold', 'shape-changing space' or 'quotient space' (because the symmetry has been 'divided out'), obtained by mapping equivariant dynamics to invariant dynamics ('image') by methods such as 'moving frames', 'cross sections', 'slices', 'freezing', 'Hilbert bases', 'quotienting', 'lowering of the degree', 'lowering the order', or 'desymmetrization'. remark 13.1

Symmetry reduction replaces a dynamical system $(\mathcal{M}, f)$ with a symmetry by a 'desymmetrized' system $(\hat{\mathcal{M}}, \hat{f})$ of figure 12.2 (b), a system where each group orbit is replaced by a point, and the action of the group is trivial, $g\hat{x} = \hat{x}$ for all $\hat{x}\in\hat{\mathcal{M}}$, $g \in G$. The reduced state space $\hat{\mathcal{M}}$ is sometimes called the 'quotient space' $\mathcal{M}/G$ because the symmetry has been 'divided out'. For a discrete symmetry, the reduced state space $\mathcal{M}/G$ is given by the fundamental domain of sect. 11.3. In presence of a continuous symmetry, the reduction to $\mathcal{M}/G$ amounts to a change of coordinates where the 'ignorable angles' $\{\phi_1, \cdots, \phi_N\}$ that parameterize N continuous coordinate transformations are separated out.

13.3 Bringing it all back home: method of slices

In the 'method of slices' the reduced state space representative $\hat{x}$ of a group orbit equivalence class is picked by slicing across the group orbits by a fixed hypersurface. We start by describing how the method works for a finite segment of a full

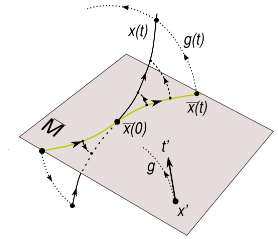


Figure 13.2: Slice $\hat{\mathcal{M}}$ is a hypersurface passing through the slice-fixing point $\hat{x}'$, and transversally to the group tangent t' at $\hat{x}'$. It intersects all group orbits (indicated by dotted lines here) in an open neighborhood of $\hat{x}'$. The full state space trajectory point $x(\tau)$ and the reduced state space trajectory point $\hat{x}(\tau)$ belong to the same group orbit $\mathcal{M}_{x(\tau)}$ and are equivalent up to a group rotation $g(\tau)$, defined in (13.1).

state space trajectory.

Definition: Equivariant state space. The full state space $\mathcal{M}$, stratified by the the action of the group G into orbits, some of which contain more than one point.

Definition: Reduced state space. A space $\mathcal{M}/G$ in which every group orbit of the equivariant state space $\mathcal{M}$ is represented by a single point.

There are many ways of constructing $\mathcal{M}/G$. One can replace equivariant coordinates $(x_1, x_2, \cdots, x_d)$ by a set of invariant polynomials $\{u_1, u_2, \cdots, u_m\}$, as in sect. 13.7. Or one can stay in the original state space, but pick a random point on each group orbit and throw away the rest. The most sensible strategy, however, is to smoothly change the coordinates in such a way that locally the symmetry group acts on N 'phase' coordinates, and leaves the smooth manifold $\hat{\mathcal{M}} = \mathcal{M}/G$ spanned by the remaining $(d-N)$ transverse coordinates invariant.

Definition: Slice. Let G act regularly on a d-dimensional manifold $\mathcal{M}$, i.e., with all group orbits N-dimensional. A *slice* through point $\hat{x}'$ is a $(d-N)$-dimensional submanifold $\hat{\mathcal{M}}$ such that all group orbits in an open neighborhood of the 'template' point $\hat{x}'$ intersect $\hat{\mathcal{M}}$ transversally once and only once (see figure 13.2).

The simplest *slice condition* defines a linear slice as a $(d-N)$-dimensional hyperplane $\hat{\mathcal{M}}$ normal to the N group rotation tangents t'_a at point $\hat{x}'$:

$$\langle \hat{x} - \hat{x}' | t'_a \rangle = 0 \,, \qquad t'_a = t_a(\hat{x}') = \mathbf{T}_a \, \hat{x}' \,. \tag{13.4}$$

In other words, 'slice' is a Poincaré section (3.14) for group orbits. Each 'big circle' –group orbit tangent to t'_a– intersects the hyperplane exactly twice, with the two solutions separated by π. As for a Poincaré section (3.4), we add an orientation condition, and select the intersection with the clockwise rotation angle into the slice.

As $\langle \hat{x}' | t'_a \rangle = 0$ by the antisymmetry of $\mathbf{T}_a$, the slice condition (13.4) fixes ϕ for a given x by

$$0 = \langle \hat{x} | t'_a \rangle = \langle x | g(\phi)^\top t'_a \rangle \,, \qquad (13.5)$$

where $g^\top$ denotes the transpose of g. The method of moving frames can be interpreted as a change of variables

$$\hat{x}(\tau) = g^{-1}(\tau)\, x(\tau)\,, \tag{13.6}$$

that is passing to a frame of reference in which condition (13.5) is identically satisfied, see example 13.1. Therefore the name 'moving frame'. A moving frame should not be confused with the comoving frame, such as the one illustrated in figure 12.7. Each relative equilibrium, relative periodic orbit and general ergodic trajectory has its own comoving frame. In the method of slices one fixes a stationary slice, and rotates all solutions back into the slice.

Moving frames can be utilized in post-processing methods; trajectories are computed in the full state space, then rotated into the slice whenever desired, with the slice condition easily implemented. The slice group tangent t' is a given vector, and $g(\phi)\, x$ is another vector, linear in x and a function of group parameters ϕ. Rotation parameters ϕ are determined numerically, by a Newton method, through the slice condition (13.5).

Figure 13.3 illustrates the method of moving frames for an SO(2) slice normal to the y_1 axis. Looks innocent, but what happens when $(x_1, y_1) = (0, 0)$? More on this in sect. 13.5.

How does one pick a slice point $\hat{x}'$? A generic point $\hat{x}'$ not in an invariant subspace should suffice to fix a slice. The rules of thumb are much like the ones for picking Poincaré sections, sect. 3.1.2. The intuitive idea is perhaps best visualized in the context of fluid flows. Suppose the flow exhibits an unstable coherent structure that –approximately– recurs often at different spatial dispositions. One can fit a 'template' to one recurrence of such structure, and describe other recurrences as its translations. A well chosen slice point belongs to such dynamically important equivalence class (i.e., group orbit). A slice is locally isomorphic to $\mathcal{M}/G$, in an open neighborhood of $\hat{x}'$. As is the case for the dynamical Poincaré sections, in general a single slice does not suffice to reduce $\mathcal{M} \to \mathcal{M}/G$ globally.

The Euclidean product of two vectors x, y is indicated in (13.4) by x-transpose times y, as in (12.6). More general bilinear norms $\langle x, y\rangle$ can be used, as long as they are G-invariant, i.e., constant on each irreducible subspace. An example is the quadratic Casimir (12.13).

example 13.1
p. 229

The slice condition (13.4) fixes N directions; the remaining vectors $(\hat{x}_{N+1} \ldots \hat{x}_d)$ span the slice hyperplane. They are $d - N$ *fundamental invariants*, in the sense that any other invariant can be expressed in terms of them, and they are functionally independent. Thus they serve to distinguish orbits in the neighborhood of the slice-fixing point $\hat{x}'$, i.e., two points lie on the same group orbit if and only if all the fundamental invariants agree.

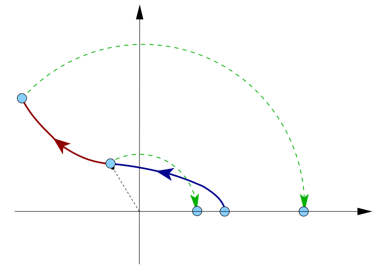


Figure 13.3: Method of moving frames for the two-modes flow SO(2)-equivariant under (12.40), with slice through $\hat{a}' = (1, 0, 0, 0)$, group tangent $t' = (0, 1, 0, 0)$. The equivariant flow is 4-dimensional: shown is the projection on the (x_1, y_1) plane. The clockwise orientation condition restricts the slice to the 3-dimensional half-hyperplane $\hat{x}_1 > 0,\ \hat{y}_1 = 0$. A trajectory started on the slice at $a(\tau_0) = \hat{a}(\tau_0)$, evolves to a state space point with a non-zero $y_1(\tau_n)$. To bring this point back to the slice, compute the polar angle ϕ_n of $a(\tau_n)$ in the (x_1, y_1) plane. Rotate $a(\tau_n)$ clockwise by ϕ_n to $\hat{a}(\tau_n) = g(-\phi_n)\, a(\tau_n)$, so that the equivalent point on the circle lies on the slice, $\hat{y}_1(\tau_n) = 0$. See sect. 13.5.

rgb]0,0,0y_1

rgb]0,0,0ϕ_2

rgb]0,0,0$a(\tau_2)$

rgb]0,0,0$a(\tau_1)$ rgb]0,0,0ϕ_1

rgb]0,0,0$x_1 = \hat{x}_1$

rgb]0,0,0$\hat{a}(\tau_1)$ rgb]0,0,0$a(\tau_0)$ rgb]0,0,0$\hat{a}(\tau_2)$

13.4 Dynamics within a slice

We made too many wrong mistakes
—Yogi Berra

So far we have taken the post-processing approach: evolve the trajectory in the full state space, than rotate all its points into the slice. You can also split up the time integration into a sequence of finite time steps, each followed by a rotation of the end point into the slice, see figure 13.3. It is tempting to see what happens if the steps are taken infinitesimal. As we shall see, we do get a flow restricted to the slice, but at a price. The relation (13.3) between the 'lab' and 'moving frame' state space velocity holds for any factorization (13.1) of the flow of form $x(\tau) = g(\tau)\,\hat{x}(\tau)$. To integrate these equations we first have to fix a particular flow factorization by imposing conditions on $\hat{x}(\tau)$, and then integrate phases $\phi(\tau)$ on a given reduced state space trajectory $\hat{x}(\tau)$.

Here we demand that the reduced state space is confined to a slice hyperplane.

Substituting (13.3) into the time derivative of the fixed slice condition (13.5),

$$\langle \hat{v}(\hat{x})|t'_a\rangle = \langle v(\hat{x})|t'_a\rangle - \dot{\phi}_b \langle t_b(\hat{x})|t'_a\rangle = 0\,,$$

yields the equation for the group phase velocities $\dot{\phi}_a$ for the slice fixed by $\hat{x}'$, together with the reduced state space $\hat{\mathcal{M}}$ flow $\hat{v}(\hat{x})$. In general, the computation of phase velocities requires inversion of the position-dependent $[N\times N]$ matrix $\langle t(\hat{x})_b|t_a\rangle$, so from now on we specialize to the simplest, $N = 1$ parameter case $G = \mathrm{SO}(2)$, where we set $\phi_a = \phi$, $t'_a = t'$:

$$\hat{v}(\hat{x}) = v(\hat{x}) - \dot{\phi}(\hat{x})t(\hat{x}), \qquad \hat{x} \in \hat{\mathcal{M}} \tag{13.7}$$
$$\dot{\phi}(\hat{x}) = \langle v(\hat{x})|t'\rangle / \langle t(\hat{x})|t'\rangle . \tag{13.8}$$

Each group orbit $\mathcal{M}_x = \{g\,x\,|\,g \in G\}$ is an equivalence class; method of slices represents the class by its single slice intersection point $\hat{x}$. By construction $\langle \hat{v}|t'\rangle = 0$, and the motion stays in the $(d-N)$-dimensional slice. We have thus replaced the original dynamical system $\{\mathcal{M}, f\}$ by a reduced system $\{\hat{\mathcal{M}}, \hat{f}\}$.

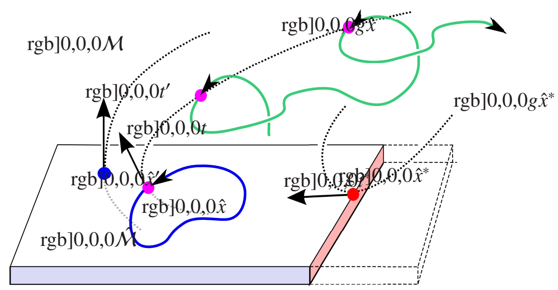


Figure 13.4: Schematic of symmetry reduction by the method of slices. The blue point is the template $\hat{x}'$. All pink points are equivalent to $\hat{x}$ up to a shift, so a relative periodic orbit (green) in the d-dimensional full state space $\mathcal{M}$ closes into a periodic orbit (blue) in the slice $\hat{\mathcal{M}} = \mathcal{M}/G$, a $(d-1)$-dimensional slab transverse to the template group tangent t'. A typical group orbit (dotted) crosses the slice hyperplane transversally, with a non-orthogonal group tangent $t = t(\hat{x})$. A slice hyperplane is almost never a global slice; it is valid up to the slice border, a $(d-2)$-dimensional hypersurface (red) of points $\hat{x}^*$ whose group orbits graze the slice, i.e. points whose tangents $t^* = t(\hat{x}^*)$ lie in $\hat{\mathcal{M}}$. Group orbits beyond the slice border do not reach the slice hyperplane: the "missing chunk" is here indicated by the dashed lines.

In the pattern recognition and 'template fitting' settings (13.8) is called the 'reconstruction equation'. Integrated together, the reduced state space trajectory (13.7) and $g(\tau) = \exp\{\phi(\tau)\cdot\mathbf{T}\}$, the integrated phase (13.8), reconstruct the full state space trajectory $x(\tau) = g(\tau)\,\hat{x}(\tau)$ from the reduced state space trajectory $\hat{x}(\tau)$, so no information about the flow is lost in the process of symmetry reduction. exercise 13.2

Slice flow equations (13.7) and (13.8) are pretty, but there is a trouble in the paradise. The slice flow encounters singularities in subsets of state space, with phase velocity $\dot{\phi}$ divergent whenever the denominator in (13.8) changes sign. We are going to refer to the set of points $\hat{x}^*$ at which the denominator of (13.8) vanishes as *slice border*

$$\langle t(\hat{x}^*)|t'\rangle = 0\,. \tag{13.9}$$

See figure 13.4 for a schematic illustration. Existence of the slice border makes the method of slices an in general local method, where one constructs a slice by picking a template on a particularly interesting solution, and then explores the dynamics nearby. However, this is only partially useful for our purposes, since we would like to explore global objects, such as symmetry-reduced chaotic attractors, interrelation of coherent solutions etc. Several attempts have been made to overcome this problem by defining multiple slices, and interconnecting them in such a way that the individual borders of different slices, are not visited by the dynamics. This, however, is a very complicated task, and requires case-by-case attention. The other option is defining a very special slice such that its border is not visited by the dynamics. In the next section, we describe such a method which is applicable to many problems that are of interest to us.

13.5 First Fourier mode slice

(N.B. Budanur)

So far, we have given a general description of the method of slices, without specifying the type of the symmetries we are reducing. We have mentioned in sect. 2.4 and sect. 12.1 that often times we are interested in dynamics of nonlinear fields in periodic cells. Such systems are generally referred to as 'spatially extended systems' and are equivariant under spatial translations. Let us assume that $u(x,\tau) = u(x+L,\tau)$ is such a scalar field over one space dimension x and time τ, expanded in Fourier series as (2.16)

$$u(x,\tau) = \sum_{k=-\infty}^{+\infty} \tilde{u}_k(\tau)\, e^{iq_k x}\,. \tag{13.10}$$

It is easy to see that spatial translations

$$u(x,\tau) \to u(x+\delta x,\tau) \tag{13.11}$$

becomes U(1) rotations for the Fourier modes

$$\tilde{u}_k \to e^{ik\theta}\tilde{u}_k\,,\ \text{where } \theta = 2\pi\delta x/L\,. \tag{13.12}$$

In the state space $a = (x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ constructed by the real and imaginary parts of a finite truncation of the Fourier modes, $(x_i, y_i) = (\mathrm{Re}\,\tilde{u}_i, \mathrm{Im}\,\tilde{u}_i)$, this symmetry is represented by SO(2) rotations (see example 12.3)

$$g(\theta) \;=\; \begin{pmatrix} R(\theta) & 0 & \cdots & 0 \\ 0 & R(2\theta) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R(N\theta) \end{pmatrix}, \tag{13.13}$$
$$\text{where } R(n\theta) = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix},$$

with the Lie algebra element

$$\mathbf{T} = \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -2 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -N \\ 0 & 0 & 0 & 0 & \cdots & N & 0 \end{pmatrix}. \tag{13.14}$$

The two-modes system is an example of a system with this kind of symmetry with modes truncated at $N = 2$. We define the first Fourier mode slice as the slice hyperplane in this state space with template

$$\hat{a}' = (1, 0, 0, \ldots, 0)\,, \tag{13.15}$$

Figure 13.5: Two-modes flow before (a) and after (b) symmetry reduction by first Fourier mode slice. Here a long trajectory (red and blue) starting on the unstable manifold of the TW_1 (red), until it falls on to the strange attractor (blue) and the shortest relative periodic orbit $\overline{1}$ (magenta). Notice that the relative equilibrium becomes an equilibriumand relative periodic orbit becomes a periodic orbit after the symmetry reduction. (N.B. Budanur)

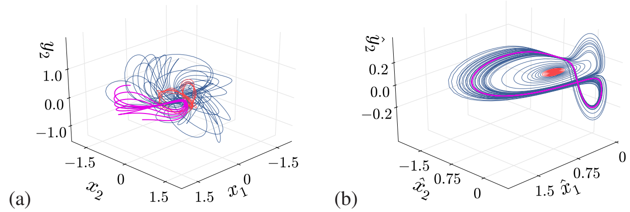
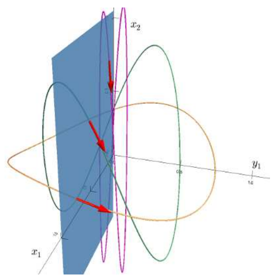


Figure 13.6: SO(2) group orbits of state space points $(0.75, 0, 0.1, 0.1)$ (orange), $(0.5, 0, 0.5, 0.5)$ (green) $(0.1, 0, 0.75, 0.75)$ (pink) and the first mode slice hyperplane (blue). The group tangents at the intersections with the slice hyperplane are shown as red arrows. As the magnitude of the first Fourier mode decreases relative to the magnitude of the second one, so does the group tangent angle to the slice hyperplane. (from ref. [13.1]).



and the directional constraint

$$\hat{x}_1 \geq 0 \tag{13.16}$$

(see figure 13.3). We can write the equation (13.7) and (13.8), which describe the dynamics within the slice hyperplane explicitly for the template (13.15) as

$$\begin{aligned} \hat{v}(\hat{a}) &= v(\hat{a}) - \frac{\dot{y}_1(\hat{a})}{\hat{x}_1} t(\hat{a}) , & (13.17) \\ \dot{\phi}(\hat{a}) &= \frac{\dot{y}_1(\hat{a})}{\hat{x}_1} . & (13.18) \end{aligned}$$

We see from (13.17) and (13.18) that they become singular when $\hat{x}_1 = 0$, i.e. when the amplitude of the first Fourier mode exactly vanishes. In sect. 13.4 we argued that the slice singularity happens when the dot product $t(\hat{a}) \cdot t'$ vanishes, in other words, when the group tangent $t(\hat{a})$ evaluated at the state space point $\hat{a}$ has no component perpendicular to the slice hyperplane. We visualize this in figure 13.6 by showing three dimensional projections of the slice hyperplane, three group orbits and group tangents for the two-modes system.

Our experience from working with spatially extended systems had been that the first Fourier mode amplitude can get very small, but it does not exactly vanish, unless a specific initial condition is set. We can deal with the situations when $\hat{x}_1$ is arbitrarily small by defining the *in-slice time* as

$$d\hat{\tau} = d\tau / \hat{x}_1 \tag{13.19}$$

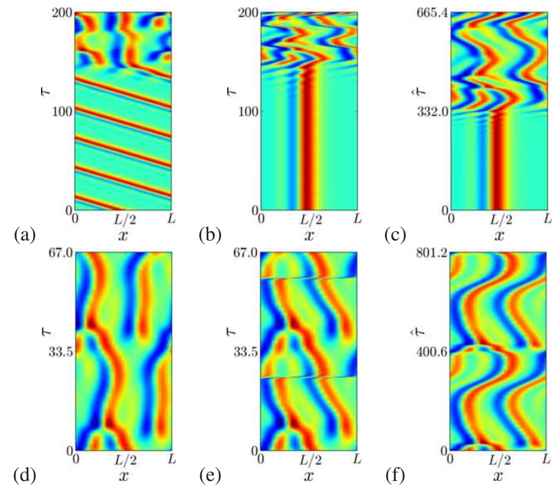


Figure 13.7: Traveling wave TW_1 with phase velocity $c = 0.737$ in configuration space: (a) the full state space solution, (b) symmetry-reduced solution with respect to the lab time, and (c) symmetry-reduced solution with respect to the in-slice time. Relative periodic orbit $T_p = 33.50$ in configuration space: (d) the full state space solution, (e) symmetry-reduced solution with respect to the lab time, and (f) symmetry-reduced solution with respect to the in-slice time. (from ref. [13.2])

and re-writing (13.17) and (13.18) in terms of $d\hat{\tau}$ as

$$\begin{aligned} d\hat{a}/d\hat{\tau} &= \hat{x}_1 v(\hat{a}) - \dot{y}_1(\hat{a})\, t(\hat{a}) , & (13.20) \\ d\theta(\hat{a})/d\hat{\tau} &= \dot{y}_1(\hat{a}) . & (13.21) \end{aligned}$$

One ensures to obtain a smooth flow by integrating (13.20) to obtain symmetry-invariant dynamics. Figure 13.7 illustrates the importance of the time rescaling on the application of first Fourier mode slice to the Kuramoto-Sivashinsky system.

13.6 Stability within a slice

(N.B. Budanur)

As we have managed to formulate a relatively simple symmetry reduction method that is applicable to many problem of interest, we can now take a step forward and ask questions such as ‘Can we compute the linear stability of a relative equilibrium within a slice?’ The answer is yes. We compute the reduced stability matrix by taking partial derivatives of (13.7)

$$
\begin{aligned}
\frac{\partial \hat{v}(\hat{x})_i}{\partial \hat{x}_j} &= \frac{\partial}{\partial \hat{x}_j}\left\{ v(\hat{x})_i - \frac{\langle v(\hat{x})|t'\rangle}{\langle t(\hat{x})|t'\rangle} t(\hat{x})_i \right\} && (13.22)\\
\hat{A}(\hat{x})_{ij} &= A(\hat{x})_{ij} - \frac{t(\hat{x})_i \,\{(\langle t(\hat{x})|t'\rangle A(\hat{x})^T - \langle v(\hat{x})|t'\rangle \mathbf{T}^T)t'\}_j}{\langle t(\hat{x})|t'\rangle^2} \\
&\quad - \frac{\langle v(\hat{x})|t'\rangle}{\langle t(\hat{x})|t'\rangle}\mathbf{T}_{ij}\,, && (13.23)
\end{aligned}
$$

which in matrix notation becomes

$$
\begin{aligned}
\hat{A}(\hat{x}) &= A(\hat{x}) - \frac{|t(\hat{x})\rangle \, \langle\, (\langle t(\hat{x})|t'\rangle A(\hat{x})^T - \langle v(\hat{x})|t'\rangle \mathbf{T}^T\,)t'|}{\langle t(\hat{x})|t'\rangle^2} \\
&\quad - \frac{\langle v(\hat{x})|t'\rangle}{\langle t(\hat{x})|t'\rangle}\mathbf{T}\,. && (13.24)
\end{aligned}
$$

How come we got this lengthy formula (13.24), while the stability of a relative equilibrium looked beautifully simple in (12.28)? Remember that (12.28) was written for the co-rotating frame of a relative equilibrium. In the language of slicing, this corresponds to picking the slice template as the relative equilibrium itself. Plug in $\hat{x} = \hat{x}' = x_{TW}$ in (13.24) and recall that $\langle v(x_{TW})|t(x_{TW})\rangle = c$. Hence the second term vanishes, and we end up with

$$
\hat{A}(x_{TW}) = A(x_{TW}) - c\mathbf{T} \qquad (13.25)
$$

as in (12.28). Equation (13.25) is only true for a relative equilibrium on its own slice. In a general slice, such as the one we described in sect. 13.5, one has to use (13.24) to compute the reduced stability matrix.

13.7 Method of images: Hilbert bases

(E. Siminos and P. Cvitanović)

Erudite reader might wonder: why all this slicing and dicing, when the problem of symmetry reduction had been solved by Hilbert and Weyl a century ago? Indeed, the most common approach to symmetry reduction is by means of a Hilbert invariant polynomial bases (11.5), motivated intuitively by existence of such nonlinear invariants as the rotationally-invariant length $r^2 = x_1^2 + x_2^2 + \cdots + x_d^2$, or, in Hamiltonian dynamics, the energy function. One trades in the equivariant state space coordinates $\{x_1, x_2, \cdots, x_d\}$ for a non-unique set of $m \geq d$ polynomials $\{u_1, u_2, \cdots, u_m\}$ invariant under the action of the symmetry group. These polynomials are linearly independent, but functionally dependent through $m - d + N$ relations called *syzygies*.

The dynamical equations follow from the chain rule

$$
\dot{u}_i = \frac{\partial u_i}{\partial x_j}\,\dot{x}_j\,, \qquad (13.26)
$$

upon substitution $\{x_1, x_2, \cdots, x_d\} \to \{u_1, u_2, \cdots, u_m\}$. One can either rewrite the dynamics in this basis or plot the 'image' of solutions computed in the original, equivariant basis in terms of these invariant polynomials.

Nevertheless we can now easily identify a suitable Poincaré section, guided by the Lorenz flow examples of chapter 11, as one that contains the z-axis and the image of the relative equilibrium TW_1, here defined by the condition $u_1 = u_4$. As in example 14.5, we construct the first return map using as coordinate the Euclidean arclength along the intersection of the unstable manifold of TW_1 with the Poincaré surface of section. Thus the goals set into the introduction to this chapter are attained: we have reduced the messy strange attractor of figure 12.1 to a 1-dimensional return map. As will be explained in example 14.5 for the Lorenz attractor, we now have the symbolic dynamics and can compute as many relative periodic orbits of the complex Lorenz flow as we wish, missing none.

Reducing dimensionality of a dynamical system by explicit elimination of variables through inclusion of syzygies introduces singularities. Such elimination of variables, however, is not needed for visualization purposes; syzygies merely guarantee that the dynamics takes place on a $(d - N)$-dimensional submanifold in the projection on the $\{u_1, u_2, \cdots, u_m\}$ coordinates. However, when one *reconstructs* the dynamics in the original space $\mathcal{M}$ from its image $\mathcal{M}/G$, the transformations have singularities at the fixed-point subspaces of the isotropy subgroups in $\mathcal{M}$.

What limits the utility of Hilbert basis methods are not such singularities, but rather the fact that the algebra needed to determine a Hilbert basis becomes computationally prohibitive as the dimension of the system and/or the symmetry group increases. Moreover, even if such basis were available, rewriting the equations in an invariant polynomial basis seems impractical, so in practice Hilbert basis computations appear not feasible beyond state space dimension of order $\approx$ ten. When the goal is to quotient continuous symmetries of high-dimensional flows, such as the Navier-Stokes flows, one needs a workable framework. The method of slices of sect. 13.2 is one such minimalist alternative.

Résumé

Here we have described how, and offered two approaches to continuous symmetry reduction. In the *method of slices* one fixes a 'slice' $\langle \hat{x} - \hat{x}'|t'\rangle = 0$, a hyperplane normal to the group tangent t' that cuts across group orbits in the neighborhood of the slice-fixing point $\hat{x}'$. Each class of symmetry-equivalent points is represented by a single point, with the symmetry-reduced dynamics in the reduced state space $\mathcal{M}/G$ given by (13.7):

$$
\hat{v} = v - \dot{\phi}\cdot t\,, \qquad \dot{\phi} = \langle v|t'\rangle / \langle t|t'\rangle\,.
$$

In practice one runs the dynamics in the full state space, and post-processes the trajectory by the method of moving frames. In the *Hilbert polynomial basis* approach one transforms the equivariant state space coordinates into invariant ones, by a nonlinear coordinate transformation

$$\{x_1, x_2, \cdots, x_d\} \to \{u_1, u_2, \cdots, u_m\} \quad + \quad \{syzygies\},$$

and studies the invariant 'image' of dynamics (13.26) rewritten in terms of invariant coordinates.

Continuous symmetry reduction is considerably more involved than the discrete symmetry reduction to a fundamental domain of chapter 11. Slices are only local sections of group orbits, and Hilbert polynomials are non-unique and difficult to compute for high-dimensional flows. However, there is no need to actually recast the dynamics in the new coordinates: either approach can be used as a visualization tool, with all computations carried out in the original coordinates, and then, when done, rotating the solutions into the symmetry reduced state space by post-processing the data. The trick is to construct a good set of symmetry invariant Poincaré sections (see sect. 3.1), and that is always a dark art, with or without a symmetry.

Relative equilibria and relative periodic orbits are the hallmark of systems with continuous symmetry. Amusingly, in this extension of 'periodic orbit' theory from unstable 1-dimensional closed periodic orbits to unstable $(N+1)$-dimensional compact manifolds $\mathcal{M}_p$ invariant under continuous symmetries, there are either no or proportionally few periodic orbits. Relative periodic orbits are almost never eventually periodic, i.e., they almost never lie on periodic trajectories in the full state space, so looking for periodic orbits in systems with continuous symmetries is a fool's errand.

However, dynamical systems are often equivariant under a combination of continuous symmetries and discrete coordinate transformations of chapter 10. An example is the orthogonal group O(n). In presence of discrete symmetries relative periodic orbits within discrete symmetry-invariant subspaces are eventually periodic. Atypical as they are (no generic chaotic orbit can ever enter these discrete invariant subspaces) they will be important for periodic orbit theory, as there the shortest orbits dominate, and they tend to be the most symmetric solutions.

The message: If a dynamical systems has a symmetry, use it!

chapter 25

Commentary

Remark 13.1 A brief history of relativity, or, 'Desymmetrization and its discontents' (after Civilization and its discontents; continued from remark 12.1).

Relative equilibria and relative periodic solutions are related by symmetry reduction to equilibria and periodic solutions of the reduced dynamics. They appear in many physical applications, such as celestial mechanics, molecular dynamics, motion of rigid

bodies, nonlinear waves, spiralling patterns, and fluid mechanics. A relative equilibrium is a solution which travels along an orbit of the symmetry group at constant speed; an introduction to them is given, for example, in Marsden [13.3]. According to Cushman, Bates [13.4] and Yoder [13.5], C. Huygens [13.6] understood the relative equilibria of a spherical pendulum many years before publishing them in 1673. A reduction of the translation symmetry was obtained by Jacobi (for a modern, symplectic implementation, see Laskar *et al.* [13.7]). In 1892 German sociologist Vierkandt [13.8] showed that on a symmetry-reduced space (the constrained velocity phase space modulo the action of the group of Euclidean motions of the plane) all orbits of the rolling disk system are periodic [13.9]. According to Chenciner [13.10], the first attempt to find (relative) periodic solutions of the N-body problem was the 1896 short note by Poincaré [13.11], in the context of the 3-body problem. Poincaré named such solutions 'relative'. Relative equilibria of the N-body problem (known in this context as the Lagrange points, stationary in the co-rotating frame) are circular motions in the inertial frame, and relative periodic orbits correspond to quasiperiodic motions in the inertial frame. For relative periodic orbits in celestial mechanics see also ref. [13.12]. A striking application of relative periodic orbits has been the discovery of "choreographies" in the N-body problems [13.13, 13.14, 13.15].

The modern story on equivariance and dynamical systems starts perhaps with S. Smale [13.16] and M. Field [13.17], and on bifurcations in presence of symmetries with Ruelle [13.18]. Ruelle proves that the stability matrix/Jacobian matrix evaluated at an equilibrium/fixed point $x \in \mathcal{M}_G$ decomposes into linear irreps of G, and that stable/unstable manifold continuations of its eigenvectors inherit their symmetry properties, and shows that an equilibrium can bifurcate to a rotationally invariant periodic orbit (i.e., relative equilibrium).

Gilmore and Lettelier monograph [13.19] offers a very clear, detailed and user friendly discussion of symmetry reduction by means of Hilbert polynomial bases (do not look for 'Hilbert' in the index, though). Vladimirov, Toronov and Derbov [13.20] use an invariant polynomial basis to study bounding manifolds of the symmetry reduced complex Lorenz flow and its homoclinic bifurcations. There is no general strategy how to construct a Hilbert basis; clever low-dimensional examples have been constructed case-by-case. The determination of a Hilbert basis appears computationally prohibitive for state space dimensions larger than ten [13.21, 13.22], and rewriting the equations of motions in invariant polynomial bases appears impractical for high-dimensional flows. Thus, by 1920's the problem of rewriting equivariant flows as invariant ones was solved by Hilbert and Weyl, but at the cost of introducing largely arbitrary extra dimensions, with the reduced flows on manifolds of lower dimensions, constrained by sets of syzygies. Cartan found this unsatisfactory, and in 1935 he introduced [13.23] the notion of a *moving frame*, a map from a manifold to a Lie group, which seeks no invariant polynomial basis, but instead rewrites the reduced $\mathcal{M}/G$ flow in terms of $d - N$ *fundamental invariants* defined by a generalization of the Poincaré section, a slice that cuts across all group orbits in some open neighborhood. Fels and Olver view the method as an alternative to the Gröbner bases methods for computing Hilbert polynomials, to compute functionally independent fundamental invariant bases for general group actions (with no explicit connection to dynamics, differential equations or symmetry reduction). 'Fundamental' here means that they can be used to generate all other invariants. Olver's monograph [13.24] is pedagogical, but does not describe the original Cartan's method. Fels and Olver papers [13.25, 13.26] are lengthy and technical. They refer to Cartan's method as method of 'moving frames' and view it as a special and less rigorous case of their 'moving coframe' method. The name 'moving coframes' arises through the use of Maurer-Cartan form which is a coframe on the Lie group G, i.e., they form a pointwise basis for the cotangent space. In refs. [13.27, 13.28] the invariant bases generated by the moving frame method are used as a basis to project a full state space trajectory to the slice (i.e., the $\mathcal{M}/G$ reduced state space).

The basic idea of the 'method of slices' is intuitive and frequently reinvented, often under a different name; for example, it is stated without attribution as the problem 1. of Sect. 6.2 of Arnol'd *Ordinary Differential Equations* [A2.1]. The factorization (13.1) is stated on p. 31 of Anosov and Arnol'd [13.30], who note, without further elaboration, that in the vicinity of a point which is not fixed by the group one can reduce the order of a system of differential equations by the dimension of the group. Ref. [13.31] refers to symmetry reduction as 'lowering the order'. For the definition of 'slice' see, for example, Chossat and Lauterbach [13.22]. Briefly, a submanifold $\mathcal{M}_{\hat{x}'}$ containing $\hat{x}'$ is called a *slice* through $\hat{x}'$ if it is invariant under isotropy $G_{\hat{x}'(\mathcal{M}_{\hat{x}'})} = \mathcal{M}_{\hat{x}'}$. If $\hat{x}'$ is a fixed point of G, than slice is invariant under the whole group. The slice theorem is explained, for example, in Encyclopaedia of Mathematics. Slices tend to be discussed in contexts much more difficult than our application - symplectic groups, sections in absence of global charts, non-compact Lie groups. We follow ref. [13.32] in referring to a local group-orbit section as a 'slice'. Refs. [13.33, 13.34] and others refer to global group-orbit sections as 'cross-sections', a term that we would rather avoid, as it already has a different and well established meaning in physics. Duistermaat and Kolk [13.35] refer to 'slices', but the usage goes back at least to Guillemin and Sternberg [13.34] in 1984, Palais [13.36] in 1961 and Mastow [13.37] in 1957 (who discusses "local cross-sections"). Bredon [13.33] discusses both cross-sections and slices. Guillemin and Sternberg [13.34] define the 'cross-section', but emphasize that finding it is very rare: "existence of a global section is a very stringent condition on a group action. The notion of 'slice' is weaker but has a much broader range of existence."

Several important fluid dynamics flows exhibit continuous symmetries which are either SO(2) or products of SO(2) groups, each of which act on different coordinates of the state space. The Kuramoto-Sivashinsky equations [A1.75, A1.76], plane Couette flow [A7.30, 30.5, 13.42, 13.43], and pipe flow [13.44, 13.45] all have continuous symmetries of this form. In the 1982 paper Rand [13.46] explains how presence of continuous symmetries gives rise to rotating and modulated rotating (quasiperiodic) waves in fluid dynamics. Haller and Mezić [13.47] reduce symmetries of three-dimensional volume-preserving flows and reinvent method of moving frames, under the name 'orbit projection map'. There is extensive literature on reduction of symplectic manifolds with symmetry; Marsden and Weinstein 1974 article [13.48] is an important early reference. Then there are studies of the reduced phase spaces for vortices moving on a sphere such as ref. [13.49], and many, many others.

Reaction-diffusion systems are often equivariant with respect to the action of a finite dimensional (not necessarily compact) Lie group. Spiral wave formation in such nonlinear excitable media was first observed in 1970 by Zaikin and Zhabotinsky [13.50]. Winfree [13.51, 13.52] noted that spiral tips execute meandering motions. Barkley and collaborators [13.53, 13.54] showed that the noncompact Euclidean symmetry of this class of systems precludes nonlinear entrainment of translational and rotational drifts and that the interaction of the Hopf and the Euclidean eigenmodes leads to observed quasiperiodic and meandering behaviors. Fiedler, in the influential 1995 talk at the Newton Institute, and Fiedler, Sandstede, Wulff, Turaev and Scheel [13.55, 13.56, 26.33, 13.58] treat Euclidean symmetry bifurcations in the context of spiral wave formation. The central idea is to utilize the semidirect product structure of the Euclidean group $E(2)$ to transform the flow into a 'skew product' form, with a part orthogonal to the group orbit, and the other part within it, as in (13.7). They refer to a linear slice $\hat{\mathcal{M}}$ near relative equilibrium as a *Palais slice*, with Palais coordinates. As the choice of the slice is arbitrary, these coordinates are not unique. According to these authors, the skew product flow was first written down by Mielke [13.59], in the context of buckling in the elasticity theory. However, this decomposition is no doubt much older. For example, it was used by Krupa [13.60, 13.22] in his local slice study of bifurcations of relative equilibria. Biktashev, Holden, and Nikolaev [13.61] cite Anosov and Arnol'd [13.30] for the 'well-known' factorization (13.1)

and write down the slice flow equations (13.7).

Neither Fiedler *et al.* [13.55] nor Biktashev *et al.* [13.61] implemented their methods numerically. That was done by Rowley and Marsden for the Kuramoto-Sivashinsky [13.32] and the Burgers [13.62] equations, and Beyn and Thümmler [13.63, 13.64] for a number of reaction-diffusion systems, described by parabolic partial differential equations on unbounded domains. We recommend the Barkley paper [13.54] for a clear explanation of how the Euclidean symmetry leads to spirals, and the Beyn and Thümmler paper [13.63] for inspirational concrete examples of how 'freezing'/'slicing' simplifies the dynamics of rotational, traveling and spiraling relative equilibria. Beyn and Thümmler write the solution as a composition of the action of a time dependent group element $g(t)$ with a 'frozen', in-slice solution $\hat{u}(t)$ (13.1). In their nomenclature, making a relative equilibrium stationary by going to a comoving frame is 'freezing' the traveling wave, and the imposition of the phase condition (i.e., slice condition (13.4)) is the 'freezing ansatz'. They find it more convenient to make use of the equivariance by extending the state space rather than reducing it, by adding an additional parameter and a phase condition. The 'freezing ansatz' [13.63] is identical to the Rowley and Marsden [13.62] and our slicing, except that 'freezing' is formulated as an additional constraint, just as when we compute periodic orbits of ODEs we add Poincaré section as an additional constraint, i.e., increase the dimensionality of the problem by 1 for every continuous symmetry (see sect. 7.2). section 7.2

Several symmetry reduction schemes are reviewed in ref. [13.28]. Here we describe the method of slices [13.32, 13.63, 13.65], the only method that we find practical for a symmetry reduction of chaotic solutions of highly nonlinear and possibly also high-dimensional flows. Derivation of sect. 13.4 follows most closely Rowley and Marsden [13.62] who, in the pattern recognition setting refer to the slice point as a 'template', and call (13.8) the 'reconstruction equation' [13.3, 13.66]. They also describe the 'method of connections' (called 'orthogonality of time and group orbit at successive times' in ref. [13.63]), for which the reconstruction equation (13.8) denominator is $\langle t(\hat{x})|t(\hat{x})\rangle$ and thus non-vanishing as long as the action of the group is regular. This avoids the spurious slice singularities, but it is not clear what the 'method of connections' buys us otherwise. It does not reduce the dimensionality of the state space, and it accrues 'geometric phases' which prevent relative periodic orbits from closing into periodic orbits. Geometric phase in laser equations, including complex Lorenz equations, has been studied in ref. [13.67, 13.68, 13.69, 13.70, 13.71]. Another theorist's temptation is to hope that a continuous symmetry would lead us to a conserved quantity. However, Noether theorem [13.76] requires that equations of motion be cast in Lagrangian form and that the Lagrangian exhibits variational symmetries [13.72, 13.73]. Such variational symmetries are hard to find for dissipative systems.

In general relativity 'symmetry reduction' is a method of finding exact solutions by imposing symmetry conditions to obtain a reduced system of equations, i.e., restricting the set of solutions considered to an invariant subspace. This is not what we mean by 'symmetry reduction' in this monograph.

References to 'cyclists' are bit of a joke in more ways than one. First, 'cyclist', 'pedestrian' throughout ChaosBook.org refer jokingly both to the title of Lipkin's *Lie groups for pedestrians* [13.74] and to our preoccupations with actual cycling. Lipkin's 'pedestrian' is fluent in Quantum Field Theory, but wobbly on Dynkin diagrams. More to the point, it is impossible to dispose of Lie groups in a page of text. As a antidote to the brevity of exposition here, consider reading Gilmore's monograph [26.8] which offers a quirky, personal and enjoyable distillation of a lifetime of pondering Lie groups. As seems to be the case with any textbook on Lie groups, it will not help you with the problem at hand, but it is the only place you can learn both what Galois actually did when he invented the theory of finite groups in 1830, and what, inspired by Galois, Lie actually did in his 1874 study of symmetries of ODEs. Gilmore also explains many things that we pass over in silence here, such as matrix groups, group manifolds, and compact groups.

One would think that with all this literature the case is shut and closed, but not so. Applied mathematicians are inordinately fond of bifurcations, and almost all of the published work focuses on equilibria, relative equilibria, and their bifurcations, and for these problems a single slice works well. Only when one tries to describe the totality of chaotic orbits does the non-global nature of slices become a serious nuisance.

(E. Siminos and P. Cvitanović)

Remark 13.2 Velocity *vs.* speed *Velocity* is a vector, the rate at which the object changes its position. *Speed*, or the magnitude of the velocity, is a scalar quantity which describes how fast an object moves. We denote the rate of change of group phases, or the *phase velocity* by the vector $c = (\dot\phi_1, \cdots, \dot\phi_N) = (c_1, \cdots, c_N)$, a component for each of the N continuous symmetry parameters. These are converted to state space velocity components along the group tangents by

$$v(x) = c(\tau) \cdot t(x)\,. \tag{13.27}$$

For rotational waves these are called 'angular velocities'.

Remark 13.3 Killing fields. The symmetry tangent vector fields discussed here are a special case of Killing vector fields of Riemannian geometry and special relativity. If this poetry warms the cockles of your heart, hang on. From wikipedia (this wikipedia might also be useful): A Killing vector field is a set of infinitesimal generators of isometries on a Riemannian manifold that preserve the metric. Flows generated by Killing fields are continuous isometries of the manifold. The flow generates a symmetry, in the sense that moving each point on an object the same distance in the direction of the Killing vector field will not distort distances on the object. A vector field X is a Killing field if the Lie derivative with respect to X of the metric g vanishes:

$$\mathcal{L}_X g = 0\,. \tag{13.28}$$

Killing vector fields can also be defined on any (possibly nonmetric) manifold $\mathcal{M}$ if we take any Lie group G acting on it instead of the group of isometries. In this broader sense, a Killing vector field is the pushforward of a left invariant vector field on G by the group action. The space of the Killing vector fields is isomorphic to the Lie algebra $\mathfrak{g}$ of G.

If the equations of motion can be cast in Lagrangian form, with the Lagrangian exhibiting variational symmetries [13.72, 13.73], Noether theorem associates a conserved quantity with each Killing vector.

13.8 Examples

Example 13.1 ***An $SO(2)$ moving frame:*** *(continued from example 12.1) The $SO(2)$ action*

$$(\hat{x}_1, \hat{y}_1) = (x_1 \cos\theta + y_1 \sin\theta,\ -x_1 \sin\theta + y_1 \cos\theta) \tag{13.29}$$

is regular on $\mathbb{R}^2 \backslash \{0\}$. Thus we can define a slice as a 'hyperplane' (here a mere line), through $\hat{x}' = (0, 1)$, with group tangent $t' = (1, 0)$, and ensure uniqueness by clockwise rotation into positive y_1 axis. Hence the reduced state space is the half-line $x_1 = 0$, $\hat{x}_2 = y_1 > 0$. The slice condition then simplifies to $\hat{x}_1 = 0$ and yields the explicit formula for the moving frame parameter

$$\theta(x_1, y_1) = \tan^{-1}(x_1/y_1)\,, \tag{13.30}$$

i.e., the angle which rotates the point (x_1, y_1) back to the slice, taking care that $\tan^{-1}$ distinguishes (x_1, y_1) plane quadrants correctly. Substituting (13.30) back to (13.29) and using $\cos(\tan^{-1} a) = (1 + a^2)^{-1/2}$, $\sin(\tan^{-1} a) = a(1 + a^2)^{-1/2}$ confirms $\hat{x}_1 = 0$. It also yields an explicit expression for the transformation to variables on the slice,

$$\hat{x}_2 = \sqrt{x_1^2 + y_1^2}\,. \tag{13.31}$$

This was to be expected as $SO(2)$ preserves lengths, $x_1^2 + y_1^2 = \hat{x}_1^2 + \hat{y}_1^2$. If dynamics is in plane and $SO(2)$ equivariant, the solutions can only be circles of radius $(x_1^2 + y_1^2)^{1/2}$, so this is the "rectification" of the harmonic oscillator by a change to polar coordinates, example A2.1. Still, it illustrates the sense in which the method of moving frames yields group invariants. *(E. Siminos)*

click to return: p. ??

Exercises

13.1. **SO(2) or harmonic oscillator slice:** Construct a moving frame slice for action of SO(2) on $\mathbb{R}^2$

$$(x, y) \mapsto (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$$

by, for instance, the positive y axis: $x = 0$, $y > 0$. Write out explicitly the group transformation that brings any point back to the slice. What invariant is preserved by this construction? (E. Siminos)

13.2. **The moving frame flow stays in the reduced state space:** Show that the flow (13.7) stays in a $(d-1)$-dimensional slice hyperplane.

13.3. **Stability of a relative equilibrium in the reduced state space:** Find an expression for the stability matrix of the system at a relative equilibrium when a linear slice is used to reduce the symmetry of the flow.

13.4. **Stability of a relative periodic orbit in the reduced state space:** Find an expression for the Jacobian matrix (monodromy matrix) of a relative periodic orbit when a linear slice is used to reduce the dynamics of the flow.

13.5. **Determination of invariants by the method of slices:** Show that the $d - N$ reduced state space coordinates determined by the method of slices are independent and invariant under group actions, and that the method of slices allows the determination of (in general non-polynomial) symmetry invariants by a simple algorithm that works well in high-dimensional state spaces.

13.6. **Invariant subspace of the two-modes system:** Show that $(0, 0, x_2, y_2)$ is a flow invariant subspace of the two-modes system (12.38), i.e., show that a trajectory with the initial point within this subspace remains within it

forever.
(N.B. Budanur)

13.7. **Slicing the two-modes system:** Choose the simplest slice template point that fixes the 1. Fourier mode,

$$\hat{x}' = (1, 0, 0, 0)\,. \tag{13.32}$$

(a) Show for the two-modes system (12.38), that the velocity within the slice (13.7), and the phase velocity (13.8) along the group orbit are

$$\begin{aligned}\hat{v}(\hat{x}) &= v(\hat{x}) - \dot{\phi}(\hat{x})t(\hat{x}) && (13.33)\\ \dot{\phi}(\hat{x}) &= -v_2(\hat{x})/\hat{x}_1 && (13.34)\end{aligned}$$

(b) Determine the chart border (the locus of point where the group tangent is either not transverse to the slice or vanishes).

(c) What is its dimension?

(d) What is its relation to the invariant subspace of exercise 13.7?

(e) Can a symmetry-reduced trajectory cross the chart border?

(N.B. Budanur and P. Cvitanović)

13.8. **The symmetry reduced two-modes flow:** Pick an initial point $\hat{x}(0)$ that satisfies the slice condition (13.4) for the template choice (13.32) and integrate (13.33) & (13.34). Plot the three dimensional slice hyperplane spanned by (x_1, x_2, y_2) to visualize the symmetry reduced dynamics. Does it look like figure 13.5 (b)?
(N.B. Budanur)

13.9. **Visualize the relative equilibrium of the two-modes system:** Starting the initial condition

$$x_0 = (0.439966, 0, -0.386267, 0.070204) \tag{13.35}$$

integrate the full state space SO(2)-equivariant (12.38) and the symmetry reduced (13.33) two-modes system for $t = 250$ time units. Plot the (x_1, x_2, y_1) projection of both trajectories. Explain your results.
(N.B. Budanur)

13.10. **Relative equilibria of the two-modes system:** Write down an expression for the reduced velocity (13.33) of the two-modes system explicitly by substituting (13.34) and solve $\hat{v} = 0$ to find the relative equilibria. Part of this might be doable analytically (you have an invariant subspace). If that does not work out for you, solve the system numerically, for the parameter values (12.37). Check that x_0 of exercise 13.35 is among your solutions. Mark the relative equilibria that you have found on the strange attractor plot of exercise 13.8, interpret the role they play in the dynamics, if any. (N.B. Budanur)

13.11. **Stability of the two-modes relative equilibrium:**

(a) Write down the stability matrix of the two-modes system in the reduced state space by computing derivatives of (13.33).

(b) Compute eigenvalues and eigenvectors of this stability matrix at the relative equilibrium (13.35)

(c) Indicate the direction along which the nearby trajectories expand.

(d) Compute the stability eigenvalues and eigenvectors of all relative equilibria of exercise 13.10

(N.B. Budanur)

13.12. **Relative periodic orbits of the two-modes system:** Initial conditions and periods for 4 relative periodic orbitof the two-modes system are listed in the table 12.1. Integrate (12.38) and (13.33) with these initial conditions for 3-4 periods, and plot the four trajectories. Explain what you see.
(N.B. Budanur)

13.13. **Poincaré section in the slice** Construct a Poincaré section for the two-modes system in the slice hyperplane, such that the relative equilibrium (13.35) and its expanding direction that you found in (13.11) is in this Poincaré section. Interpolate this Poincaré section with a smooth curve, and compute the arclengths positions of each crossing of the symmetry-reduced flow with the Poincaré section. (N.B. Budanur)

13.14. **Finding relative periodic orbits from a Poincaré return map.** Produce a return map of the arclengths that you found in exercise 13.13. Plot this return map. Note that its derivative is discontinuous at its critical point - why? Interpolate to this return map in two pieces and find its fixed point. Take the fixed point as the initial point to integrate the reduced two-modes system (13.33) for $t = 3.7$. What do you see? (N.B. Budanur)

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