11. Solutions of an equivariant system can satisfy all of system’s symmetries, a subgroup of them, or have no symmetry at all. For a generic ergodic orbit \( f(x) \) the trajectory and any of its images under action of \( g \in G \) are distinct with probability one, \( f(x) \cap g f(x) = \emptyset \) for all \( t, t' \). For example, a typical turbulent trajectory is very different. For example, the symmetry of the laminar solution of the plane Couette flow is the full symmetry of its Navier-Stokes equations. In between we find solutions whose symmetries are subgroups of the full symmetry of dynamics.

**Definition: Isotropy subgroup.** The maximal set of group actions which maps a state space point \( x \) into itself,

\[
G_x = \{ g \in G : gx = x \},
\]

is called the isotropy group (or stability subgroup or little group) of \( x \). Think of a point \((0,0,z)\), \(z \neq 0\) on \(z\) axis in 3 dimensions. Its isotropy group is the \(O(2)\) group of rotations in the \((x,y)\) plane.

A solution usually exhibits less symmetry than the equations of motion. The symmetry of a solution is thus a subgroup of the symmetry group of dynamics. We thus also need a notion of set-wise invariance, as opposed to the above point-wise invariance under \( G_x \).

**Definition: Symmetry of a solution.** We shall refer to the maximal subgroup \( G_p \subseteq G \) of actions on state space points within a compact set \( M_p \), which leave no point fixed but leave the set invariant, as the symmetry \( G_p \) of the solution labelled \( p \).

\[
G_p = \{ g \in G_p | gx \in M_p, gx \neq x \text{ for } g \neq e \},
\]

and reserve the notion of ‘isotropy’ of a set \( M_p \) for the subgroup \( G_p \) that leaves each point in it fixed.

A cycle \( p \) is \( G_p \)-symmetric (set-wise symmetric, self-dual) if the action of elements of \( G_p \) on the set of periodic points \( M_p \) reproduces the set. \( g \in G_p \) acts as a shift in time, mapping the periodic point \( x \in M_p \) into another periodic point.

**Definition: Multiplicity.** For a finite discrete group, the multiplicity of orbit \( p \) is \( m_p = |G|/|G_p^e| \).

---

*Even the butterfly that started the hurricane flapped its wings for a reason.*

— Louis Menand, *Thinking Sideways*, New Yorker, 30 March 2015
Definition: Stratum. A stratum is the union of group orbits of the same type; two orbits, \( p, p' \), belong to the same stratum if and only if their symmetries \( G_p, G_{p'} \) are conjugate. In other words, a stratum is to state space what a class is to the set of all group elements in \( G \).

Definition: \( G_p \)-fixed orbits: An equilibrium \( x_p \) or a compact solution \( p \) is point-wise or \( G_p \)-fixed if it lies in the invariant points subspace \( \text{Fix}(G_p) \), \( gx = x \) for all \( g \in G_p \), and \( x = x_p \) or \( x \in M_p \). A solution that is \( G \)-invariant under all group \( G \) operations has multiplicity 1. Stability of such solutions will have to be examined with care, as they lie on the boundaries of domains related by the action of the symmetry group.

In the literature the symmetry group of a solution is often called stabilizer or isotropy subgroup. Saying that \( G_p \) is the symmetry of the solution \( p \), or that the orbit \( M_p \) is \( G_p \)-invariant, accomplishes as much without confusing you with all these names (see remark 10.1). In what follows we say “the symmetry of the periodic orbit \( p \) is \( C_2 = \{ e, R \} \),” rather than bandy about ‘stabilizers’ and such.

The key concept in the classification of dynamical orbits is their symmetry. We note three types of solutions: (i) fully asymmetric solutions \( a \), (ii) subgroup \( G_i \)-set-wise invariant cycles \( x \) built by repeats of relative cycle segments \( \tilde{a} \), and (iii) isotropy subgroup \( G_{EQ} \)-invariant equilibria or point-wise \( G_p \)-fixed cycles \( b \).

Definition: Asymmetric (or fully asymmetric) orbits. An orbit (in particular, an equilibrium or periodic orbit) has no symmetry if \( \{x_a\} \cap \{gx_a\} = \emptyset \) for any \( g \in G \), where \( \{x_a\} \) is the set of periodic points belonging to the cycle \( a \). Thus \( g \in G \) generate \( |G| \) distinct orbits with the same number of points and the same stability properties.

In example 11.4, we illustrate the non-abelian, noncommutative group structure of the 3-disk game of pinball of sect. 1.3, which has symmetry group elements that do not commute.

Consider next perhaps the simplest 3-dimensional flow with a symmetry, the iconic flow of Lorenz. The example is long but worth working through: the symmetry-reduced dynamics is much simpler than the original Lorenz flow.

**Figure 11.1:** The \( D_1 \)-equivariant bimodal sawtooth map of figure 10.2 has three types of periodic orbits: (a) \( D_1 \)-fixed fixed point \( \tilde{a} \), asymmetric fixed points pair \( (L, R) \), (b) \( D_1 \)-symmetric (setwise invariant) 2-cycle \( LR \), (c) Asymmetric 2-cycles pair \( (LC, CR) \) (study example 11.3; continued in figure 11.5) (E.Siminos)

Note: nonlinear coordinate transformations such as the doubled-polar angle representation (11.10) are not required to implement the symmetry quotienting \( M/G \). We deploy them only as a visualization aid that might help the reader disentangle 2-dimensional projections of higher-dimensional flows. All numerical calculations can still be carried in the initial, full state space formulation of a flow, with symmetry-related points identified by linear symmetry transformations.

**Figure 11.2:** Lorenz attractor of figure 3.4, the full state space coordinates \( \{x, y, z\} \), with the unstable manifold orbits \( W^u(E_s) \). (Green) is a continuation of the unstable \( e^{\pi} \) of \( E_s \), and (brown) is its \( \pi \)-rotated symmetry partner. Compare with figure 11.3. (E. Siminos)

In depth: appendix A7, p. 868

**11.2 Relative periodic orbits**

So far we have demonstrated that symmetry relates classes of orbits. Now we show that a symmetry reduces computation of periodic orbits to repeats of shorter, "relative periodic orbit" segments.
Equivariance of a flow under a symmetry means that the symmetry image of a cycle is again a cycle, with the same period and stability. The new orbit may be topologically distinct (in which case it contributes to the multiplicity of the cycle) or it may be the same cycle.

A cycle \( p \) is \( G_p \)-symmetric under symmetry operation \( g \in G_p \) if the operation acts on it as a shift in time, advancing a cycle point to a cycle point on the symmetry related segment. The cycle \( p \) can thus be subdivided into \( m_p \) repeats of a relative periodic orbit segment, ‘prime’ in the sense that the full state space cycle is built from its repeats. Thus, in the presence of a discrete symmetry, the notion of a periodic orbit is replaced by the notion of the shortest segment of the cycle which tiles the cycle under the action of the group. In what follows we refer to this segment as a relative periodic orbit. In the literature this is sometimes referred to as a short periodic orbit, or, for finite symmetry groups, as a pre-periodic orbit.

The relative periodic orbit \( p \) (or its equivariant periodic orbit) is the orbit \( x(t) \) in state space \( M \) which exactly recurs

\[
x(t) = g_p \cdot x(t + T_p)
\]

(11.3)

for the shortest fixed relative period \( T_p \) and a fixed group action \( g \in G_p \). These group actions are referred to as ‘shifts’ or, in the case of continuous symmetries, as ‘phases.’ For a discrete group \( G = e \) and finite \( m \) (10.3), the period of the corresponding full state space orbit is given by the \( m_p \) \( x \) (period of the relative periodic orbit), \( T_p = |G_p|T_p \), and the \( i \)th Floquet multiplier \( \lambda_{ei} \) is given by \( \lambda_{ei} = e^{i2\pi m_p} \) of the relative periodic orbit. The elements of the quotient space \( b \in G/G_p \) generate the copies \( bp \), so the multiplicity of the full state space cycle \( p \) is \( m_p = |G|/|G_p| \).

11.3 Dynamics reduced to fundamental domain

I submit my total lack of apprehension of fundamental concepts.

—John F. Gibson

So far we have used symmetry to effect a reduction in the number of independent cycles, by separating them into classes, and slicing them into ‘prime’ relative orbit segments. The next step achieves much more: it replaces each class by a single (typically shorter) prime cycle segment.

1. Discrete symmetry tessellates the state space into dynamically equivalent domains, and thus induces a natural partition of state space: If the dynamics is invariant under a discrete symmetry, the state space \( M \) can be completely tiled by a fundamental domain \( \tilde{M} \) and its symmetry images \( M_a = a \cdot \tilde{M}, \quad M_b = b \cdot \tilde{M}, \ldots \) under the action of the symmetry group \( G = \{ e, a, b, \ldots \} \).

\[
M = \tilde{M} \cup M_a \cup M_b \cdots \cup M_G.
\]

(11.4)

2. Discrete symmetry can be used to restrict all computations to the fundamental domain \( \tilde{M} = \tilde{M}/G \), the reduced state space quotient of the full state space \( M \) by the group actions of \( G \).

We can use the invariance condition (10.4) to move the starting point \( x \) into the fundamental domain \( \tilde{x} = a \cdot \tilde{x} \), and then use the relation \( a^{-1} b = h^{-1} \) to also relate the endpoint \( \tilde{y} = a \cdot \tilde{y} \) to its image in the fundamental domain \( \tilde{M} \). While the global trajectory runs over the full space \( M \), the restricted trajectory is brought back into the fundamental domain \( \tilde{M} \) any time it exits into an adjoining tile; the two trajectories are related by the symmetry operation \( h \) which maps the global endpoint into its fundamental domain image.

3. Cycle multiplicities induced by the symmetry are removed by reduction of the full dynamics to the dynamics on a fundamental domain. Each symmetry-related set of global cycles \( p \) corresponds to precisely one fundamental domain (or relative) cycle \( \tilde{p} \).

4. Conversely, each fundamental domain cycle \( \tilde{p} \) traces out a segment of the global cycle \( p \), with the end point of the cycle \( \tilde{p} \) mapped into the irreducible segment of \( p \) with the group element \( h_p \). A relative periodic orbit segment in the full state space is thus a periodic orbit in the fundamental domain.
11.4 Invariant polynomials

All invariants are expressible in terms of a finite number among them. We cannot claim its validity for every group \( G \); rather, it will be our chief task to investigate for each particular group whether a finite integrity basis exists or not; the answer, to be sure, will turn out affirmative in the most important cases.

—Hermann Weyl, a motivational quote on the “so-called first main theorem of invariant theory”

Physical laws should have the same form in symmetry-equivalent coordinate frames, so they are often formulated in terms of functions (Hamiltonians, Lagrangians, ...) invariant under a given set of symmetries. The key result of the representation theory of invariant functions is:

**Hilbert-Weyl theorem.** For a compact group \( G \) there exists a finite \( G \)-invariant homogenous polynomial basis \( \{u_1, u_2, \ldots, u_m\} \), \( m \geq d \), such that any \( G \)-invariant polynomial can be written as a multinomial

\[
h(x) = p(u_1(x), u_2(x), \ldots, u_m(x)), \quad x \in \mathcal{M}.
\]

These polynomials are linearly independent, but can be functionally dependent through nonlinear relations called syzygies.

In practice, explicit construction of \( G \)-invariant basis can be a laborious undertaking, and we will not take this path except for a few simple low-dimensional cases, such as the 5-dimensional example of sect. 13.7. We prefer to apply the symmetry to the system as given, rather than undertake a series of nonlinear coordinate transformations that the theorem suggests. (What ‘compact’ in the above refers to will become clearer after we have discussed continuous symmetries. For now, it suffices to know that any finite discrete group is compact.)
CHAPTER 11. WORLD IN A MIRROR

Commentary

11.1 Remarks

Remark 11.1 Symmetries of the Lorenz equation: (continued from remark 2.3) After having studied example 11.5 you will appreciate why ChaosBook.org starts out with the symmetry-less Rössler flow (2.27), instead of the better known Lorenz flow (2.22). Indeed, getting rid of symmetry was one of Rössler’s motivations. He threw the baby out with the water; for Lorenz flow dimensionalities of stable/unstable manifolds make possible a robust heteroclinic connection absent from Rössler flow, with unstable manifold of an equilibrium flowing into the stable manifold of another equilibrium. How such connections are forced upon us is best grasped by perusing the chapter 13 ‘Heteroclinic tangles’ of the inimitable Abraham and Shaw illustrated classic [14.23]. Their beautiful hand-drawn sketches elucidate the origin of heteroclinic connections in the Lorenz flow (and its high-dimensional Navier-Stokes relatives) better than any computer simulation. Miranda and Stone [11.2] were first to quotient the $C_2$ symmetry and explicitly construct the desymmetrized, ‘proto-Lorenz system,’ by a nonlinear coordinate transformation into the Hilbert-Weyl polynomial basis invariant under the action of the symmetry group [11.3]. For in-depth discussion of symmetry-reduced (‘images’) and symmetry-extended (‘covers’) topology, symbolic dynamics, periodic orbits, invariant polynomial bases etc., of Lorenz, Rössler and many other low-dimensional systems there is no better reference than the Gilmore and Letellier monograph [13.19]. They interpret [11.5] the proto-Lorenz and its ‘double cover’ Lorenz as ‘intensities’ being the squares of ‘amplitudes,’ and call quotiented flows such as (Lorenz)/$C_2$ ‘images.’ Our ‘doubled-polar angle’ visualization figure 14.8 is a proto-Lorenzo in disguise; we, however, integrate the flow and construct Poincaré sections and return maps in the original Lorenz $(z,x,\dot{z})$ coordinates, without any nonlinear coordinate transformations. The Poincaré return map figure 14.9 is reminiscent in shape both of the one given by Lorenz in his original paper, and the one plotted in a radial coordinate by Gilmore and Letellier. Nevertheless, it is profoundly different: our return maps are from unstable manifold $\to$ itself, and thus intrinsic and coordinate independent. In this we follow ref. [A1.79]. This construction is necessary for high-dimensional flows in order to avoid problems such as double-valuedness of return map projections on arbitrary 1-dimensional coordinates encountered already in the Rössler example of figure 3.3. More importantly, as we know the embedding of the unstable manifold into the full state space, a periodic point of our return map is $\to$ regardless of the length of the cycle - the periodic point in the full state space, so no additional Newton search are needed. In homage to Lorenz, we note that his return map was already symmetry-reduced: as $z$ belongs to the symmetry invariant Fix $(G)$ subspace, one can replace dynamics in the full space by $\dot{z}$, $\ddot{z}$, $\cdots$. That is $G$-invariant by construction [13.19].

Remark 11.2 Examples of systems with discrete symmetries. Almost any flow of interest is symmetric in some way or other: the list of examples is endless, we list here a handful that we found interesting. One has a $C_2$ symmetry in the Lorenz system (remark 2.3), the Ising model, and in the 3-dimensional anisotropic Kepler potential [A39.17, 38.18, A1.24], a $D_3 \cong C_3$ symmetry in quartic oscillators [11.10, 11.11], in the pure $eV^2$ potential [11.12, 11.13] and in hydrogen in a magnetic field [11.14], and a $D_4 \cong C_4$ symmetry in Vlasov [23.2]. A very nice non-trivial desymmetrization is carried out in ref. [23.11]. An example of a system with $D_3 \cong C_3$ symmetry is provided by the motion of a particle in the Hénon-Heiles potential [11.17, 11.18, 11.19, 25.10]

$$V(r,\theta) = \frac{1}{2}r^2 + \frac{1}{3!}\sin(3\theta).$$

Our 3-disk coding is insufficient for this system because of the existence of elliptic islands and because the three orbits that run along the symmetry axis cannot be labeled in our code. As these orbits run along the boundary of the fundamental domain, they require the special treatment. A partial classification of the 67 possible symmetries of solutions of the plane Cosette flow of example 12.9, and their reduction 5 conjugate classes is given in ref. [13.42].

11.5 Examples

Example 11.1 $D_3$-symmetric cycles: For $D_3$, the period of a set-wise symmetric cycle is even ($n = 2m$), and the mirror image of the $s_j$ periodic point is reached by traversing the relative periodic orbit segment $\cdot s_j s^\cdot$ of length $n_j$, $f^n(s_j) = s_j s^-$, see figure 11.1 (b).

Example 11.2 $D_3$-invariant cycles: In the example at hand there is only one $G$-invariant (point-wise invariant) orbit, the fixed point $C$ at the origin, see figure 11.1 (a). As reflection symmetry is the only discrete symmetry that a map of the interval can have, this example completes the group-theoretic analysis of 1-dimensional maps. We shall continue analysis of this system in example 11.7, and work out the symbolic dynamics of such reflection symmetric systems in example 15.7.

Example 11.3 Group $D_3$ - a reflection symmetric l.c. map: Consider the bimodal ‘sawtooth’ map of example 10.5, with the state space $M = [1, 1]$ split into three regions $M = (M_L, M_C, M_R)$ which we label with a 3-letter alphabet $L$ (left), $C$ (center), and $R$ (right). The symbolic dynamics is complete ternary dynamics, with any sequence of letters $A = (L.C.R)$ corresponding to an admissible trajectory (‘complete’ means no additional grammar rules required, see example 14.7 below). The $D_3$-equivariance of the map, $D_3 = \{e, r, e\}$, implies that if $(s_x, \ldots)$ is a trajectory, so is $(s_{x+\pi/3})$.

Fix $(G)$, the set of points invariant under group action of $D_3$, $M \cap rM$, is just this fixed point $x = 0$, the reflection symmetry point. If $x$ is an asymmetric cycle, $r$ maps it into the reflected cycle $r_x$, with the same period and the same stability properties, see the fixed points pair $[L, R]$ and the 2-cycles pair $[L.C, C.R]$ in figure 11.1 (c).

Example 11.4 2-disk game of pinball - cycle symmetries: (continued from example 10.8) The $C_3$ subgroup $G_3 = \{e, C^{(1)}, C^{(2)}\}$ invariance is exemplified by the two cycles $T_{12}$ and $T_{23}$ which are invariant under rotations by $2\pi/3$ and $4\pi/3$, but are mapped into each other by any reflection, figure 11.4 (a), and have multiplicity $|G_3|/|G_1| = 3$.

The $C_3^o$ type of a subgroup is exemplified by the symmetries of $p = 1213$. This cycle is invariant under reflection $\sigma_{12} = T_{12} = 1213$, so the invariant subgroup

$$G_3 = \{e, \sigma_{12}\},$$

with multiplicity $|G_3|/|G_1| = 3$; the cycles in this class, $T_{12}$, $T_{23}$ and $T_{31}$, are related by $2\pi/3$ rotations, figure 11.4 (b).
A cycle of no symmetry, such as $T = \mathbb{T}$, has $G(x) = \{x\}$ and contributes in all six copies (the remaining cycles in the class are $T = \mathbb{T}$, $T = \mathbb{T}$, $T = \mathbb{T}$, $T = \mathbb{T}$ and $T = \mathbb{T}$), figure 11.4(c).

Besides the above spatial symmetries, for Hamiltonian systems cycles may be related by time reversal symmetry. An example are the cycles $T = \mathbb{T}$, $T = \mathbb{T}$, $T = \mathbb{T}$, $T = \mathbb{T}$ and $T = \mathbb{T}$ which have the same periods and stabilities, but are related by no space symmetry, see figure 11.4. (continued in example 11.8)

Example 11.5 Desymmetrization of Lorenz flow: (continuation of example 10.6) Lorenz equation (2.22) is equivariant under (10.15), the action of order-2 group $C_2 = \{e, C_2\}$, where $C_1^{1/2}$ is the $[x, y]$-plane, half-cycle rotation by $\pi$ about the $z$-axis:

$$ C_1^{1/2}(x, y, z) = (-x, -y, z) \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} $$

(11.6)

which has the same periods and stabilities, but are related by no space symmetry, see figure 11.4. (continued in example 11.8)

Example 11.6 Relative periodic orbits of Lorenz flow: (continuation of example 11.5) The relation between the full state space periodic orbits, and the fundamental domain (11.10) reduced relative periodic orbits of the Lorenz flow: an asymmetric full state space cycle pair $p_R$, $p_P$ maps into a single cycle $\tilde{p}$ in the fundamental domain, and any self-dual cycle $p = p_P = \tilde{p}R\tilde{p}$ is a repeat of a relative periodic orbit $\tilde{p}$.

Example 11.7 Group $D_2$ and reduction to the fundamental domain: Consider again the reflection-symmetric Ulam sawtooth map $f(x) = f(\tau x)$ of example 11.3, with symmetry group $D_2 = \{e, \sigma\}$. The state space $M = [-1, 1]$ can be tiled by half-line $M = [0, 1]$, and $\sigma M = [-1, 0]$, its image under a reflection across $x = 0$ point. The dynamics can then be restricted to the fundamental domain $x_0 = 0$, every time a trajectory leaves this interval, it is mapped back using $\sigma$.

In figure 11.5 the fundamental domain map $f(\tilde{x})$ is obtained by reflecting $x < 0$ segments of the global map $f(x)$ into the upper right quadrant. $f$ is also bimodal and piecewise-linear, with $[0, 1]$ split into three regions $M = [M_0, M_1, M_2]$ which we label with a 3-letter alphabet $\mathcal{A} = [0, 1, 2]$. The symbolic dynamics is again complete ternary dynamics, with any sequence of letters $[0, 1, 2]$ admissible.

However, the interpretation of the ‘desymmetrized’ dynamics is quite different - the multiplicity of every periodic orbit is now 1, and relative periodic segments of the full state space dynamics are all periodic orbits in the fundamental domain. Consider figure 11.5:

In (a) the boundary fixed point $\vec{1}$ is also the fixed point $\vec{0}$.

The asymmetric fixed point pair $\{\vec{1}, \vec{0}\}$ is reduced to the fixed point $\vec{0}$, and the full state space symmetric 2-cycle $\vec{1} \vec{0}$ is reduced to the fixed point $\vec{0}$. The asymmetric 2-cycle pair $\vec{1} \vec{0}$ is reduced to the 2-cycle $\mathcal{1} \mathcal{0}$. Finally, the symmetric 4-cycle $\mathcal{1} \mathcal{0} \mathcal{1} \mathcal{0}$ is reduced to the 2-cycle $\mathcal{1} \mathcal{0}$. This completes the conversion from the full state space for all fundamental domain fixed points and 2-cycles, figure 11.5(c).

Example 11.8 3-disk game of pinball in the fundamental domain:

If the dynamics is equivariant under interchanges of disks, the absolute disk labels $e_i = 1, 2, \cdots, N$ can be replaced by the symmetry-invariant relative disk labels $g_i$, where $g_i$ is the discrete group element that maps disk $i$ into disk $j$. For 3-disk system $g_i$ is either reflection $\sigma$ or twist $\alpha$ rotation.
by $C$ to the next disk (symbol '1'). An immediate gain arising from symmetry invariant relabeling is that $N$-disk symbolic dynamics becomes $(N-1)$-nary, with no restrictions on the admissible sequences.

An irreducible segment corresponds to a periodic orbit in the fundamental domain, a one-sixth slice of the full 3-disk system, with the symmetry axes acting as reflecting mirrors (see figure 10.3(d)). A set of orbits related in the full space by discrete symmetries maps onto a single fundamental domain orbit. The reduction to the fundamental domain desymmetrizes the dynamics and removes all global discrete symmetry-induced degeneracies: rotationally symmetric global orbits (such as the 3-cycles $\{123, 231\}$) have multiplicity 2, reflection symmetric ones (such as the 2-cycles $12, 33$) have multiplicity 3, and global orbits with no symmetry are 6-fold degenerate. Table 15.2 lists some of the shortest binary symbol sequences and orbit symmetries. Some examples of such orbits are shown in figures 11.4 and 11.6. (continued in example 15.8)

### Exercises

**11.1. Reduction of 3-disk symbolic dynamics to binary.**

(continued from exercise 11.1)

(a) Verify that the 3-disk cycles $\{123, 231, 312, 132\}$ have multiplicity 2, reflection symmetric ones (such as the 2-cycles $12, 33$) have multiplicity 3, and global orbits with no symmetry are 6-fold degenerate. Table 15.2 lists some of the shortest binary symbol strings, together with the corresponding full 3-disk symbol sequences and orbit symmetries. Some examples of such orbits are shown in figures 11.4 and 11.6. (continued in example 15.8)

(b) Check the reduction for short cycles in table 15.2 by drawing them both in the full 3-disk system and in the fundamental domain, as in figure 11.6.

(c) Optional: Can you see how the group elements listed in table 15.2 relate irreducible segments to the fundamental domain periodic orbits?

### 11.2. C$_3$-equivariance of Lorenz system.**

Verify that the vector field in Lorenz equations (2.22)

$$\dot{x} = y(z - x), \quad \dot{y} = r - x y - x z, \quad \dot{z} = b x - x y$$

(11.11)

is equivariant under the action of cyclic group $C_3 = \{e, C, C^2\}$ acting on $\mathbb{R}^3$ by a $\pi$ rotation about the $z$ axis,

$$C^{1/2}(x, y, z) = (-x, y, z),$$

as claimed in example 10.6. (continued in exercise 11.3)

### 11.3. Lorenz system in polar coordinates: group theory.

Use (A2.3), (A2.4) to rewrite the Lorenz equation (11.11) in polar coordinates $(r, \theta, z)$, where $x = r \cos \theta, y = r \sin \theta$.

1. Show that in the polar coordinates Lorenz flow takes form

$$\dot{r} = r \frac{\sin \theta (r + z) - (r + z) \sin \theta}{(r + z) \cos \theta},$$

$$\dot{\theta} = \frac{1}{2} \left( -r \frac{\sin \theta (r + z) - (r + z) \sin \theta}{(r + z) \cos \theta} + (r + z) \cos \theta \right),$$

$$\dot{z} = -r \frac{\sin \theta (r + z) - (r + z) \sin \theta}{(r + z) \cos \theta}.$$

(11.12)

2. Argue that the transformation to polar coordinates is invertible almost everywhere. Where does the inverse not exist? What is group-theoretically special about the subspace on which the inverse not exist?

### 11.4. Proto-Lorenz system.

Here we quotient out the cyclic symmetry by constructing an explicit "intensity" representation of the desymmetrized Lorenz flow, following Miranda and Stone [11.2].

1. Rewrite the Lorenz equation (2.22) in terms of the variables

$$u, v, z = (x^2 - y^2, 2xy, z),$$

(11.13)

show that it takes form

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -r (u + v) + (r - u)N \\ -r (v + u) + (r + u)N - u - v/2 - z \end{bmatrix},$$

$$N = \sqrt{u^2 + v^2}.$$

2. Show that this is the (Lorenz)/$C_2$ quotient map for the Lorenz flow, i.e., that it identifies points related by the $\pi$ rotation in the $[x, y]$ plane.

3. Show that (11.11) is invertible. Where does the inverse not exist?

4. Compute the equilibria of proto-Lorenz and their stabilities. Compare with the equilibria of the Lorenz flow.

5. Plot the strange attractor both in the original form (2.22) and in the proto-Lorenz form (11.14)
for the Lorenz parameter values $\sigma = 10$, $b = 8/3$, $\rho = 28$. Topologically, does it resemble more the Lorenz, or the R"ossler attractor, or neither? (plot by J. Halcrow)

7. Show that a periodic orbit of the proto-Lorenz is either a periodic orbit or a relative periodic orbit of the Lorenz flow.
8. Show that if a periodic orbit of the proto-Lorenz is also periodic orbit of the Lorenz flow, their Floquet multipliers are the same. How do the Floquet multipliers of relative periodic orbits of the Lorenz flow relate to the Floquet multipliers of the proto-Lorenz?
10. Show that the coordinate change (11.13) is the same as rewriting (11.12) in variables

$$(u, v) = (r^2 \cos 2\theta, r^2 \sin 2\theta),$$

i.e., squaring a complex number $z = x + iy$, $z^2 = u + iv$.
11. How is (11.14) related to the invariant polynomial basis of example 11.9 and exercise 11.12?

References