Chapter 33

Noise

> He who establishes his argument by noise and command
> shows that his reason is weak.
> —M. de Montaigne

THIS CHAPTER (which reader can safely skip on the first reading) is about noise, how it affects classical dynamics, and the ways it mimics quantum dynamics.

Why - in a monograph on deterministic and quantum chaos - start discussing noise? First, in physical settings any dynamics takes place against a noisy background, and whatever prediction we might have, we have to check its robustness to noise. Second, as we show in this chapter, to the leading order in noise strength, the semiclassical Hamilton-Jacobi formalism applies to weakly stochastic flows in toto. As classical noisy dynamics is more intuitive than quantum dynamics, understanding effects of noise helps demystify some of the formal machinery of semiclassical quantization. Surprisingly, symplectic structure emerges here not as a deep principle of mechanics, but an artifact of the leading approximation to quantum/noisy dynamics. Third, the variational principle derived here turns out to be a powerful tool for determining periodic orbits, see chapter 34. And, last but not least, upon some reflection, the whole enterprize of replacing deterministic trajectories by deterministic evolution operators, chapters 19 to 23, seems fatally flawed; if we have given up infinite precision in specifying initial conditions, why do we alow ourselves the infinite precision in the specification of evolution laws, i.e., define the evolution operator by means of the Dirac delta function $\delta(y - f^t(x))$? It will be comforting to learn that the deterministic evolution operators survive unscathed, as the leading approximation to the noisy ones in the limit of weak noise.

Another key result derived here is the evolution law (33.45) for the covariance matrix Q_a of a linearly evolved Gaussian density,

$$Q_{a+1} = M_a Q_a M_a^\top + \Delta_a .$$

To keep things simple we shall describe covariance evolution in the discrete time dynamics context, but the results apply both to the continuous and discrete time

flows. The most important lesson, however, is that physicist's Brownian diffusion intuition -that the effect of the noise is to spread out the trajectory as $\sqrt{t}$- is *wrong*: In nonlinear dynamics the noise is always *local*, determined by balancing local nonlinear dynamics against the memory of the noise past.

section 33.5

We start by deriving the continuity equation for purely deterministic, noiseless flow, and then incorporate noise in stages: diffusion equation, Langevin equation, Fokker-Planck equation, stochastic path integrals, Hamilton-Jacobi formulation.

33.1 Deterministic transport

(E.A. Spiegel and P. Cvitanović)

The large body of accrued wisdom on the subject of flows called fluid dynamics is about physical flows of media with continuous densities. On the other hand, the flows in state spaces of dynamical systems frequently require more abstract tools. To sharpen our intuition about those, it is helpful to outline the more tangible fluid dynamical vision.

Consider first the simplest property of a fluid flow called *material invariant*. A material invariant $I(x)$ is a property attached to each point x that is preserved by the flow, $I(x) = I(f^t(x))$; for example, at point $x(t) = f^t(x))$ a green particle (more formally: a *passive scalar*) is embedded into the fluid. As $I(x)$ is invariant, its total time derivative vanishes, $\dot{I}(x) = 0$. Written in terms of partial derivatives this is the *conservation equation* for the material invariant

$$\partial_t I + v \cdot \partial I = 0 . \tag{33.1}$$

Let the *density* of representative points be $\rho(x, t)$. The manner in which the flow redistributes $I(x)$ is governed by a partial differential equation whose form is relatively simple because the representative points are neither created nor destroyed. This conservation property is expressed in the integral statement

$$\partial_t \int_V dx\, \rho I = - \int_{\partial V} d\sigma\, \hat{n}_i v_i \rho I ,$$

where V is an arbitrary volume in the state space $\mathcal{M}$, ∂V is its surface, $\hat{n}$ is its outward normal, and repeated indices are summed over throughout. The divergence theorem turns the surface integral into a volume integral,

$$\int_V [\partial_t(\rho I) + \partial_i(v_i \rho I)]\, dx = 0 ,$$

where ∂_i is the partial derivative operator with respect to x_i. Since the integration is over an arbitrary volume, we conclude that

$$\partial_t(\rho I) + \partial_i(\rho I v_i) = 0 . \tag{33.2}$$

The choice $I \equiv 1$ yields the *continuity equation* for the density:

$$\partial_t \rho + \partial_i(\rho v_i) = 0 . \tag{33.3}$$

Here we have used the language of fluid mechanics to ease the visualization, but, as we have seen in our previous derivation of the continuity equation (19.22), any deterministic state space flow satisfies the continuity equation in any dimension.

Why -even though the dynamics is nonlinear- is this equation linear? As each deterministic orbit is distinct and intersects no other orbit, no 'particles' are created or destroyed, they are non-interacting, hence description in terms of linear evolution operators possible.

33.2 Brownian diffusion

Consider tracer molecules, let us say big, laggardly green molecules, embedded in a denser gas of light molecules. Assume that the density of tracer molecules ρ compared to the background gas density is low, so we can neglect green-green collisions. Each green molecule, jostled by frequent collisions with the background gas, executes its own Brownian motion. The molecules are neither created nor destroyed, so their number within an arbitrary volume V changes with time only by the current density j_i flow through its surface ∂V (with $\hat{n}$ its outward normal):

$$\partial_t \int_V dx\, \rho = - \int_{\partial V} d\sigma\, \hat{n}_i j_i \,. \tag{33.4}$$

The divergence theorem turns this into the conservation law for tracer density:

$$\partial_t \rho + \partial_i j_i = 0 \,. \tag{33.5}$$

The tracer density ρ is defined as the average density of a 'material particle,' averaged over a subvolume large enough to contain many green (and still many more background) molecules, but small compared to the macroscopic observational scales. What is j? If the density is constant, on the average as many molecules leave the material particle volume as they enter it, so a reasonable phenomenological assumption is that the *average* current density (*not* the individual particle current density ρv_i in (33.3)) is driven by the density gradient

$$j_i = -D \frac{\partial \rho}{\partial x_i} \,. \tag{33.6}$$

This is the *Fick law*, with the diffusion constant D a phenomenological parameter. Substituting this current into (33.5) yields the *diffusion* or *heat* equation,

$$\frac{\partial}{\partial t} \rho(x,t) = D \frac{\partial^2}{\partial x^2} \rho(x,t) \,. \tag{33.7}$$

For sake of streamlining the argument we have assumed in (33.6) that diffusion in d dimensions is time-invariant, homogenous and isotropic, $\Delta(x,t) = 2D\mathbf{1}$. More generally, diffusion is described by a state space- (Toto, remember? we're not in 3 spatial dimensions anymore) and time-dependent symmetric *diffusion tensor* $\Delta_{ij} = \Delta_{ji}$, with $j_i = -\frac{1}{2}\Delta_{ij}\partial_j \rho$, leading to the anisotropic diffusion equation

$$\partial_t \rho(x,t) = \frac{1}{2}\partial_i \left(\Delta_{ij}(x,t)\, \partial_j \rho(x,t) \right) \,. \tag{33.8}$$

In practice, the diffusion tensor is almost always anisotropic. For example, physicist's Brownian diffusion is a flow in the 6-dimensional {configuration, velocity} phase space, with white noise probability distribution $\exp(-\mathbf{v}^2/2k_BT)$, modeling random force kicks applied only to the 3 velocity variables $\mathbf{v}$. In this case one thinks of diffusion coefficient $D = k_BT/2$ as temperature.

33.2.1 Heat kernel

The heat equation we can solve analytically. Fourier transforming (33.7),

$$\frac{\partial}{\partial t}\tilde{\rho}(k,t) = -Dk^2\tilde{\rho}(k,t)\,, \qquad \rho(x,t) = \int \frac{dk}{2\pi}\tilde{\rho}(k,t)\,e^{ikx}\,, \tag{33.9}$$

substituting $\dot{\tilde{\rho}}$, integrating over time,

$$\rho(x,t) = \int \frac{dk}{2\pi}\,\tilde{\rho}(k,0)\,e^{ikx-Dk^2t}\,,$$

and Fourier transforming back we obtain an exact solution of the heat equation in terms of an initial Dirac delta density distribution, $\rho(x,0) = \delta(x - x_0)$,

$$\rho(x,t) = \frac{1}{(4\pi Dt)^{d/2}}e^{-\frac{(x-x_0)^2}{4Dt}}\,. \tag{33.10}$$

A field theorist will see this as reminiscent of the quantum free particle propagation (see sect. 38.2.2). The solution of the general, anisotropic case (33.8) (here exact for all times) is

$$\mathcal{L}_{FP}(x,t;x_0,0) = \frac{1}{\sqrt{\det(2\pi t\,\Delta)}}\exp\left[-\frac{1}{2\,t}(x^\top\frac{1}{\Delta}x)\right]\,. \tag{33.11}$$

The average distance covered in time t obeys the diffusion formula

$$\langle(x-x_0)^2\rangle_t = \int dx\,\rho(x,t)\,(x-x_0)^2 = 2dDt\,. \tag{33.12}$$

The Einstein Brownian diffusion formula describes the 3-dimensional Brownian motion of heavy molecules jostled by thermal motions of the light molecules of the fluid. Here the diffusion is something much more general: it takes place in the dynamical state space of dimension d, and is described by diffusion tensor $D \to \Delta_{ij}(x,t)$.

33.2.2 Random walks

More insight into the evolution of the density of tracer molecules is obtained considering a d-dimensional random walk of an individual tracer molecule kicked by a stochastic term,

$$\frac{dx}{dt} = \hat{\xi}(t)\,. \tag{33.13}$$

A way to make sense of $\hat{\xi}(t)$ is to first construct the probability distribution for additive noise ξ at short but finite time steps $\delta\tau$, with $t_{n+1} = t_n + \delta\tau$, and the particle $x_n = x(t_n)$ at time t_n executing a random walk, $x_{n+1} = x_n + \xi(t_n)$, where x is a d-dimensional state vector, and $x_{n,j}$ is its jth component at time n. The natural choice is that probability that the trajectory reaches x_{n+1} is given by a normalized Gaussian

$$\mathcal{L}_{FP}(x_{n+1}, t_{n+1}; x_n, t_n) = \frac{1}{\sqrt{\det(2\pi\delta\tau\,\Delta)}} \exp\left[-\frac{1}{2\,\delta\tau}(\xi_n{}^\top \frac{1}{\Delta}\xi_n)\right], \qquad (33.14)$$

$\xi_n = x_{n+1} - x_n$, characterized by zero mean and the diffusion tensor (covariance matrix),

$$\langle \xi_j(t_n)\rangle = 0, \qquad \langle \xi_i(t_m)\,\xi_j{}^\top(t_n)\rangle = \delta\tau\Delta_{ij}\,\delta_{nm}, \qquad (33.15)$$

where $\langle\cdots\rangle$ stands for ensemble average over many realizations of the noise, and the superfix $^\top$ indicates a transpose. As the time discretization $\delta\tau$ is arbitrary, the diffusing cloud of noisy trajectories should be described by a distribution that keeps its form as $\delta\tau \to 0$. Indeed, the semigroup property of a Gaussian kernel,

$$\mathcal{L}_{FP}(x, t; x'', t'') = \int dx'\,\mathcal{L}_{FP}(x, t; x', t')\,\mathcal{L}_{FP}(x', t'; x'', t''), \qquad (33.16)$$

ensures that the distribution keeps its form under successive diffusive steps. $\mathcal{L}_{FP}(x, t; x_0, 0)$ describes the diffusion at any time, including the integer time increments $\{t_n\} = \{\delta\tau, 2\delta\tau, \cdots, n\delta\tau, \cdots\}$, and thus provides a bridge between the continuous and discrete time formulations of noisy evolution.
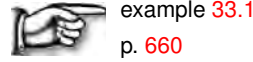
example 33.1
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33.3 Noisy trajectories

> The connection between path integration and Brownian motion is so close that they are nearly indistinguishable. Unfortunately though, like a body and its mirror image, the sum over paths for Brownian motion is a theory having substance, while its path integral image exists mainly in the eye of the beholder.
> —L. S. Schulman

(P. Cvitanović and D. Lippolis)

So far we have considered tracer molecule dynamics which is purely Brownian, with no deterministic 'drift'. Consider next a d-dimensional deterministic flow $\dot{x} = v(x)$ perturbed by a stochastic term $\hat{\xi}$,

$$\frac{dx}{dt} = v(x) + \hat{\xi}(t), \qquad (33.17)$$

where the deterministic velocity field $v(x)$ is called 'drift' in the stochastic literature, and $\hat{\xi}(t)$ is additive noise, uncorrelated in time. We shall refer to equations

of this type as *Langevin equations*. The more general case of a tensor $\Delta(x)$ which is a state space position dependent but time independent can be treated along the same lines. In this case the stochastic flow (33.17) is written as

$$dx = v(x)\,dt + \sigma(x)\,d\hat{\xi}(t)\,, \qquad \langle \xi_n\,\xi_m^\top \rangle = \mathbf{1}\,\delta_{nm}\,, \quad \Delta = \sigma\,\sigma^\top\,. \qquad (33.18)$$

$\sigma(x)$ is called the 'diffusion matrix', and the noise is referred to as 'multiplicative'. Explicit time dependence in $\Delta(x, t)$ would take us into world of non-autonomous, externally driven flows, beyond the comfort zone of ChaosBook.org.

As in (33.13), a way to make sense of (33.17) is to first construct the probability distribution for additive noise ξ at a short but finite time $\delta\tau$. In time $\delta\tau$ the deterministic trajectory advances by $v(x_n)\,\delta\tau$. As $\delta\tau$ is arbitrary, it is desirable that the diffusing cloud of noisy trajectories is given by a distribution that keeps its form as $\delta\tau \to 0$. This holds if the noise is Brownian, i.e., the probability that the trajectory reaches x_{n+1} is given by a normalized Gaussian (33.14),

$$\mathcal{L}_{FP}(x_{n+1}, \delta\tau; x_n, 0) = \frac{1}{N}\exp\left[-\frac{1}{2\,\delta\tau}(\xi_n^\top \frac{1}{\Delta_n}\xi_n)\right]\,. \qquad (33.19)$$

Here $\xi_n = \delta x_n - v(x_n)\,\delta\tau$, the deviation of the noisy trajectory from the deterministic one, can be viewed either in terms of velocities $\{\dot{x}, v(x)\}$ (continuous time formulation), or finite time maps $\{x_n \to x_{n+1}, x_n \to f^{\delta\tau}(x_n)\}$ (discrete time formulation),

$$\delta x_n = x_{n+1} - x_n \simeq \dot{x}_n\,\delta\tau\,, \qquad f^{\delta\tau}(x_n) - x_n \simeq v(x_n)\,\delta\tau\,, \qquad (33.20)$$

where

$$\{x_0, x_1, \cdots, x_n, \cdots, x_k\} = \{x(0), x(\delta\tau), \cdots, x(n\delta\tau), \cdots, x(t)\} \qquad (33.21)$$

is a sequence of $k + 1$ points $x_n = x(t_n)$ along the noisy trajectory, separated by time increments $\delta\tau = t/k$.

The phenomenological Fick law current (33.6) is now a sum of two components, the material particle deterministic drift $v(x)$ and the weak noise term

$$j_i = v_i\rho - D\frac{\partial\rho}{\partial x_i}\,, \qquad \left[= v_i\rho - \frac{1}{2}\Delta_{ij}(x)\,\partial_j\rho(x, t)\right]\,, \qquad (33.22)$$

with the full, anisotropic and space-dependent version indicated in $[\cdots]$, and sum over repeated index j implied. Substituting this j into (33.5) yields

$$\partial_t\rho + \partial_i(\rho v_i) = D\,\partial^2\rho\,. \qquad (33.23)$$

In the general, anisotropic case (33.8) this equation is known as the *Fokker-Planck* or *forward Kolmogorov* equation

$$\partial_t\rho + \partial_i(\rho v_i) = \frac{1}{2}\partial_i\left(\Delta_{ij}(x)\,\partial_j\rho(x, t)\right)\,. \qquad (33.24)$$

The left hand side, $d\rho/dt = \partial_t\rho + \partial\cdot(\rho v)$, is deterministic, with the continuity equation (33.3) recovered in the weak noise limit $D \to 0$. The right hand side

describes the diffusive transport in or out of the material particle volume. If the density is lower than in the immediate neighborhood, the local curvature is positive, $\partial^2 \rho > 0$, and the density grows. Conversely, for negative curvature diffusion lowers the local density, thus smoothing the variability of ρ. Where is the density going globally?

If the system is bound, the probability density vanishes sufficiently fast outside the central region, $\rho(x, t) \to 0$ as $|x| \to \infty$, and the total probability is conserved

$$\int dx \rho(x, t) = 1 \,.$$

Any initial density $\rho(x, 0)$ is smoothed by diffusion and with time tends to the natural measure, the invariant density

$$\rho_0(x) = \lim_{t \to \infty} \rho(x, t) \,, \tag{33.25}$$

an eigenfunction $\rho(x, t) = e^{st} \rho_0(x)$ of the time-independent Fokker-Planck equation

$$\left(\partial_i v_i - D \partial^2 + s_\alpha \right) \rho_\alpha = 0 \,, \tag{33.26}$$

with vanishing eigenvalue $s_0 = 0$. Provided the noiseless classical flow is hyperbolic, in the vanishing noise limit the leading eigenfunction of the Fokker-Planck equation tends to natural measure (19.14) of the corresponding deterministic flow, the leading eigenvector of the Perron-Frobenius operator.

If the system is open, there is a continuous outflow of probability from the region under study, the leading eigenvalue is contracting, $s_0 < 0$, and the density of the system tends to zero. In this case the leading eigenvalue s_0 of the time-independent Fokker-Planck equation (33.26) can be interpreted by saying that a finite density can be maintained by pumping back probability into the system at a constant rate $\gamma = -s_0$. The value of γ for which any initial probability density converges to a finite stationary equilibrium density is called the *escape rate*. In the noiseless limit this coincides with the deterministic escape rate (20.35).

The distribution (33.14) describes how an initial density of particles concentrated in a Dirac delta function at x_n spreads in time $\delta\tau$. In the Fokker-Planck description individual noisy Langevin trajectories (33.17) are replaced by the evolution of the density of noisy trajectories. The finite time Fokker-Planck evolution $\rho(x, t) = \left[\mathcal{L}_{FP}^t \circ \rho \right] (x, 0)$ of an initial density $\rho(x_0, 0)$ is obtained by a sequence of consecutive short-time steps (33.14)

$$\mathcal{L}_{FP}(x_k, t; x_0, 0) = \int [dx] \exp \left\{ -\frac{1}{4D\delta\tau} \sum_{n=0}^{k-1} [x_{n+1} - f^{\delta\tau}(x_n)]^2 \right\} \,, \tag{33.27}$$

where $t = k\,\delta\tau$, and the Gaussian normalization factor in (33.14) is absorbed into intermediate integrations by defining

$$
\begin{aligned}
[dx] \quad &= \quad \prod_{n=0}^{k-1} \frac{dx_n^d}{N_n} \\
N_n \quad &= \quad [\det (2\pi\delta\tau\Delta(x_n))]^{1/2} \qquad \text{(anisotropic diffusion tensor } \Delta) \\
&= \quad (4D\delta\tau)^{d/2} \qquad\qquad \text{(isotropic diffusion, } \Delta(x) = 2\,D\,\mathbf{1}) \,. \tag{33.28}
\end{aligned}
$$

As $D \to 0$, the distribution tends to the noiseless, deterministic Dirac delta function Perron-Frobenius operator (19.10). The stochastic flow (33.17) can now be understood as the continuous time, $\delta\tau \to 0$ limit, with the velocity noise $\hat{\xi}(t)$ a Gaussian random variable of zero mean and covariance matrix

$$\langle \hat{\xi}_j(t) \rangle = 0, \qquad \langle \hat{\xi}_i(t) \hat{\xi}_j(t') \rangle = \Delta_{ij} \delta(t - t'). \tag{33.29}$$

It is worth noting that the continuous time flow noise $\hat{\xi}(t)$ in (33.17) and (33.29) is dimensionally a velocity $[x]/[t]$, as $\mathcal{L}_{FP}(x_{n+1}, \delta\tau; x_n, 0)$ is a probability density for velocity ξ, while the discrete time noise ξ_n in (33.14), (33.15) is dimensionally a length $[x]$, as $\rho(x, t)$ is a probability density for position x. The important point is that the same diffusion tensor $\Delta(x)$ describes the diffusion both in the configuration space and the velocity space.

The continuous time limit of (33.27), $\delta\tau = t/k \to 0$, defines formally the Fokker-Planck evolution operator

$$\mathcal{L}_{FP}(x, t; x_0, 0) = \int [dx] \exp\left\{-\frac{1}{4D} \int_0^t [\dot{x}(\tau) - v(x(\tau))]^2 d\tau\right\} \tag{33.30}$$

as a stochastic path (or Wiener) integral for a noisy flow, and the associated continuous time Fokker-Planck (or forward Kolmogorov) equation (33.24) describes the time evolution of a density of noisy trajectories. We have introduced noise phenomenologically, and used the weak noise assumption in retaining only the first derivative of ρ in formulating the Fick law (33.6) and including noise additively in (33.22). The $\delta\tau \to 0$ limit and the proper definition of $\dot{x}(\tau)$ are delicate issues of no import for the applications studied here. A full theory of stochastic ODEs is much subtler, but this will do for our purposes.

The exponent

$$-\frac{1}{4D\,\delta\tau}\left[x_{n+1} - f^{\delta\tau}(x_n)\right]^2 \simeq -\frac{\delta\tau}{4D}\left[\dot{x}(\tau) - v(x(\tau))\right]^2 \tag{33.31}$$

can be interpreted as a cost function which penalizes deviation of the noisy trajectory δx from its deterministic prediction $v\,\delta\tau$, or, in the continuous time limit, the deviation of the noisy trajectory tangent $\dot{x}$ from the deterministic velocity field v. Its minimization is one of the most important tools of the optimal control theory, with velocity $\dot{x}(\tau)$ along a trial path varied with aim of minimizing its distance to the target $v(x(\tau))$.

33.4 Noisy maps

(P. Cvitanović and D. Lippolis)

For pedagogical reasons we shall often find it convenient to consider a noisy *discrete time* dynamical system

$$x_{n+1} = f(x_n) + \xi_n, \tag{33.32}$$

where x is a d-dimensional state vector, and $x_{n,j}$ is its jth component at time n. In the Fokker-Planck description individual noisy trajectories are replaced by the evolution of the density of noisy trajectories, with the $\xi_n = x_{n+1} - f(x_n)$ probability distribution of zero mean and diffusion tensor, and the time increment in (33.15) set to $\delta\tau = 1$,

$$\langle \xi_{n,j} \rangle = 0, \qquad \langle \xi_{n,i}\,\xi_{m,j}^\top \rangle = \Delta_{ij}(x_n)\,\delta_{nm}. \tag{33.33}$$

As we shall show, in nonlinear dynamics the noise is *never* isotropic and/or homogeneous. Even if the infinitesimal time step noise (33.14) covariance matrix in (33.18) were independent of the state space position x, this cannot be true of $\Delta(x)$ for the discrete time flow (33.32) obtained by the Poincaré section reduction method of sect. 3.1, as the return times (3.1) and the noise accumulated along the corresponding trajectory segments depend on the starting Poincaré section point. Indeed, as we shall argue in sect. 33.5, in nonlinear dynamics *all noise is local*. As long as the noise distribution at x is autonomous (not explicitly dependent on time) the stochastic flow (33.32) can be written as $x_{n+1} = x_n + \sigma(x_n)\,\xi_n$, where $\Delta = \sigma\sigma^\top$, and $\sigma(x)$ is the multiplicative noise diffusion matrix defined in (33.18).

The action of discrete one-time step *Fokker-Planck evolution operator* on the density distribution ρ at time k,

$$\rho_{k+1}(y) = [\mathcal{L}_{FP}\rho_k](y) = \int dx\,\mathcal{L}_{FP}(y,x)\,\rho_k(x)$$

$$\mathcal{L}_{FP}(y,x) = \frac{1}{N(x)}\,e^{-\frac{1}{2}(y-f(x))^\top \frac{1}{\Delta(x)}(y-f(x))}, \tag{33.34}$$

is centered on the deterministic step $f(x)$ and smeared out diffusively by noise. Were diffusion uniform and isotropic, $\Delta(x) = 2D\mathbf{1}$, the Fokker-Planck evolution operator would be proportional to $\exp\left(-\{y - f(x)\}^2/2\Delta\right)$, i.e., the penalty for straying from the deterministic path is just a quadratic error function. The kth iterate of $\mathcal{L}_{FP}^k(x_k; x_0) = \mathcal{L}_{FP}(x_k, t; x_0, 0)$ is a d-dimensional path integral over the $k-1$ intermediate noisy trajectory points,

$$\mathcal{L}_{FP}^k(x_k; x_0) = \int [dx]\,e^{-\frac{1}{2}\sum_n (x_{n+1}-f(x_n)^\top)\frac{1}{\Delta(x_n)}(x_{n+1}-f(x_n))}, \tag{33.35}$$

where the Gaussian normalization factor in (33.34) is absorbed into intermediate integrations by defining

$$[dx] = \prod_{n=1}^{k-1} \frac{dx_n^d}{N_n}, \qquad N_n = \sqrt{(2\pi)^d \det\Delta(x_n)}. \tag{33.36}$$

We shall also need to determine the effect of noise accumulated along the trajectory points *preceding* x. As the noise is additive forward in time, one cannot simply invert the Fokker-Planck evolution operator; instead, the past is described by the *adjoint Fokker-Planck evolution operator*,

$$\tilde{\rho}_{k-1}(x) = [\mathcal{L}_{FP}^\dagger \tilde{\rho}_k](x) = \int [dy]\,e^{-\frac{1}{2}(y-f(x))^\top \frac{1}{\Delta}(y-f(x))}\,\tilde{\rho}_k(y), \tag{33.37}$$

which transports a density concentrated around the point $f(x)$ to a density concentrated around the previous point x and adds noise to it. In the deterministic, vanishing noise limit this is the Koopman operator (31.1).

The Fokker-Planck evolution operator (33.34) is non-hermitian and non-unitary. For example, if the deterministic flow is contracting, the natural measure (the leading right eigenvector of the evolution operator) will be concentrated and peaked, but then the corresponding left eigenvector has to be broad and flat, as backward in time the deterministic flow is expanding. We shall denote by ρ_α the right eigenvectors of $\mathcal{L}_{FP}$, and by $\tilde{\rho}_\alpha$ its left eigenvectors, i.e., the right eigenvectors of the adjoint operator $\mathcal{L}_{FP}^\dagger$.

33.5 All nonlinear noise is local

> I ain't gonna work for Maggie's pa no more
> No, I ain't gonna work for Maggie's pa no more
> Well, he puts his cigar
> Out in your face just for kicks
> — Bob Dylan, *Maggie's Farm*

(P. Cvitanović and D. Lippolis)

Our main goal in this section is to convince the reader that the diffusive dynamics of nonlinear flows is *fundamentally different from Brownian motion*, with the flow inducing a local, history dependent noise. In order to accomplish this, we generalize here the notion of invariant deterministic recurrent solutions, such as fixed points and periodic orbits, to noisy flows. While a Langevin trajectory (33.32) can never return exactly to the initial point and thus cannot ever be periodic, in the Fokker-Planck formulation (33.35) a recurrent motion can be defined as one where a peaked distribution returns to the initial neighborhood after time n. Recurrence so defined not only coincides with the classical notion of a recurrent orbit in the vanishing noise limit, but it also enables us to derive exact formulas for how this local, history dependent noise is to be computed.

As the function $x_{n+1} - f(x_n)$ is a nonlinear function, in general the path integral (33.35) can only be evaluated numerically. In the vanishing noise limit the Gaussian kernel sharpens into the Dirac δ-function, and the Fokker-Planck evolution operator reduces to the deterministic Perron-Frobenius operator (19.10). For weak noise the Fokker-Planck evolution operator can be evaluated perturbatively as an asymptotic series in powers of the diffusion constant, centered on the deterministic trajectory. Here we retain only the linear term in this series, which has a particulary simple dynamics given by a covariance matrix evolution formula (see (33.45) below) that we now derive.

We shift local coordinates labeled at time 'a'' to the deterministic trajectory $\{\ldots, x_{-1}, x_0, x_1, x_2, \ldots,\}$ centered coordinate frame $x = x_a + z_a$, Taylor expand $f(x) = f_a(z_a) = x_{a+1} + M_a z_a + \cdots$, and approximate the noisy map (33.32) by its

linearization,

$$z_{a+1} = M_a z_a + \xi_a, \quad M_{ij}(x) = \partial f_i / \partial x_j, \tag{33.38}$$

with the deterministic trajectory points at $z_a = z_{a+1} = 0$, and $M_a = M(x_a)$ the one time step Jacobian matrix. The corresponding linearized Fokker-Planck evolution operator (33.34) action on density $\rho_a(z_a) = \rho(x_a + z_a, a)$ is given in the local coordinates by

$$\rho_{a+1}(z_{a+1}) = \int dz_a \, \mathcal{L}_{FP}^a(z_{a+1}, z_a) \rho_a(z_a) \tag{33.39}$$

by the linearization (33.38) centered on the deterministic trajectory

$$\mathcal{L}_{FP}^a(z_{a+1}, z_a) = \frac{1}{N} e^{-\frac{1}{2}(z_{a+1} - M_a z_a)^\top \frac{1}{\Delta_a}(z_{a+1} - M_a z_a)}. \tag{33.40}$$

The superscript 'a' in $\mathcal{L}_{FP}^a$ distinguishes the local, linearized Fokker-Planck evolution operator coordinate frame $z_a = x - x_a$ centered on the deterministic trajectory point x_a from the full global evolution operator (33.35), in global coordinate system x.

The kernel of the linearized Fokker-Planck evolution operator (33.40) is a Gaussian. As a convolution of a Gaussian with a Gaussian is again a Gaussian, we investigate the action of the linearized Fokker-Planck evolution operator on a normalized, cigar-shaped Gaussian density distribution

$$\rho_a(z) = \frac{1}{C_a} e^{-\frac{1}{2} z^\top \frac{1}{Q_a} z}, \qquad C_a = [\det(2\pi Q_a)]^{1/2}, \tag{33.41}$$

and the action of the linearized adjoint Fokker-Planck evolution operator on density

$$\tilde{\rho}_a(z) = \frac{1}{C_a} e^{-\frac{1}{2} z^\top \frac{1}{\tilde{Q}_a} z}, \qquad C_a = \left[\det(2\pi \tilde{Q}_a)\right]^{1/2}, \tag{33.42}$$

also centered on the deterministic trajectory, with strictly positive $[d \times d]$ covariance matrices $Q, \tilde{Q}$. Label 'a' plays a double role, and $\{a+1, a\}$ stands both for the {next, initial} space partition and for the times the trajectory lands in these partitions. The linearized Fokker-Planck evolution operator (33.40) maps the Gaussian $\rho_a(z_a)$ into the Gaussian

$$\rho_{a+1}(z_{a+1}) = \frac{1}{C_a} \int [dz_a] \, e^{-\frac{1}{2}\left[(z_{a+1} - M_a z_a)^\top \frac{1}{\Delta_a}(z_{a+1} - M_a z_a) + z_a^\top \frac{1}{Q_a} z_a\right]} \tag{33.43}$$

one time step later. Likewise, linearizing the adjoint Fokker-Planck evolution operator (33.37) around the x_a trajectory point yields:

$$\tilde{\rho}_a(z_a) = \frac{1}{C_{a+1}} \int [dz_{a+1}] \, e^{-\frac{1}{2}\left[(z_{a+1} - M_a z_a)^\top \frac{1}{\Delta_a}(z_{a+1} - M_a z_a) + z_{a+1}^\top \frac{1}{\tilde{Q}_{a+1}} z_{a+1}\right]} \tag{33.44}$$

Completing the squares, integrating and substituting (33.41), respectively (33.42) we obtain the formula for covariance matrix evolution forward in time,

$$Q_{a+1} = M_a Q_a M_a^\top + \Delta_a . \tag{33.45}$$

In the adjoint case, the evolution of the $\tilde{Q}$ is given by

$$M_a \tilde{Q}_a M_a^\top = \tilde{Q}_{a+1} + \Delta_a . \tag{33.46}$$

The two covariance matrices differ, as the adjoint evolution $\tilde{Q}_a$ is computed by going backwards along the trajectory. These *covariance evolution* rules are the basis of all that follows.

Think of the initial covariance matrix (33.41) as an error correlation matrix describing the precision of the initial state, a cigar-shaped probability distribution $\rho_a(z_a)$. In one time step this density is deterministically advected and deformed into density with covariance $MQM^\top$, and then the noise Δ is added: the two kinds of independent uncertainties add up as sums of squares, hence the covariance evolution law (33.45), resulting in the Gaussian ellipsoid whose widths and orientation are given by the singular values and singular vectors (6.14) of the covariance matrix. After n time steps, the variance Q_a is built up from the deterministically propagated $M_a^n Q_{a-n} M_a^{nT}$ initial distribution, and the sum of noise kicks at intervening times, $M_a^k \Delta_{a-k} M_a^{kT}$, also propagated deterministically.

The pleasant surprise is that the evaluation of this noise requires no Fokker-Planck PDE formalism. The width of a Gaussian packet centered on a trajectory is fully specified by a deterministic computation that is already a pre-computed byproduct of the periodic orbit computations; the deterministic orbit and its linear stability. We have attached label 'a' to $\Delta_a = \Delta(x_a)$ in (33.45) to account for the noise distributions that are inhomogeneous, state space dependent, but time independent multiplicative noise.

33.6 Weak noise: Hamiltonian formulation

> All imperfection is easier to tolerate if served up in small doses.
> — Wislawa Szymborska

(G. Vattay and P. Cvitanović)

In the spirit of the WKB approximation (to be fully developed in chapter 37), we shall now study the evolution of the probability distribution by rewriting it as

$$\rho(x, t) = e^{\frac{1}{2D} R(x,t)} . \tag{33.47}$$

The time evolution of R is given by

$$\partial_t R + v \partial R + (\partial R)^2 = D \partial v + D \partial^2 R .$$

Consider now the weak noise limit and drop the terms proportional to D. The remaining equation

$$\partial_t R + H(x, \partial R) = 0$$

is known as the Hamilton-Jacobi equation . The function R can be interpreted as the Hamilton's principal function, corresponding to the Hamiltonian

$$H(x, p) = p\, v(x) + p^2/2\,,$$

with the Hamilton's equations of motion

$$
\begin{aligned}
\dot{x} &= \partial_p H = v + p \\
\dot{p} &= -\partial_x H = -A^\top p\,,
\end{aligned}
\tag{33.48}
$$

where A is the stability matrix (4.3)

$$A_{ij}(x) = \frac{\partial v_i(x)}{\partial x_j}\,.$$

The noise Lagrangian is then

$$L(x, \dot{x}) = \dot{x} \cdot p - H = \frac{1}{2}[\dot{x} - v(x)]^2\,.
\tag{33.49}$$

We have come the full circle - the Lagrangian is the exponent of our assumed Gaussian distribution (33.31) for noise $\xi^2 = [\dot{x} - v(x)]^2$. What is the meaning of this Hamiltonian, Lagrangian? Consider two points x_0 and x. Which noisy path is the most probable path that connects them in time t? The probability of a given path $\mathcal{P}$ is given by the probability of the noise sequence $\xi(t)$ which generates the path. This probability is proportional to the product of the noise probability functions (33.31) along the path, and the total probability for reaching x from x_0 in time t is given by the sum over all paths, or the stochastic path integral (Wiener integral)

$$
\begin{aligned}
P(x, x_0, t) \quad &\sim \quad \sum_{\mathcal{P}} \prod_j p(\xi(\tau_j), \delta\tau_j) = \int \prod_j d\xi_j \left(\frac{\delta\tau_j}{4\pi D}\right)^{d/2} e^{-\frac{\xi(\tau_j)^2}{4D}\delta\tau_i} \\
&\rightarrow \quad \frac{1}{Z} \sum_{\mathcal{P}} \exp\left(-\frac{1}{4D}\int_0^t d\tau\, \xi^2(\tau)\right),
\end{aligned}
\tag{33.50}
$$

where $\delta\tau_i = \tau_i - \tau_i$, and the normalization constant is

$$\frac{1}{Z} = \lim \prod_i \left(\frac{\delta\tau_i}{2\pi D}\right)^{d/2}\,.$$

The most probable path is the one maximizing the integral inside the exponential. If we express the noise (33.17) as

$$\xi(t) = \dot{x}(t) - v(x(t))\,,$$

the probability is maximized by the variational principle

$$\min \int_0^t d\tau [\dot{x}(\tau) - v(x(\tau))]^2 = \min \int_0^t L(x(\tau), \dot{x}(\tau)) d\tau \,.$$

By the standard arguments, for a given x, x' and t the probability is maximized by a solution of Hamilton's equations (33.48) that connects the two points $x_0 \to x'$ in time t. The solution is a bit boring: $\dot{x} = v$, $p = 0$, and lives in the initial, d-dimensional state space, so not much is to be made of this surprising appearance of Hamiltonians.

Résumé

When a deterministic trajectory is smeared out under the influence of Gaussian noise of strength D, the deterministic dynamics is recovered in the weak noise limit $D \to 0$. The effect of the noise can be taken into account by adding noise corrections to the classical trace formula.

Symplectic structure emerges here not as a deep principle of mechanics, but an artifact of the leading approximation to quantum/noisy dynamics, not respected by higher order corrections. The same is true of semiclassical quantum dynamics; higher corrections do not respect canonical invariance.

Commentary

Remark 33.1. A brief history of noise.   The theory of stochastic processes is a vast subject, starting with the Laplace 1810 memoir [29], spanning over centuries, and over disciplines ranging from pure mathematics to impure finance. The presentation given here is based on the Cvitanović and Lippolis 2012 Maribor lectures [11]. The material reviewed is standard [2, 26, 36], but needed in order to set the notation for what is new here, the role that local Fokker-Planck operators play in defining stochastic neighborhoods of periodic orbits. We enjoyed reading van Kampen classic [26], especially his railings against those who blunder carelessly into nonlinear landscapes. Having committed this careless chapter to print, we shall no doubt be cast to a special place on the long list of van Kampen's sinners (and not for the first time, either). A more specialized monograph like Risken's [36] will do just as well. Schulman's monograph [38] contains a very readable summary of Kac's [24] exposition of Wiener's integral over stochastic paths. The standard Langevin equation [7] is a stochastic equation for a Brownian particle, in which one replaces the Newton's equation for force by two counter-balancing forces: random accelerations $\xi(t)$ which tend to smear out a particle trajectory, and a damping term which drives the velocity to zero. In this context D is Einstein diffusion constant, and (33.12) is the Einstein diffusion formula [16]. Here we denote by 'Langevin equation' a more general family of stochastic differential equations (33.17) with additive or multiplicative [15, 28] weak noise. Noisy *discrete time* dynamical systems are discussed in refs. [6, 17, 27].

In probabilist literature [4] the differential operator $-\nabla \cdot (v(x)\rho(x,t)) + D \nabla^2 \rho(x,t)$ is called 'Fokker-Planck operator;' here we reserve the term 'Fokker-Planck evolution operator' for the finite time, 'Green's function' integral operator (33.30), i.e., the stochastic

path (Wiener) integral [31, 36, 39] for a noisy flow, with the associated continuous time Fokker-Planck [26, 32, 36] (or forward Kolmogorov) equation (33.24).

The cost function (33.31) appears to have been first introduced by Wiener as the exact solution for a purely diffusive Wiener-Lévy process in one dimension, see (33.51). Onsager and Machlup [20, 33] use it in their variational principle to study thermodynamic fluctuations in a neighborhood of single, linearly attractive equilibrium point (i.e., without any dynamics). It plays important role in the optimal control theory [3, 34]. Gaussians are often rediscovered, so Onsager-Machlup seminal paper, which studies the same attractive linear fixed point, is in literature often credited for being the first to introduce a variational method -the 'principle of least dissipation'- based on the Lagrangian of form (33.49). They, in turn, credit Rayleigh [35] with introducing the least dissipation principle in hydrodynamics. Onsager-Machlup paper deals only with a finite set of linearly damped thermodynamic variables, and not with a nonlinear flow or unstable periodic orbits.

Gaspard [21] derives a trace formula for the Fokker-Planck equation associated with Itó stochastic differential equations describing noisy time-continuous nonlinear dynamical systems. In the weak-noise limit, the trace formula provides estimations of the eigenvalues of the Fokker-Planck operator on the basis of the Pollicott-Ruelle resonances of the noiseless deterministic system, which is assumed to be non-bifurcating. At first order in the noise amplitude, the effect of noise on a periodic orbit is given in terms of the period and the derivative of the period with respect to the pseudo-energy of the Onsager-Machlup-Freidlin-Wentzell scheme [20]. The dynamical 'action' Lagrangian in the exponent of (33.30), and the associated symplectic Hamiltonian were first written down in 1970's by Freidlin and Wentzell [20], whose formulation of the 'large deviation principle' was inspired by the Feynman quantum path integral [18]. Feynman, in turn, followed Dirac [14] who was the first to discover that in the short-time limit the quantum propagator (imaginary time, quantum sibling of the Wiener stochastic distribution (33.51)) is exact. Gaspard [21] thus refers to the 'pseudo-energy of the Onsager-Machlup-Freidlin-Wentzell scheme.' M. Roncadelli [13, 37] refers to the Fokker-Planck exponent in (33.30) as the 'Wiener-Onsager-Machlup Lagrangian,' constructs weak noise saddle-point expansion and writes transport equations for the higher order coefficients. In our exposition the setting is more general: we study fluctuations over a state space-varying velocity field $v(x)$.

Remark 33.2. Weak noise perturbation theory.   DasBuch omits any discussion of the Martin-Siggia-Rose [31] type weak noise corrections. For an overview of possible ways for improvement of diagrammatic summation in noisy field theories, see *Chaotic Field Theory: a Sketch* [8]. The details are in the three papers on trace formulas for stochastic evolution operators (see also ref. [37]): Weak noise perturbation theory [9], smooth conjugation method [10], and local matrix representation approach [12]. Such corrections have not been evaluated before, probably because one is so unsure about nature of the noise itself that nth order correction is beyond the point. Doing continuous time flows requires the same kind of corrections, with diagrams standing for integrals rather than sums, though no one ever tried weakly stochastic flows in continuous time.

Remark 33.3. Covariance evolution.   In quantum mechanics the linearized evolution operator corresponding to the linearized Fokker-Planck evolution operator (33.40) is known as the Van Vleck propagator, the basic block in the semi-classical periodic orbit quantization [22], see chapter 38. Q covariance matrix composition rule (33.45) or its continuous time version is called 'covariance evolution' (for example, in ref. [41]), but it

goes all the way back to Lyapunov's 1892 thesis [30]. In the Kalman filter literature [1, 25] it is called 'prediction'.

Remark 33.4. Operator ordering. According to L. Arnold [2] review of the original literature, the derivations are much more delicate than what is presented here: the noise is *colored* rather than Dirac delta function in time. He refers only to the linear case as the 'Langevin equation'. The $\delta\tau \to 0$ limit and the proper definition of $\dot{x}(\tau)$ are delicate issues [2, 19, 23, 40] of no import for the applications of stochasticity studied here: Itó and Stratanovich operator ordering issues arise in the order beyond the leading approximation considered here.

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Example 33.1. Random walk in one dimension The white noise $\xi_n = x_{n+1} - x_n$ for a 1-dimensional diffusion process is a normally distributed random variable, with standard normal (i.e., Gaussian) probability distribution function,

$$\mathcal{L}_{FP}(x, t; x', t') = \frac{1}{\sqrt{4\pi D(t - t')}} \exp\left[-\frac{(x - x')^2}{4D(t - t')}\right],\tag{33.51}$$

of mean 0, variance $2D(t - t')$, and standard deviation $\sqrt{2D(t - t')}$, uncorrelated in time:

$$\langle x_{n+1} - x_n \rangle = 0, \qquad \langle (x_{m+1} - x_m)(x_{n+1} - x_n) \rangle = 2D\,\delta\tau\,\delta_{mn}.\tag{33.52}$$

section 33.4

Exercises

33.1. **Who ordered $\sqrt{\pi}$?**   Derive the Gaussian integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, e^{-\frac{x^2}{2a}} = \sqrt{a}, \qquad a > 0.$$

assuming only that you know to integrate the exponential function e^{-x}. Hint: x^2 is a radius-squared of something. π is related to the area or circumference of something.

33.2. **d-dimensional Gaussian integrals.**   Show that the Gaussian integral in d-dimensions is given by

$$\begin{aligned} Z[J] &= \int d^d x \, e^{-\frac{1}{2} x^{\top} \cdot M^{-1} \cdot x + x^{\top} \cdot J} \\ &= (2\pi)^{d/2} |\det M|^{\frac{1}{2}} \, e^{\frac{1}{2} J^{\top} \cdot M \cdot J}, \qquad (33.53) \end{aligned}$$

where M is a real positive definite $[d \times d]$ matrix, i.e., a matrix with strictly positive eigenvalues, x and J are d-dimensional vectors, and $(\cdots)^{\top}$ denotes the transpose.

This integral you will see over and over in statistical mechanics and quantum field theory: it's called 'free field theory', 'Gaussian model', 'Wick expansion', etc.. This is the starting, 'propagator' term in any perturbation expansion.

Here we require that the real symmetric matrix M in the exponent is strictly positive definite, otherwise the integral is infinite. Negative eigenvalues can be accommodated by taking a contour in the complex plane [5], see exercise 33.4 *Fresnel integral*. Zero eigenvalues require stationary phase approximations that go beyond the Gaussian saddle point approximation, typically to the Airy-function type stationary points, see exercise 37.4 *Airy function for large arguments*.

33.3. **Convolution of Gaussians.**

(a) Show that the Fourier transform of the convolution

$$[f * g](x) = \int d^d y \, f(x - y) g(y)$$

corresponds to the product of the Fourier transforms

$$[f * g](x) = \frac{1}{(2\pi)^d} \int d^d k \, F(k) G(k) e^{-ik \cdot x}, \quad (33.54)$$

where

$$F(k) = \int \frac{d^d x}{(2\pi)^{d/2}} f(x) \, e^{-ik \cdot x}, \quad G(k) = \int \frac{d^d x}{(2\pi)^{d/2}} g(x) \, e^{-ik \cdot x}.$$

(b) Consider two normalized Gaussians

$$\begin{aligned} f(x) &= \frac{1}{N_1} e^{-\frac{1}{2} x^{\top} \cdot \frac{1}{\Delta_1} \cdot x}, \quad N_1 = \sqrt{\det(2\pi\Delta_1)} \\ g(x) &= \frac{1}{N_2} e^{-\frac{1}{2} x^{\top} \cdot \frac{1}{\Delta_2} \cdot x}, \quad N_2 = \sqrt{\det(2\pi\Delta_2)} \\ 1 &= \int d^d k \, f(x) = \int d^d k \, g(x). \end{aligned}$$

Evaluate their Fourier transforms

$$F(k) = \frac{1}{(2\pi)^{d/2}} e^{\frac{1}{2} k^{\top} \cdot \Delta_1 \cdot k}, \qquad G(k) = \frac{1}{(2\pi)^{d/2}} e^{\frac{1}{2} k^{\top} \cdot \Delta_2 \cdot k}.$$

Show that the convolution of two normalized Gaussians is a normalized Gaussian

$$[f * g](x) = \frac{(2\pi)^{-d/2}}{\sqrt{\det(\Delta_1 + \Delta_2)}} e^{-\frac{1}{2} x^{\top} \cdot \frac{1}{\Delta_1 + \Delta_2} \cdot x}.$$

In other words, covariances Δ_j add up. This is the d-dimenional statement of the familiar fact that cumulative error squared is the sum of squares of individual errors. When individual errors are small, and you are adding up a sequence of them in time, you get Brownian motion. If the individual errors are small and added independently to a solution of a deterministic equation, you get Langevin and Fokker-Planck equations.

33.4. **Fresnel integral.**

(a) Derive the Fresnel integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, e^{-\frac{x^2}{2ia}} = \sqrt{ia} = |a|^{1/2} e^{i\frac{\pi}{4} \frac{a}{|a|}}.$$

Consider the contour integral $I_R = \int_{C(R)} \exp(iz^2) \, dz$, where $C(R)$ is the closed circular sector in the upper half-plane with boundary points 0, R and $R \exp(i\pi/4)$. Show that $I_R = 0$ and that $\lim_{R \to \infty} \int_{C_1(R)} \exp(iz^2) \, dz = 0$, where $C_1(R)$ is the contour integral along the circular sector from R to $R \exp(i\pi/4)$. [Hint: use $\sin x \geq (2x/\pi)$ on $0 \leq x \leq \pi/2$.] Then, by breaking up the contour $C(R)$ into three components, deduce that

$$\lim_{R \to \infty} \left(\int_0^R e^{ix^2} dx - e^{i\pi/4} \int_0^R e^{-r^2} dr \right)$$

vanishes, and, from the real integration $\int_0^{\infty} \exp(-x^2) \, dx = \sqrt{\pi}/2$, deduce that

$$\int_0^{\infty} e^{ix^2} dx = e^{i\pi/4} \sqrt{\pi}/2.$$

Now rescale x by real number $a \neq 0$, and complete the derivation of the Fresnel integral.

(b) In exercise 33.2 the exponent in the d-dimensional Gaussian integrals was real, so the real symmetric matrix M in the exponent had to be strictly positive definite. However, in quantum physics one often has to evaluate the d-dimenional Fresnel integral

$$\frac{1}{(2\pi)^{d/2}} \int d^d\phi \, e^{-\frac{1}{2i}\phi^\top \cdot M^{-1} \cdot \phi + i\,\phi \cdot J} \, ,$$

with a hermitian matrix M. Evaluate it. What are conditions on its spectrum in order that the integral be well defined?

33.5. **Airy function for large arguments.** Important contributions as stationary phase points may arise from extremal points where the first non-zero term in a Taylor expansion of the phase is of third or higher order. Such situations occur, for example, at bifurcation points or in diffraction effects, (such as waves near sharp corners, waves creeping around obstacles, etc.). In such calculations, one meets Airy functions integrals of the form

$$Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dy \, e^{i(xy - \frac{y^3}{3})} \, . \tag{33.55}$$

Calculate the Airy function $Ai(x)$ using the stationary phase approximation. What happens when considering the limit $x \to 0$. Estimate for which value of x the stationary phase approximation breaks down.

33.6. **Solving the Lyapunov differential equation.** Continuous time Lyapunov evolution equation for a covariance matrix $Q(t)$ is given by

$$\dot{Q} = A\,Q + Q\,A^\top + \Delta \, , \tag{33.56}$$

where $\{Q, A, \Delta\}$ are $[d \times d]$ matrices. The superscript $(\ )^\top$ indicates the transpose of the matrix. The stability matrix $A = A(x)$ and the noise covariance matrix $\Delta = \Delta(x)$ are given. They are evaluated on a trajectory $x(t)$, and thus vary in time, $A(t) = A(x(t))$ and $\Delta(t) = \Delta(x(t))$. Determine the covariance matrix $Q(t)$ for a given initial condition $Q(0)$, by taking the following steps:

(a) Write the solution in the form

$$Q(t) = J(t)[Q(0) + W(t)]J^\top(t) \, ,$$

with the Jacobian matrix $J(t)$ satisfying

$$\dot{J}(t) = A(t)\,J(t) \, , \qquad J(0) = I \, , \tag{33.57}$$

with I the $[d \times d]$ identity matrix. The Jacobian matrix at time t,

$$J(t) = \hat{T} e^{\int_0^t d\tau\, A(\tau)} \, , \tag{33.58}$$

where $\hat{T}$ denotes the 'time-ordering' operation, can be evaluated by integrating (33.57).

(b) Show that $W(t)$ satisfies

$$\dot{W} = \frac{1}{J}\,\Delta\,\frac{1}{J^\top} \, , \qquad W(0) = 0 \, . \tag{33.59}$$

(c) Integrate (33.56) to obtain

$$Q(t) = J(t)\left[Q(0) + \int_0^t d\tau\, \frac{1}{J(\tau)}\,\Delta(\tau)\,\frac{1}{J^\top(\tau)}\right]J^\top(t) \, . \tag{33.60}$$

(d) If $A(t)$ commutes with itself throughout the interval $0 \le \tau \le t$, the time-ordering operation is redundant, and we have the explicit solution $J(t, t') = \exp\int_{t'}^t d\tau\, A(\tau)$. Show that in this case the solution reduces to

$$
\begin{aligned}
Q(t) \ = \ & J(t)\, Q(0)\, J(t)^\top \tag{33.61} \\
& + \int_0^t d\tau'\, J(t, \tau')\,\Delta(\tau')\,J(t, \tau')^\top \, .
\end{aligned}
$$

(e) It is hard to imagine a time dependent $A(t) = A(x(t))$ that would be commuting. However, in the neighborhood of an equilibrium point x^* one can approximate the stability matrix with its time-independent linearization, $A = A(x^*)$. Show that in that case (33.58) reduces to

$$J(t, t') = e^{(t-t')A} \, ,$$

and (33.61) to what?

33.7. **Solving the Lyapunov differential equation.** Prove that if A is stable, the continuous Lyapunov equation

$$AQ + QA^\top = -\Delta < 0 \, .$$

has a solution

$$Q = \int_0^\infty dt\, e^{tA}\Delta e^{tA^\top} \, , \tag{33.62}$$

and that this solution is unique. (P. Cvitanović)

33.8. **Solving the discrete Lyapunov equation.** Prove that if M is contracting, the discrete Lyapunov equation

$$Q - MQM^\top = \Delta > 0$$

has a solution

$$Q = \sum_{k=0}^\infty M^k \Delta M^{k\top} \, , \tag{33.63}$$

and that this solution is unique. (P. Cvitanović)

33.9. **Continuous vs. discrete time Lyapunov equation.** Show that the continuous Lyapunov equation solution (suited to a Laplace transform),

$$Q = \int_0^\infty dt\, e^{tA} \Delta e^{tA^\top}, \qquad A < 0,$$

is equivalent to the discrete Lyapunov equation solution (suited to a Z-transform),

$$Q = \sum_{k=0}^\infty M^k \bar\Delta M^{k\top}, \qquad |M| < 1,$$

where

$$M = e^A, \quad \bar\Delta = \int_0^1 dt\, e^{tA} \Delta e^{tA^\top}.$$

Parenthetically, often the notation does not distinguish $\bar\Delta$ from Δ. It should. (P. Cvitanović)

33.10. **Lyapunov differential equation in resolvent form.** Show that the continuous Lyapunov equation solution,

$$Q = \int_0^\infty dt\, e^{tA} \Delta e^{tA^\top}, \qquad A < 0,$$

is equivalent to

$$Q = \frac{1}{2\pi} \int_{-\infty}^\infty d\omega\, \frac{1}{i\omega - A} \Delta \frac{1}{-i\omega - A^\top}.$$

(P. Cvitanović)

33.11. **Discrete time Lyapunov equation in the resolvent form.** Show that the continuous Lyapunov equation solution,

$$Q = \int_0^\infty dt\, e^{tA} \Delta e^{tA^\top}, \qquad A < 0,$$

is equivalent to the discrete Lyapunov equation solution in resolvent form,

$$Q = \frac{1}{2\pi} \int_0^{2\pi} d\omega\, \frac{1}{1 - e^{-i\omega} M} \bar\Delta \frac{1}{1 - e^{i\omega} M^\top}.$$

(P. Cvitanović)

33.12. **Noise covariance matrix for a discrete time periodic orbit.**

(a) Prove that the covariance matrix at a periodic point x_a on a limit cycle p,

$$Q_a = M_{p,a} Q_a M_{p,a}{}^\top + \Delta_{p,a}, \tag{33.64}$$

where

$$\Delta_{p,a} = \Delta_a + M_{a-1} \Delta_{a-1} M_{a-1}{}^\top + M_{a-2}^2 \Delta_{a-2} (M_{a-2}^2)^\top$$
$$+ \cdots + M_{a-n_p+1}^{n_p-1} \Delta_{a-n_p+1} (M_{a-n_p+1}^{n_p-1})^\top \tag{33.65}$$

is the noise accumulated per a single transversal of the periodic orbit, $M_{p,a} = M_p(x_a)$ is the cycle Jacobian matrix (4.5) evaluated on the periodic point x_a, and we have used the periodic orbit condition $x_{a+n_p} = x_a$.

(b) Derive the analogous formulas for the adjoint Fokker-Planck covariance matrix at a periodic point x_a on a repelling cycle p.