Chapter 15

Counting

I'm gonna close my eyes
And count to ten
I'm gonna close my eyes
And when I open them again
Everything will make sense to me then
—Tina Dico, 'Count To Ten'

We are now in a position to apply the periodic orbit theory to the first and the easiest problem in theory of chaotic systems: cycle counting. This is the simplest illustration of the raison d'etre of periodic orbit theory; we derive a duality transformation that relates *local* information - in this case the next admissible symbol in a symbol sequence - to *global* averages, in this case the mean rate of growth of the number of cycles with increasing cycle period. In chapter 14 we have transformed, by means of the transition matrices / graphs, the topological dynamics of chapter 11 into a multiplicative operation. Here we show that the nth power of a transition matrix counts all itineraries of length n. The asymptotic growth rate of the number of admissible itineraries is therefore given by the leading eigenvalue of the transition matrix; the leading eigenvalue is in turn given by the leading zero of the characteristic determinant of the transition matrix, which is - in this context - called the *topological zeta function*.

For flows with finite transition graphs this determinant is a finite *topological polynomial* which can be read off the graph. However, (a) even something as humble as the quadratic map generically requires an infinite partition (sect. 15.5), but (b) the finite partition approximants converge exponentially fast.

The method goes well beyond the problem at hand, and forms the core of the entire treatise, making tangible the abstract notion of "spectral determinants" yet to come.

15.1 How many ways to get there from here?

In the 3-disk system of example 11.1 the number of admissible trajectories doubles with every iterate: there are $K_n = 3 \cdot 2^n$ distinct itineraries of length n. If disks are too close and a subset of trajectories is pruned, this is only an upper bound and explicit formulas might be hard to discover, but we still might be able to establish a lower exponential bound of the form $K_n \geq Ce^{n\hat{h}}$. Bounded exponentially by $3e^{n \ln 2} \geq K_n \geq Ce^{n\hat{h}}$, the number of trajectories must grow exponentially as a function of the itinerary length, with rate given by the *topological entropy*:

$$h = \lim_{n\to\infty} \frac{1}{n} \ln K_n \ . \tag{15.1}$$

We shall now relate this quantity to the spectrum of the transition matrix, with the growth rate of the number of topologically distinct trajectories given by the leading eigenvalue of the transition matrix.

The transition matrix element $T_{ij} \in \{0, 1\}$ in (14.1) indicates whether the transition from the starting partition j into partition i in one step is allowed or not, and the (i, j) element of the transition matrix iterated n times

exercise 15.1

$$(T^n)_{ij} = \sum_{k_1,k_2,\ldots,k_{n-1}} T_{ik_1} T_{k_1k_2} \ldots T_{k_{n-1}j} \tag{15.2}$$

receives a contribution 1 from every admissible sequence of transitions, so $(T^n)_{ij}$ is the number of admissible n symbol itineraries starting with j and ending with i.

Example 15.1 ***3-disk itinerary counting.*** *The* $(T^2)_{13} = T_{12}T_{23} = 1$ *element of* T^2 *for the 3-disk transition matrix (14.8)*

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} . \tag{15.3}$$

corresponds to path $3 \to 2 \to 1$*, the only 2-step path from 3 to 1, while* $(T^2)_{33} = T_{31}T_{13} + T_{32}T_{23} = 2$ *counts the two returning, periodic paths* $\overline{31}$ *and* $\overline{32}$*. Note that the trace* $\mathrm{tr}\, T^2 = (T^2)_{11} + (T^2)_{22} + (T^2)_{33} = 2T_{13}T_{31} + 2T_{21}T_{12} + 2T_{32}T_{23}$ *has a contribution from each 2-cycle* $\overline{12}$, $\overline{13}$, $\overline{23}$ *twice, one contribution from each periodic point.*

The total number of admissible itineraries of n symbols is

$$K_n = \sum_{ij} (T^n)_{ij} = (\, 1, 1, \ldots, 1 \,)\ T^n \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} . \tag{15.4}$$

We can also count the number of prime cycles and pruned periodic points, but in order not to break up the flow of the argument, we relegate these pretty results to sect. 15.7. Recommended reading if you ever have to compute lots of cycles.

A finite $[N \times N]$ matrix T has eigenvalues $\{\lambda_0, \lambda_1, \cdots, \lambda_{m-1}\}$ and (right) eigenvectors $\{\varphi_0, \varphi_1, \cdots, \varphi_{m-1}\}$ satisfying $T\varphi_\alpha = \lambda_\alpha \varphi_\alpha$. Expressing the initial vector in (15.4) in this basis (which might be incomplete, with $m \leq N$ eigenvectors),

$$T^n \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = T^n \sum_{\alpha=0}^{m-1} b_\alpha \varphi_\alpha = \sum_{\alpha=0}^{m-1} b_\alpha \lambda_\alpha^n \varphi_\alpha \,,$$

and contracting with $(1, 1, \ldots, 1)$, we obtain

$$K_n = \sum_{\alpha=0}^{m-1} c_\alpha \lambda_\alpha^n \,.$$

exercise 15.3

The constants c_α depend on the choice of initial and final partitions: In this example we are sandwiching T^n between the vector $(1, 1, \ldots, 1)$ and its transpose, but any other pair of vectors would do, as long as they are not orthogonal to the leading eigenvector φ_0. In an experiment the vector $(1, 1, \ldots, 1)$ would be replaced by a description of the initial state, and the right vector would describe the measurement time n later.

Perron theorem states that a Perron-Frobenius matrix has a nondegenerate (isolated) positive real eigenvalue $\lambda_0 > 1$ (with a positive eigenvector) which exceeds the moduli of all other eigenvalues. Therefore as n increases, the sum is dominated by the leading eigenvalue of the transition matrix, $\lambda_0 > |\mathrm{Re}\,\lambda_\alpha|$, $\alpha = 1, 2, \cdots, m-1$, and the topological entropy (15.1) is given by

$$\begin{aligned} h &= \lim_{n\to\infty} \frac{1}{n} \ln c_0 \lambda_0^n \left[1 + \frac{c_1}{c_0} \left(\frac{\lambda_1}{\lambda_0} \right)^n + \cdots \right] \\ &= \ln \lambda_0 + \lim_{n\to\infty} \left[\frac{\ln c_0}{n} + \frac{1}{n} \frac{c_1}{c_0} \left(\frac{\lambda_1}{\lambda_0} \right)^n + \cdots \right] \\ &= \ln \lambda_0 \,. \end{aligned} \qquad (15.5)$$

What have we learned? The transition matrix T is a one-step, *short time* operator, advancing the trajectory from one partition to the next admissible partition. Its eigenvalues describe the rate of growth of the total number of trajectories at the *asymptotic times*. Instead of painstakingly counting $K_1, K_2, K_3, \ldots$ and estimating (15.1) from a slope of a log-linear plot, we have the *exact* topological entropy if we can compute the leading eigenvalue of the transition matrix T. This is reminiscent of the way free energy is computed from transfer matrices for 1-dimensional lattice models with finite range interactions. Historically, it is this analogy with statistical mechanics that led to introduction of evolution operator methods into the theory of chaotic systems.

15.2 Topological trace formula

There are two standard ways of computing eigenvalues of a matrix - by evaluating the trace $\mathrm{tr}\, T^n = \sum \lambda_\alpha^n$, or by evaluating the determinant $\det(1 - zT)$. We start by evaluating the trace of transition matrices. The main lesson will be that the trace receives contributions only from itineraries that return to the initial partition, i.e., periodic orbits.

Consider an M-step memory transition matrix, like the 1-step memory example (14.10). The trace of the transition matrix counts the number of partitions that map into themselves. More generally, each closed walk through n concatenated entries of T contributes to $\mathrm{tr}\, T^n$ the product (15.2) of the matrix entries along the walk. Each step in such a walk shifts the symbolic string by one symbol; the trace ensures that the walk closes on a periodic string c. Define t_c to be the *local trace*, the product of matrix elements along a cycle c, each term being multiplied by a book keeping variable z. In chapters that follow, the 'local trace' t_c will take a continuum of values, so for the remainder of this chapter we stick to the 't_c' notation rather than to the 0 or z^n values specific to the counting problem.

The quantity $z^n \mathrm{tr}\, T^n$ is then the sum of t_c for all cycles of period n. The t_c = (product of matrix elements along cycle c is manifestly cyclically invariant, $t_{100} = t_{010} = t_{001}$, so a prime cycle p of period n_p contributes n_p times, once for each periodic point along its orbit. For the purposes of periodic orbit counting, the local trace takes values

$$t_p = \begin{cases} z^{n_p} & \text{if } p \text{ is an admissible cycle} \\ 0 & \text{otherwise,} \end{cases} \tag{15.6}$$

i.e., (setting $z = 1$) the local trace is $t_p = 1$ if the cycle is admissible, and $t_p = 0$ otherwise.

Example 15.2 ***Traces for binary symbolic dynamics.*** *For example, for the [8×8] transition matrix $T_{s_1 s_2 s_3, s_0 s_1 s_2}$ version of (14.10), or any refined partition [$2^n \times 2^n$] transition matrix, n arbitrarily large, the periodic point $\overline{100}$ contributes $t_{100} = z^3 T_{\overline{100},\overline{010}} T_{\overline{010},\overline{001}} T_{\overline{001},\overline{100}}$ to $z^3 \mathrm{tr}\, T^3$. This product is manifestly cyclically invariant, $t_{100} = t_{010} = t_{001}$, so a prime cycle $p = \overline{001}$ of period 3 contributes 3 times, once for each periodic point along its orbit.* exercise 11.7

For the binary labeled non–wandering set the first few traces are given by (consult tables 15.1 and 15.2)

$$\begin{aligned} z\, \mathrm{tr}\, T &= t_0 + t_1, \\ z^2 \mathrm{tr}\, T^2 &= t_0^2 + t_1^2 + 2t_{10}, \\ z^3 \mathrm{tr}\, T^3 &= t_0^3 + t_1^3 + 3t_{100} + 3t_{101}, \\ z^4 \mathrm{tr}\, T^4 &= t_0^4 + t_1^4 + 2t_{10}^2 + 4t_{1000} + 4t_{1001} + 4t_{1011}. \end{aligned} \tag{15.7}$$

In the binary case the trace picks up only two contributions on the diagonal, $T_{0\cdots0,0\cdots0} + T_{1\cdots1,1\cdots1}$, no matter how much memory we assume. We can even take infinite memory

Table 15.1: Prime cycles for the binary symbolic dynamics up to length 9. The numbers of prime cycles are given in table 15.3.

n_p	p	n_p	p	n_p	p	n_p	p	n_p	p
1	0	7	0001001	8	00001111	9	000001101	9	001001111
	1		0000111		00010111		000010011		001010111
2	01		0001011		00011011		000010101		001011011
3	001		0001101		00011101		000011001		001011101
	011		0010011		00100111		000100011		001100111
4	0001		0010101		00101011		000100101		001101011
	0011		0001111		00101101		000101001		001101101
	0111		0010111		00110101		000001111		001110101
5	00001		0011011		00011111		000010111		010101011
	00011		0011101		00101111		000011011		000111111
	00101		0101011		00110111		000011101		001011111
	00111		0011111		00111011		000100111		001101111
	01011		0101111		00111101		000101011		001110111
	01111		0110111		01010111		000101101		001111011
6	000001		0111111		01011011		000110011		001111101
	000011	8	00000001		00111111		000110101		010101111
	000101		00000011		01011111		000111001		010110111
	000111		00000101		01101111		001001011		010111011
	001011		00001001		01111111		001001101		001111111
	001101		00000111	9	000000001		001010011		010111111
	001111		00001011		000000011		001010101		011011111
	010111		00001101		000000101		000011111		011101111
	011111		00010011		000001001		000101111		011111111
7	0000001		00010101		000010001		000110111		
	0000011		00011001		000000111		000111011		
	0000101		00100101		000001011		000111101		

Table 15.2: The total numbers N_n of periodic points of period n for binary symbolic dynamics. The numbers of contributing prime cycles illustrates the preponderance of long prime cycles of period n over the repeats of shorter cycles of periods n_p, where $n = rn_p$. Further enumerations of binary prime cycles are given in tables 15.1 and 15.3. (L. Rondoni)

n	N_n	# of prime cycles of period n_p									
		1	2	3	4	5	6	7	8	9	10
1	2	2									
2	4	2	1								
3	8	2		2							
4	16	2	1		3						
5	32	2				6					
6	64	2	1	2			9				
7	128	2						18			
8	256	2	1		3				30		
9	512	2		2						56	
10	1024	2	1			6					99

$M \to \infty$, *in which case the contributing partitions are shrunk to the fixed points,* $\mathrm{tr}\,T = T_{\overline{0},\overline{0}} + T_{\overline{1},\overline{1}}$.

If there are no restrictions on symbols, the symbolic dynamics is complete, and all *binary sequences are* admissible *(or* allowable*) itineraries. As this type of symbolic dynamics pops up frequently, we list the shortest binary prime cycles in table 15.1.* exercise 11.2

Hence $\mathrm{tr}\,T^n = N_n$ counts the number of *admissible periodic points* of period n. The nth order trace (15.7) picks up contributions from all repeats of prime cycles, with each cycle contributing n_p periodic points, so N_n, the total number of periodic points of period n is given by

$$z^n N_n = z^n \mathrm{tr}\,T^n = \sum_{n_p|n} n_p t_p^{n/n_p} = \sum_p n_p \sum_{r=1}^{\infty} \delta_{n,n_p r} t_p^r \,. \tag{15.8}$$

Here $m|n$ means that m is a divisor of n. An example is the periodic orbit counting in table 15.2.

In order to get rid of the awkward divisibility constraint $n = n_p r$ in the above sum, we introduce the generating function for numbers of periodic points

$$\sum_{n=1}^{\infty} z^n N_n = \mathrm{tr}\,\frac{zT}{1-zT} \,. \tag{15.9}$$

The right hand side is the geometric series sum of $N_n = \mathrm{tr}\,T^n$. Substituting (15.8) into the left hand side, and replacing the right hand side by the eigenvalue sum $\mathrm{tr}\,T^n = \sum \lambda_\alpha^n$, we obtain our first example of a trace formula, the *topological trace formula*

$$\sum_{\alpha=0} \frac{z\lambda_\alpha}{1-z\lambda_\alpha} = \sum_p \frac{n_p t_p}{1-t_p} \,. \tag{15.10}$$

A trace formula relates the spectrum of eigenvalues of an operator - here the transition matrix - to the spectrum of periodic orbits of a dynamical system. It is a statement of duality between the short-time, local information - in this case the next admissible symbol in a symbol sequence - to long-time, global averages, in this case the mean rate of growth of the number of cycles with increasing cycle period.

The z^n sum in (15.9) is a discrete version of the Laplace transform (see sect. 18.1.2), and the resolvent on the left hand side is the antecedent of the more sophisticated trace formulas (18.10) and (18.23).We shall now use this result to compute the spectral determinant of the transition matrix.

15.3 Determinant of a graph

Our next task is to determine the zeros of the *spectral determinant* of an $[m \times m]$ transition matrix

$$\det(1 - zT) = \prod_{\alpha=0}^{m-1} (1 - z\lambda_\alpha) \,. \tag{15.11}$$

We could now proceed to diagonalize T on a computer, and get this over with. It pays, however, to dissect $\det(1-zT)$ with some care; understanding this computation in detail will be the key to understanding the cycle expansion computations of chapter 20 for arbitrary dynamical averages. For T a finite matrix, (15.11) is just the characteristic polynomial for T. However, we shall be able to compute this object even when the dimension of T and other such operators becomes infinite, and for that reason we prefer to refer to (15.11) loosely as the "spectral determinant."

There are various definitions of the determinant of a matrix; we will view the determinant as a sum over all possible permutation cycles composed of the traces $\mathrm{tr}\, T^k$, in the spirit of the determinant–trace relation (1.16): exercise 4.1

$$\begin{aligned}
\det(1 - zT) &= \exp(\mathrm{tr}\, \ln(1 - zT)) = \exp\left(-\sum_{n=1} \frac{z^n}{n} \mathrm{tr}\, T^n\right) \\
&= 1 - z\, \mathrm{tr}\, T - \frac{z^2}{2}\left((\mathrm{tr}\, T)^2 - \mathrm{tr}\, T^2\right) - \ldots
\end{aligned} \tag{15.12}$$

This is sometimes called a *cumulant* expansion. Formally, the right hand is a Taylor series in z about $z = 0$. If T is an $[m \times m]$ finite matrix, then the characteristic polynomial is at most of order m. In that case the coefficients of z^n must vanish *exactly* for $n > m$.

We now proceed to relate the determinant in (15.12) to the corresponding transition graph of chapter 14: toward this end, we start with the usual textbook expression for a determinant as the sum of products of all permutations

$$\det M = \sum_{\{\pi\}} (-1)^\pi M_{1,\pi_1} M_{2,\pi_2} \cdots M_{m,\pi_m} \tag{15.13}$$

where $M = 1 - zT$ is a $[m \times m]$ matrix, $\{\pi\}$ denotes the set of permutations of m symbols, π_k is the permutation π applied to k, and $(-1)^\pi = \pm 1$ is the parity of permutation π. The right hand side of (15.13) yields a polynomial in T of order m in z: a contribution of order n in z picks up $m - n$ unit factors along the diagonal, the remaining matrix elements yielding

$$(-z)^n (-1)^\pi T_{s_1 \pi s_1} \cdots T_{s_n \pi s_n} \tag{15.14}$$

where π is the permutation of the subset of n distinct symbols $s_1 \cdots s_n$ indexing T matrix elements. As in (15.7), we refer to any combination $t_c = T_{s_1 s_k} T_{s_3 s_2} \cdots T_{s_2 s_1}$, for a given itinerary $c = s_1 s_2 \cdots s_k$, as the *local trace* associated with a closed loop c on the transition graph. Each term of the form (15.14) may be factored in terms of local traces $t_{c_1} t_{c_2} \cdots t_{c_k}$, that is loops on the transition graph. These loops are non-intersecting, as each node may only be reached by *one* link, and they are indeed loops, as if a node is reached by a link, it has to be the starting point of another *single* link, as each s_j must appear exactly *once* as a row and column index.

So the general structure is clear, a little more thinking is only required to get the sign of a generic contribution. We consider only the case of loops of length 1 and 2, and leave to the reader the task of generalizing the result by induction. Consider first a term in which only loops of unit length appear in (15.14), i.e., only the diagonal elements of T are picked up. We have $k = m$ loops and an even permutation π so the sign is given by $(-1)^k$, where k is the number of loops. Now take the case in which we have i single loops and j loops of length $n = 2j + i$. The parity of the permutation gives $(-1)^j$ and the first factor in (15.14) gives $(-1)^n = (-1)^{2j+i}$. So once again these terms combine to $(-1)^k$, where $k = i + j$ is the number of loops. Let f be the maximal number of non-intersecting loops. We may summarize our findings as follows: exercise 15.4

> The characteristic polynomial of a transition matrix is given by the sum of all possible partitions π of the corresponding transition graph into products of k non-intersecting loops, with each loop trace t_p carrying a minus sign:

$$\det(1 - zT) = \sum_{k=0}^{f} \sum_{\pi}{}' (-1)^k t_{p_1} \cdots t_{p_k} \qquad (15.15)$$

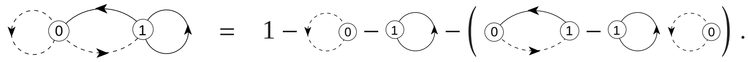
Any self-intersecting loop is *shadowed* by a product of two loops that share the intersection point. As both the long loop t_{ab} and its shadow $t_a t_b$ in the case at hand carry the same weight $z^{n_a+n_b}$, the cancelation is exact, and the loop expansion (15.15) is finite. In the case that the local traces count prime cycles (15.6), $t_p = 0$ or z^n, we refer to $\det(1 - zT)$ as the *topological polynomial*.

We refer to the set of all non-self-intersecting loops $\{t_{p_1}, t_{p_2}, \cdots t_{p_f}\}$ as the *fundamental cycles* (for an explicit example, see the loop expansion of example 15.6). This is not a very good definition, as transition graphs are not unique –the most we know is that for a given finite-grammar language, there exist transition graph(s) with the minimal number of loops. Regardless of how cleverly a transition graph is constructed, it is always true that for any finite transition graph the number of fundamental cycles f is finite. If the graph has m nodes, no fundamental cycle is of period longer than m, as any longer cycle is of necessity self-intersecting.

The above loop expansion of a determinant in terms of traces is most easily grasped by working through a few examples. The complete binary dynamics transition graph of figure 14.4 is a little bit too simple, but let us start humbly and consider it anyway.

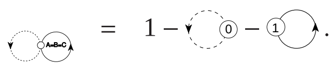
Example 15.3 ***Topological polynomial for complete binary dynamics:*** *(continuation of example 14.2) There are only two non-intersecting loops, yielding*

$$\det(1 - zT) = 1 - t_0 - t_1 - (t_{01} - t_0 t_1) = 1 - 2z \qquad (15.16)$$



Due to the symmetry under $0 \leftrightarrow 1$ *interchange, this is a redundant graph (the 2-cycle* t_{01} *is exactly shadowed by the 1-cycles). Another way to see is that itineraries are labeled by the* $\{0, 1\}$ *links, node labels can be omitted. As both nodes have 2 in-links and 2 out-links, they can be identified, and a more economical presentation is in terms of the* $[1\times1]$ *adjacency matrix (14.12)*

$$\det(1 - zA) = 1 - t_0 - t_1 = 1 - 2z \qquad (15.17)$$



The leading (and only) zero of this characteristic polynomial yields the topological entropy $e^h = 2$. *As there are* $K_n = 2^n$ *binary strings of length* N, *this comes as no surprise.*

Similarly, for the complete symbolic dynamics of N symbols the transition graph has one node and N links, yielding

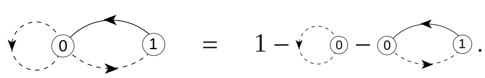
$$\det(1 - zT) = 1 - Nz, \qquad (15.18)$$

which gives the topological entropy $h = \ln N$.

Example 15.4 ***Golden mean pruning:*** *The "golden mean" pruning of example 14.5 has one grammar rule: the substring* $\_11\_$ *is forbidden. The corresponding transition graph non-intersecting loops are of length 1 and 2, so the topological polynomial is given by*

exercise 15.5

$$\det(1 - zT) = 1 - t_0 - t_{01} = 1 - z - z^2 \qquad (15.19)$$



The leading root of this polynomial is the golden mean, so the entropy (15.5) is the logarithm of the golden mean, $h = \ln \frac{1+\sqrt{5}}{2}$.

fast track:
sect. 15.4, p. 295

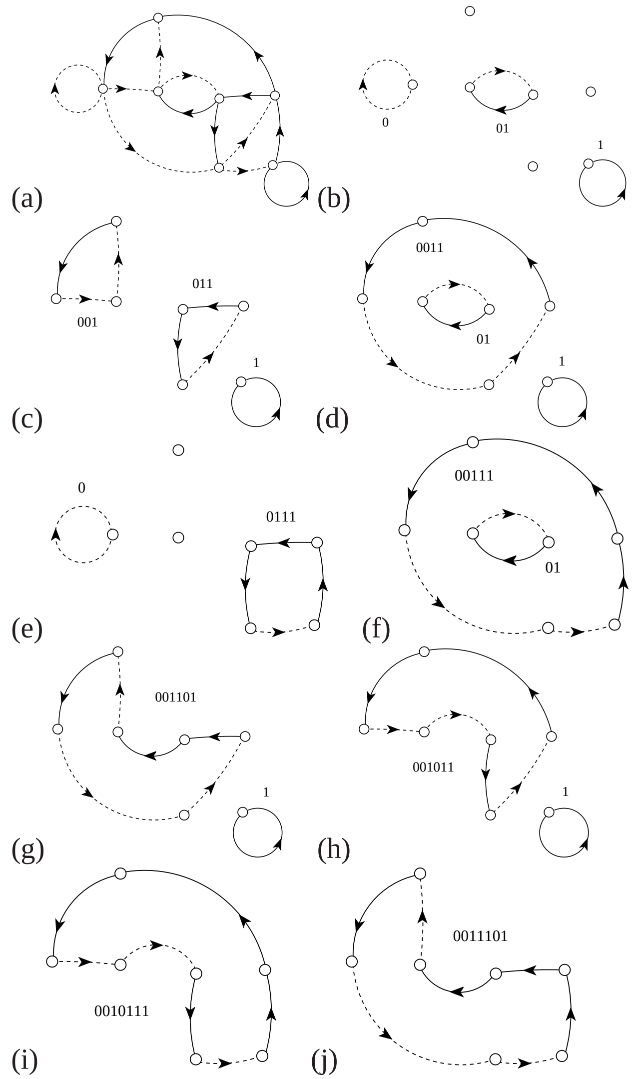


Figure 15.1: (a) The region labels in the nodes of transition graph figure 14.3 can be omitted, as the links alone keep track of the symbolic dynamics. (b)-(j) The fundamental cycles (15.23) for the transition graph (a), i.e., the set of its non-self-intersecting loops. Each loop represents a local trace t_p, as in (14.5).

Example 15.5 ***Nontrivial pruning:*** *The non-self-intersecting loops of the transition graph of figure 14.6 (d) are indicated in figure 14.6 (e). The determinant can be written down by inspection, as the sum of all possible partitions of the graph into products of non-intersecting loops, with each loop carrying a minus sign:*

$$\det(1 - zT) = 1 - t_0 - t_{0011} - t_{0001} - t_{00011} + t_0 t_{0011} + t_{0011} t_{0001} \,. \tag{15.20}$$

With $t_p = z^{n_p}$, where n_p is the period of the p-cycle, the smallest root of

$$0 = 1 - z - 2z^4 + z^8 \tag{15.21}$$

yields the topological entropy $h = -\ln z$, $z = 0.658779\ldots$, $h = 0.417367\ldots$, significantly smaller than the entropy of the covering symbolic dynamics, the complete binary shift with topological entropy $h = \ln 2 = 0.693\ldots$ exercise 15.9

Example 15.6 ***Loop expansion of a transition graph.*** *(continued from example 14.7) Consider a state space covered by 7 neighborhoods (14.11), with the topological time evolution given by the transition graph of figure 14.3.*

The determinant $\det(1-zT)$ *of the transition graph in figure 14.3 can be read off the graph, and expanded as a polynomial in* z*, with coefficients given by products of non-intersecting loops (traces of powers of* T*) of the transition graph figure 15.1:*

$$\begin{aligned}\det(1-zT) = 1 - &(t_0+t_1)z - (t_{01}-t_0t_1)z^2 - (t_{001}+t_{011}-t_{01}t_0-t_{01}t_1)z^3 \\ &-(t_{0011}+t_{0111}-t_{001}t_1-t_{011}t_0-t_{011}t_1+t_{01}t_0t_1)z^4 \\ &-(t_{00111}-t_{0111}t_0-t_{0011}t_1+t_{011}t_0t_1)z^5 \\ &-(t_{001011}+t_{001101}-t_{0011}t_{01}-t_{001}t_{011})z^6 \\ &-(t_{0010111}+t_{0011101}-t_{001011}t_1-t_{001101}t_1-t_{00111}t_{01}+t_{0011}t_{01}t_1+t_{001}t_{011}t_1)z^7 .\end{aligned} \tag{15.22}$$

Twelve cycles up to period 7 are fundamental cycles:

$$\overline{0}, \overline{1}, \overline{01}, \overline{001}, \overline{011}, \overline{0011}, \overline{0111}, \overline{00111}, \overline{001011}, \overline{001101}, \overline{0010111}, \overline{0011101}, \tag{15.23}$$

out of the total of 41 prime cycles (listed in table 15.1) up to cycle period 7. The topological polynomial $t_p \to z^{n_p}$

$$1/\zeta_{top}(z) = 1 - 2z + z^7$$

is interesting; the shadowing fails first at the cycle length $n = 7$*, so the topological entropy is only a bit smaller than the binary* $h = \ln 2$*. Not exactly obvious from the partition (14.11).*

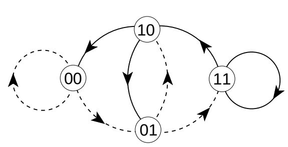
15.4 Topological zeta function

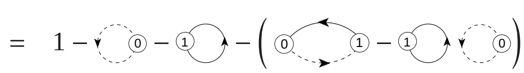
What happens if there is no finite-memory transition matrix, if the transition graph is infinite? If we are never sure that looking further into the future will reveal no further forbidden blocks? There is still a way to define the determinant, and this idea is central to the whole treatise: the determinant is then defined by its *cumulant* expansion (15.12)

exercise 4.1

$$\det(1-zT) = 1 - \sum_{n=1}^{\infty} \hat{c}_n z^n . \tag{15.24}$$

Example 15.7 ***Complete binary*** $\det(1-zT)$ ***expansion.*** *(continuation of example 14.6) consider the loop expansion of the binary 1-step memory transition graph (14.10)*





$$\begin{aligned} &= 1 - t_0 - t_1 - [(t_{01} - t_1t_0)] - [(t_{001} - t_{01}t_0) + (t_{011} - t_{01}t_1)] \\ &\quad -[(t_{0001} - t_0t_{001}) + (t_{0111} - t_{011}t_1) \\ &\qquad +(t_{0011} - t_{001}t_1 - t_0t_{011} + t_0t_{01}t_1)] \\ &= 1 - \sum_f t_f - \sum_n \hat{c}_n = 1 - 2z . \end{aligned} \tag{15.25}$$

For finite dimensional matrices the expansion is a finite polynomial, and (15.24) is an identity; however, for infinite dimensional operators the cumulant expansion coefficients $\hat{c}_n$ *define* the determinant.

Let us now evaluate the determinant in terms of traces for an arbitrary transition matrix. In order to obtain an expression for the spectral determinant (15.11) in terms of cycles, substitute (15.8) into (15.24) and sum over the repeats of prime cycles using $\ln(1-x) = -\sum_r x^r/r$,

$$\begin{aligned} \det(1 - zT) &= \exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{t_p^r}{r}\right) = \exp\left(\sum_p \ln(1 - t_p)\right) \\ &= \prod_p (1 - t_p)\,, \end{aligned} \tag{15.26}$$

where for the topological entropy the weight assigned to a prime cycle p of period n_p is $t_p = z^{n_p}$ if the cycle is admissible, or $t_p = 0$ if it is pruned. This determinant is called the *topological* or the *Artin-Mazur* zeta function, conventionally denoted by

$$1/\zeta_{\mathrm{top}}(z) = \prod_p (1 - z^{n_p}) = 1 - \sum_{n=1} \hat{c}_n z^n \;. \tag{15.27}$$

Counting cycles amounts to giving each admissible prime cycle p weight $t_p = z^{n_p}$ and expanding the Euler product (15.27) as a power series in z. As the precise expression for the coefficients $\hat{c}_n$ in terms of local traces t_p is more general than the current application to counting, we shall postpone its derivation to chapter 20.

The topological entropy h can now be determined from the leading zero $z = e^{-h}$ of the topological zeta function. For a finite $[m \times m]$ transition matrix, the number of terms in the characteristic equation (15.15) is finite, and we refer to this expansion as the *topological polynomial* of order $\leq m$. The utility of defining the determinant by its cumulant expansion is that it works even when the partition is infinite, $m \to \infty$; an example is given in sect. 15.5, and many more later on.

fast track:
sect. 15.5, p. 297

15.4.1 Topological zeta function for flows

We now apply the method that we shall use in deriving (18.23) to the problem of deriving the topological zeta functions for flows. The time-weighted density of prime cycles of period t is

$$\Gamma(t) = \sum_p \sum_{r=1} T_p\, \delta(t - rT_p)\,. \tag{15.28}$$

The Laplace transform smooths the sum over Dirac delta spikes (see (18.22)) and yields the *topological trace formula*

$$\sum_p \sum_{r=1} T_p \int_{0_+}^{\infty} dt\, e^{-st}\, \delta(t - rT_p) = \sum_p T_p \sum_{r=1}^{\infty} e^{-sT_p r} \qquad (15.29)$$

and the *topological zeta function* for flows:

$$1/\zeta_{\mathrm{top}}(s) \;=\; \prod_p \left(1 - e^{-sT_p}\right) , \qquad (15.30)$$

related to the trace formula by

$$\sum_p T_p \sum_{r=1}^{\infty} e^{-sT_p r} \;=\; -\frac{\partial}{\partial s} \ln 1/\zeta_{\mathrm{top}}(s) \, .$$

This is the continuous time version of the discrete time topological zeta function (15.27) for maps; its leading zero $s = -h$ yields the topological entropy for a flow.

15.5 Topological zeta function for an infinite partition

(K.T. Hansen and P. Cvitanović)

To understand the need for topological zeta function (15.24), we turn a dynamical system with (as far as we know - there is no proof) an infinite partition, or an infinity of ever-longer pruning rules. Consider the $1-dimensional$ quadratic map (11.3)

$$f(x) = Ax(1 - x) \, , \qquad A = 3.8 \, .$$

Numerically the kneading sequence (the itinerary of the critical point $x = 1/2$ (11.13)) is exercise 15.20

$$K = 1011011110110111101011110111110 \ldots$$

where the symbolic dynamics is defined by the partition of figure 11.12. How this kneading sequence is converted into a series of pruning rules is a dark art.For the moment it suffices to state the result, to give you a feeling for what a "typical" infinite partition topological zeta function looks like. For example, approximating the dynamics by a transition graph corresponding to a repeller of the period 29

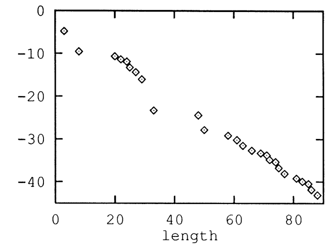


Figure 15.2: The logarithm $\ln|z_0^{(n)} - z_0|$ of the difference between the leading zero of the n-th polynomial approximation to topological zeta function and our best estimate (15.33), as a function of order of the polynomial n (the topological zeta function evaluated for the closest value of A to $A = 3.8$ for which the quadratic map has a stable cycle of period n). (from K.T. Hansen [12.20])

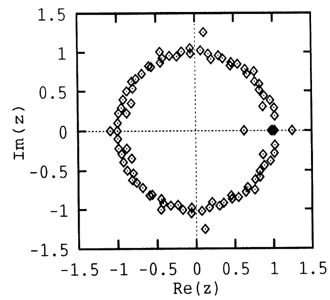


Figure 15.3: The 90 zeroes of the topological zeta function for the quadratic map for $A = 3.8$ approximated by the nearest topological zeta function with a stable cycle of length 90. (from K.T. Hansen [12.20])

attractive cycle close to the $A = 3.8$ strange attractor yields a transition graph with 29 nodes and the characteristic polynomial

$$\begin{aligned} 1/\zeta_{\text{top}}^{(29)} &= 1 - z^1 - z^2 + z^3 - z^4 - z^5 + z^6 - z^7 + z^8 - z^9 - z^{10} \\ &\quad + z^{11} - z^{12} - z^{13} + z^{14} - z^{15} + z^{16} - z^{17} - z^{18} + z^{19} + z^{20} \\ &\quad - z^{21} + z^{22} - z^{23} + z^{24} + z^{25} - z^{26} + z^{27} - z^{28} \,. \end{aligned} \tag{15.31}$$

The smallest real root of this approximate topological zeta function is

$$z = 0.62616120 \ldots \tag{15.32}$$

Constructing finite transition graphs of increasing length corresponding to $A \to 3.8$ we find polynomials with better and better estimates for the topological entropy. For the closest stable period 90 orbit we obtain our best estimate of the topological entropy of the repeller:

$$h = -\ln 0.62616130424685 \ldots = 0.46814726655867 \ldots . \tag{15.33}$$

Figure 15.2 illustrates the convergence of the truncation approximations to the topological zeta function as a plot of the logarithm of the difference between the zero of a polynomial and our best estimate (15.33), plotted as a function of the period of the stable periodic orbit. The error of the estimate (15.32) is expected to be of order $z^{29} \approx e^{-14}$ because going from period 28 to a longer truncation typically yields combinations of loops with 29 and more nodes giving terms $\pm z^{29}$ and of higher order in the polynomial. Hence the convergence is exponential, with an exponent of $-0.47 = -h$, the topological entropy itself. In figure 15.3

we plot the zeroes of the polynomial approximation to the topological zeta function obtained by accounting for all forbidden strings of length 90 or less. The leading zero giving the topological entropy is the point closest to the origin. Most of the other zeroes are close to the unit circle; we conclude that for infinite state space partitions the topological zeta function has a unit circle as the radius of convergence. The convergence is controlled by the ratio of the leading to the next-to-leading eigenvalues, which is in this case indeed $\lambda_1/\lambda_0 = 1/e^h = e^{-h}$.

15.6 Shadowing

The topological zeta function is a pretty function, but the infinite product (15.26) should make you pause. For finite transition matrices the left hand side is a determinant of a finite matrix, therefore a finite polynomial; so why is the right hand side an infinite product over the infinitely many prime periodic orbits of all periods?

The way in which this infinite product rearranges itself into a finite polynomial is instructive, and crucial for all that follows. You can already take a peek at the full cycle expansion (20.7) of chapter 20; all cycles beyond the fundamental t_0 and t_1 appear in the shadowing combinations such as

$$t_{s_1 s_2 \cdots s_n} - t_{s_1 s_2 \cdots s_m} t_{s_{m+1} \cdots s_n} .$$

For subshifts of finite type such shadowing combinations cancel *exactly*, if we are counting cycles as we do in (15.16) and (15.25), or if the dynamics is piecewise linear, as in exercise 19.3. As we argue in sect. 1.5.4, for nice hyperbolic flows whose symbolic dynamics is a subshift of finite type, the shadowing combinations *almost* cancel, and the spectral determinant is dominated by the fundamental cycles from (15.15), with longer cycles contributing only small “curvature” corrections.

These exact or nearly exact cancelations depend on the flow being smooth and the symbolic dynamics being a subshift of finite type. If the dynamics requires an infinite state space partition, with pruning rules for blocks of increasing length, most of the shadowing combinations still cancel, but the few corresponding to new forbidden blocks do not, leading to a finite radius of convergence for the spectral determinant, as depicted in figure 15.3.

One striking aspect of the pruned cycle expansion (15.31) compared to the trace formulas such as (15.9) is that coefficients are not growing exponentially - indeed they all remain of order 1, so instead having a radius of convergence e^{-h}, in the example at hand the topological zeta function has the unit circle as the radius of convergence. In other words, exponentiating the spectral problem from a trace formula to a spectral determinant as in (15.24) increases the *analyticity domain*: the pole in the trace (15.10) at $z = e^{-h}$ is promoted to a smooth zero of the spectral determinant with a larger radius of convergence.

This sensitive dependence of spectral determinants on whether or not the symbolic dynamics is a subshift of finite type is bad news. If the system is generic and not structurally stable (see sect. 12.2), a smooth parameter variation is in no sense a smooth variation of topological dynamics - infinities of periodic orbits are created or destroyed, and transition graphs go from being finite to infinite and back. That will imply that the global averages that we intend to compute are generically nowhere differentiable functions of the system parameters, and averaging over families of dynamical systems can be a highly nontrivial enterprise; a simple illustration is the parameter dependence of the diffusion constant computed in a remark in chapter 25.

You might well ask: What is wrong with computing the entropy from (15.1)? Does all this theory buy us anything? An answer: If we count K_n level by level, we ignore the self-similarity of the pruned tree - examine for example figure 14.5, or the cycle expansion of (15.35) - and the finite estimates of $h_n = \ln K_n/n$ converge nonuniformly to h, and on top of that with a slow rate of convergence, $|h - h_n| \approx O(1/n)$ as in (15.5). The determinant (15.11) is much smarter, as by construction it encodes the self-similarity of the dynamics, and yields the asymptotic value of h with no need for any finite n extrapolations.

fast track:
sect. 16, p. 310

15.7 Counting cycles

In what follows, we shall occasionally need to compute all cycles up to topological period n, so it is important to know their exact number. The formulas are fun to derive, but a bit technical for plumber on the street, and probably best skipped on the first reading.

15.7.1 Counting periodic points

The number of periodic points of period n is denoted N_n. It can be computed from (15.24) and (15.9) as a logarithmic derivative of the topological zeta function

$$\begin{aligned} \sum_{n=1} N_n z^n &= \operatorname{tr}\left(-z\frac{d}{dz}\ln(1-zT)\right) = -z\frac{d}{dz}\ln\det\,(1-zT) \\ &= \frac{-z\frac{d}{dz}(1/\zeta_{\mathrm{top}})}{1/\zeta_{\mathrm{top}}}\,. \end{aligned} \tag{15.34}$$

Observe that the trace formula (15.10) diverges at $z \to e^{-h}$, because the denominator has a simple zero there.

Table 15.3: Number of prime cycles for various alphabets and grammars up to period 10. The first column gives the cycle period, the second gives the formula (15.37) for the number of prime cycles for complete N-symbol dynamics, and columns three through five give the numbers of prime cycles for $N = 2, 3$ and 4.

n	$M_n(N)$	$M_n(2)$	$M_n(3)$	$M_n(4)$
1	N	2	3	4
2	$N(N-1)/2$	1	3	6
3	$N(N^2-1)/3$	2	8	20
4	$N^2(N^2-1)/4$	3	18	60
5	$(N^5-N)/5$	6	48	204
6	$(N^6-N^3-N^2+N)/6$	9	116	670
7	$(N^7-N)/7$	18	312	2340
8	$N^4(N^4-1)/8$	30	810	8160
9	$N^3(N^6-1)/9$	56	2184	29120
10	$(N^{10}-N^5-N^2+N)/10$	99	5880	104754

Example 15.8 ***Complete N-ary dynamics:*** *To check formula (15.34) for the finite-grammar situation, consider the complete N-ary dynamics (14.7) for which the number of periodic points of period n is simply $\mathrm{tr}\, T_c^n = N^n$. Substituting*

$$\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}\, T_c^n = \sum_{n=1}^{\infty} \frac{(zN)^n}{n} = -\ln(1-zN)\,,$$

into (15.24) we verify (15.18). The logarithmic derivative formula (15.34) in this case does not buy us much either, it simply recovers

$$\sum_{n=1} N_n z^n = \frac{Nz}{1-Nz}\,.$$

Example 15.9 ***Nontrivial pruned dynamics:*** *Consider the pruning of figure 14.6 (e). Substituting (15.34) we obtain*

$$\sum_{n=1} N_n z^n = \frac{z+8z^4-8z^8}{1-z-2z^4+z^8}\,. \qquad (15.35)$$

The topological zeta function is not merely a tool for extracting the asymptotic growth of N_n; it actually yields the exact numbers of periodic points. In case at hand it yields a nontrivial recursive formula $N_1 = N_2 = N_3 = 1$, $N_n = 2n+1$ for $n = 4,5,6,7,8$, and $N_n = N_{n-1} + 2N_{n-4} - N_{n-8}$ for $n > 8$.

15.7.2 Counting prime cycles

Having calculated the number of periodic points, our next objective is to evaluate the number of *prime* cycles M_n for a dynamical system whose symbolic dynamics is built from N symbols. The problem of finding M_n is classical in combinatorics (counting necklaces made out of n beads of N different kinds) and is easily solved. There are N^n possible distinct strings of length n composed of N letters. These N^n strings include all M_d prime d-cycles whose period d equals or divides n. A prime cycle is a non-repeating symbol string: for example, $p = \overline{011} = \overline{101} = \overline{110} = \ldots 011011 \ldots$ is prime, but $\overline{0101} = 010101 \ldots = \overline{01}$ is not. A prime d-cycle contributes d strings to the sum of all possible strings, one for each cyclic permutation. The total number of possible periodic symbol sequences of period n is therefore related to the number of prime cycles by

$$N_n = \sum_{d|n} d M_d \,, \tag{15.36}$$

where N_n equals $\mathrm{tr}\, T^n$. The number of prime cycles can be computed recursively

$$M_n = \frac{1}{n}\left(N_n - \sum_{d|n}^{d<n} d M_d \right) ,$$

or by the *Möbius inversion formula*

exercise 15.10

$$M_n = n^{-1} \sum_{d|n} \mu\left(\frac{n}{d}\right) N_d \,. \tag{15.37}$$

where the Möbius function $\mu(1) = 1$, $\mu(n) = 0$ if n has a squared factor, and $\mu(p_1 p_2 \ldots p_k) = (-1)^k$ if all prime factors are different.

We list the number of prime cycles up to period 10 for 2-, 3- and 4-letter complete symbolic dynamics in table 15.3, obtained by Möbius inversion (15.37).

exercise 15.11

Example 15.10 ***Counting N-disk periodic points:*** *A simple example of pruning is the exclusion of "self-bounces" in the N-disk game of pinball. The number of points that are mapped back onto themselves after n iterations is given by $N_n = \mathrm{tr}\, T^n$. The pruning of self-bounces eliminates the diagonal entries, $T_{N-disk} = T_c - \mathbf{1}$, so the number of the N-disk periodic points is*

$$N_n = \mathrm{tr}\, T^n_{N-disk} = (N-1)^n + (-1)^n (N-1) \,. \tag{15.38}$$

Here T_c is the complete symbolic dynamics transition matrix (14.7). For the N-disk pruned case (15.38), Möbius inversion (15.37) yields

$$\begin{aligned} M_n^{N-disk} &= \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) (N-1)^d + \frac{N-1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) (-1)^d \\ &= M_n^{(N-1)} \quad \text{for} \quad n > 2 \,. \end{aligned} \tag{15.39}$$

There are no fixed points, so $M_1^{N-disk} = 0$. The number of periodic points of period 2 is $N^2 - N$, hence there are $M_2^{N-disk} = N(N-1)/2$ prime cycles of period 2; for periods $n > 2$, the number of prime cycles is the same as for the complete $(N-1)$-ary dynamics of table 15.3.

Table 15.4: List of 3-disk prime cycles up to period 10. Here n is the cycle period, M_n is the number of prime cycles, N_n is the number of periodic points, and S_n the number of distinct prime cycles under D_3 symmetry (see chapter 21 for further details). Column 3 also indicates the splitting of N_n into contributions from orbits of periods that divide n. The prefactors in the fifth column indicate the degeneracy m_p of the cycle; for example, 3·12 stands for the three prime cycles $\overline{12}$, $\overline{13}$ and $\overline{23}$ related by $2\pi/3$ rotations. Among symmetry-related cycles, a representative $\hat{p}$ which is lexically lowest is listed. The cycles of period 9 grouped with parentheses are related by time reversal symmetry, but not by any D_3 transformation.

n	M_n	N_n	S_n	$m_p \cdot \hat{p}$
1	0	0	0	
2	3	6=3·2	1	3·12
3	2	6=2·3	1	2·123
4	3	18=3·2+3·4	1	3·1213
5	6	30=6·5	1	6·12123
6	9	66=3·2+2·3+9·6	2	6·121213 + 3·121323
7	18	126=18·7	3	6·1212123 + 6·1212313 + 6·1213123
8	30	258=3·2+3·4+30·8	6	6·12121213 + 3·12121313 + 6·12121323
				+ 6·12123123 + 6·12123213 + 3·12132123
9	56	510=2·3+56·9	10	6·121212123 + 6·(121212313 + 121212323)
				+ 6·(121213123 + 121213213) + 6·121231323
				+ 6·(121231213 + 121232123) + 2·121232313
				+ 6·121321323
10	99	1022	18	

Example 15.11 ***Pruning individual cycles:*** *Consider the 3-disk game of pinball. The prohibition of repeating a symbol affects counting only for the fixed points and the 2-cycles. Everything else is the same as counting for a complete binary dynamics (15.39). To obtain the topological zeta function, just divide out the binary 1- and 2-cycles* $(1 - zt_0)(1 - zt_1)(1 - z^2 t_{01})$ *and multiply with the correct 3-disk 2-cycles* $(1 - z^2 t_{12})(1 - z^2 t_{13})(1 - z^2 t_{23})$*:* exercise 15.14 exercise 15.15

$$
\begin{aligned}
1/\zeta_{3-disk} &= (1-2z)\frac{(1-z^2)^3}{(1-z)^2(1-z^2)} \\
&= (1-2z)(1+z)^2 = 1 - 3z^2 - 2z^3 \,. \qquad (15.40)
\end{aligned}
$$

The factorization reflects the underlying 3-disk symmetry; we shall rederive it in (21.25). As we shall see in chapter 21, symmetries lead to factorizations of topological polynomials and topological zeta functions.

Example 15.12 ***Alphabet*** $\{a, cb^k;\ \overline{b}\}$***:*** *(continuation of exercise 15.16) In the cycle counting case, the dynamics in terms of* $a \to z$, $cb^k \to z + z^2 + z^3 + \cdots = z/(1-z)$ *is a complete binary dynamics with the explicit fixed point factor* $(1 - t_b) = (1 - z)$*:* exercise 15.19

$$
1/\zeta_{top} = (1-z)\left(1 - z - \frac{z}{1-z}\right) = 1 - 3z + z^2 \,.
$$

Table 15.5: The 4-disk prime cycles up to period 8. The symbols is the same as shown in table 15.4. Orbits related by time reversal symmetry (but no C_{4v} symmetry) already appear at cycle period 5. Cycles of period 7 and 8 have been omitted.

n	M_n	N_n	S_n	$m_p \cdot \hat{p}$
1	0	0	0	
2	6	12=6·2	2	4·12 + 2·13
3	8	24=8·3	1	8·123
4	18	84=6·2+18·4	4	8·1213 + 4·1214 + 2·1234 + 4·1243
5	48	240=48·5	6	8·(12123 + 12124) + 8·12313
				+ 8·(12134 + 12143) + 8·12413
6	116	732=6·2+8·3+116·6	17	8·121213 + 8·121214 + 8·121234
				+ 8·121243 + 8·121313 + 8·121314
				+ 4·121323 + 8·(121324 + 121423)
				+ 4·121343 + 8·121424 + 4·121434
				+ 8·123124 + 8·123134 + 4·123143
				+ 4·124213 + 8·124243
7	312	2184	39	
8	810	6564	108	

Résumé

The main result of this chapter is the cycle expansion (15.27) of the topological zeta function (i.e., the spectral determinant of the transition matrix):

$$1/\zeta_{\mathrm{top}}(z) = 1 - \sum_{k=1} \hat{c}_k z^k .$$

For subshifts of finite type, the transition matrix is finite, and the topological zeta function is a finite polynomial evaluated by the loop expansion (15.15) of $\det(1 - zT)$. For infinite grammars the topological zeta function is defined by its cycle expansion. The topological entropy h is given by the leading zero $z = e^{-h}$. This expression for the entropy is *exact*; in contrast to the initial definition (15.1), no $n \to \infty$ extrapolations of $\ln K_n/n$ are required.

What have we accomplished? We have related the number of topologically distinct paths from one state space region to another region to the leading eigenvalue of the transition matrix T. The spectrum of T is given by topological zeta function, a certain sum over traces $\mathrm{tr}\, T^n$, and in this way the periodic orbit theory has entered the arena through the trace formula (15.10), already at the level of the topological dynamics.

The main lesson of learning how to count well, a lesson that will be constantly reaffirmed, is that while trace formulas are a conceptually essential step in deriving and understanding periodic orbit theory, the spectral determinant is the right object to use in actual computations. Instead of summing all of the exponentially many periodic points required by trace formulas at each level of truncation, spectral determinants incorporate only the small incremental corrections to what is already known - and that makes them a more powerful tool for computations.

Contrary to claims one all too often encounters in the literature, "exponential proliferation of trajectories" is not the problem; what limits the convergence of cycle expansions is the proliferation of the grammar rules, or the "algorithmic complexity," as illustrated by sect. 15.5, and figure 15.3 in particular. Nice, finite grammar leads to nice, discrete spectrum; infinite grammar leads to analyticity walls in the complex spectral plane.

Historically, these topological zeta functions were the inspiration for applying the transfer matrix methods of statistical mechanics to the problem of computation of dynamical averages for chaotic flows. The key result was the dynamical zeta function to be derived in chapter 18, a weighted generalization of the topological zeta function.

Commentary

Remark 15.1 Artin-Mazur zeta functions. Motivated by A. Weil's zeta function for the Frobenius map [15.8], Artin and Mazur [19.11] introduced the zeta function (15.27) that counts periodic points for diffeomorphisms (see also ref. [15.9] for their evaluation for maps of the interval). Smale [15.10] conjectured rationality of the zeta functions for Axiom A diffeomorphisms, later proved by Guckenheimer [15.11] and Manning [15.12]. See remark 19.4 on page 375 for more zeta function history.

Remark 15.2 "Entropy." The ease with which the topological entropy can be motivated obscures the fact that our construction does not lead to an invariant characterization of the dynamics, as the choice of symbolic dynamics is largely arbitrary: the same caveat applies to other entropies.In order to obtain invariant characterizations we will have to work harder. Mathematicians like to define the (impossible to evaluate) supremum over all possible partitions. The key point that eliminates the need for such searches is the existence of *generators*, i.e., partitions that under the dynamics are able to probe the whole state space on arbitrarily small scales. A generator is a finite partition $\mathcal{M} = \{\mathcal{M}_1 \ldots \mathcal{M}_N\}$ with the following property: consider the partition built upon all possible intersections of sets $f^n(\mathcal{M}_i)$, where f is dynamical evolution and n takes all possible integer values (positive as well as negative), then the closure of such a partition coincides with the 'algebra of all measurable sets.' For a thorough (and readable) discussion of generators and how they allow a computation of the Kolmogorov entropy, see ref. [15.1].

Remark 15.3 Perron-Frobenius matrices. For a proof of the Perron theorem on the leading eigenvalue see ref. [1.26]. Appendix A4.1 of ref. [15.2] offers a clear discussion of the spectrum of the transition matrix.

Remark 15.4 Determinant of a graph. Many textbooks offer derivations of the loop expansions of characteristic polynomials for transition matrices and their transition graphs, see for example refs. [15.3, 15.4, 15.5].

Remark 15.5 Ordering periodic orbit expansions. In sect. 20.5 we will introduce an alternative way of hierarchically organizing cumulant expansions, in which the order is dictated by stability rather than cycle period: such a procedure may be better suited to perform computations when the symbolic dynamics is not well understood.

Remark 15.6 T is not trace class. Note to the erudite reader: the transition matrix T (in the infinite partition limit (15.24)) is *not* trace class. Still the trace is well defined in the $n \to \infty$ limit.

Remark 15.7 Counting prime cycles. Duval has an efficient algorithm for generating Lyndon words (non-periodic necklaces, i.e., prime cycle itineraries).

Exercises

15.1. **A transition matrix for 3-disk pinball.**

a) Draw the transition graph corresponding to the 3-disk ternary symbolic dynamics, and write down the corresponding transition matrix corresponding to the graph. Show that iteration of the transition matrix results in two coupled linear difference equations, - one for the diagonal and one for the off diagonal elements. (Hint: relate $\mathrm{tr}\, T^n$ to $\mathrm{tr}\, T^{n-1} + \ldots$.)

b) Solve the above difference equation and obtain the number of periodic orbits of length n. Compare your result with table 15.4.

c) Find the eigenvalues of the transition matrix $\mathbf{T}$ for the 3-disk system with ternary symbolic dynamics and calculate the topological entropy. Compare this to the topological entropy obtained from the binary symbolic dynamics $\{0, 1\}$.

15.2. **3-disk prime cycle counting.** A *prime* cycle p of length n_p is a single traversal of the orbit; its label is a non-repeating symbol string of n_p symbols. For example, $\overline{12}$ is prime, but $\overline{2121}$ is not, since it is $\overline{21} = \overline{12}$ repeated.

Verify that a 3-disk pinball has 3, 2, 3, 6, 9, $\cdots$ prime cycles of length 2, 3, 4, 5, 6, $\cdots$.

15.3. **Sum of A_{ij} is like a trace.** Let A be a matrix with eigenvalues λ_k. Show that

$$\Gamma_n := \sum_{i,j} [A^n]_{ij} = \sum_k c_k \lambda_k^n \,.$$

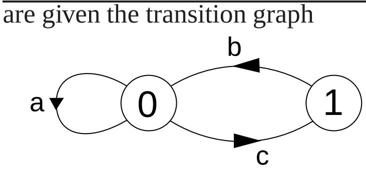
(a) Under what conditions do $\ln |\mathrm{tr}\, A^n|$ and $\ln |\Gamma_n|$ have the same asymptotic behavior as $n \to \infty$, i.e., their ratio converges to one?

(b) Do eigenvalues λ_k need to be distinct, $\lambda_k \neq \lambda_l$ for $k \neq l$? How would a degeneracy $\lambda_k = \lambda_l$ affect your argument for (a)?

15.4. **Loop expansions.** Prove by induction the sign rule in the determinant expansion (15.15):

$$\det(1 - z\mathbf{T}) = \sum_{k \geq 0} \sum_{p_1 + \cdots + p_k} (-1)^k t_{p_1} t_{p_2} \cdots t_{p_k} \,.$$

15.5. **Transition matrix and cycle counting.** Suppose you are given the transition graph



This diagram can be encoded by a matrix T, where the entry T_{ij} means that there is a link connecting node i to node j. The value of the entry is the weight of the link.

a) Walks on the graph are given a weight that is the product of the weights of all links crossed by the walk. Convince yourself that the transition matrix for this graph is:

$$T = \begin{bmatrix} a & c \\ b & 0 \end{bmatrix} .$$

b) Enumerate all the walks of length three on the transition graph. Now compute T^3 and look at the entries. Is there any relation between the terms in T^3 and all the walks?

c) Show that T^n_{ij} is the number of walks from point i to point j in n steps. (Hint: one might use the method of induction.)

d) Estimate the number K_n of walks of length n for this simple transition graph.

e) The topological entropy h measures the rate of exponential growth of the total number of walks K_n as a function of n. What is the topological entropy for this transition graph?

15.6. **Alphabet {0,1}, prune $\_00\_$.** The transition graph example 14.9 implements this pruning rule which implies that "0" must always be bracketed by "1"s; in terms of a new symbol $2 := 10$, the dynamics becomes unrestricted symbolic dynamics with with binary alphabet {1,2}. The cycle expansion (15.15) becomes

$$\begin{aligned} 1/\zeta &= (1-t_1)(1-t_2)(1-t_{12})(1-t_{112})\dots \\ &= 1-t_1-t_2-(t_{12}-t_1t_2) \\ &\quad -(t_{112}-t_{12}t_1)-(t_{122}-t_{12}t_2)\dots \end{aligned} \tag{15.41}$$

In the original binary alphabet this corresponds to:

$$\begin{aligned} 1/\zeta &= 1-t_1-t_{10}-(t_{110}-t_1t_{10}) \\ &\quad -(t_{1110}-t_{110}t_1)-(t_{11010}-t_{110}t_{10})\dots \end{aligned} \tag{15.42}$$

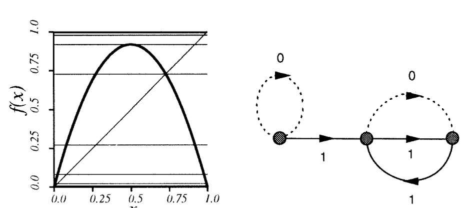
This symbolic dynamics describes, for example, circle maps with the golden mean winding number. For unimodal maps this symbolic dynamics is realized by the tent map of exercise 11.6.

15.7. **"Golden mean" pruned map.** (continuation of exercise 11.6) Show that the total number of periodic orbits of length n for the "golden mean" tent map is

$$\frac{(1+\sqrt{5})^n+(1-\sqrt{5})^n}{2^n}\,.$$

Continued in exercise 19.2. See also exercise 15.8.

15.8. **A unimodal map with golden mean pruning.** Consider the unimodal map



for which the critical point maps into the right hand fixed point in three iterations, $S^+ = 100\overline{1}$. Show that the admissible itineraries are generated by the above transition graph, with transient neighborhood of $\overline{0}$ fixed point, and $\_00\_$ pruned from the recurrent set. (K.T. Hansen)

15.9. **Glitches in shadowing.** (medium difficulty) Note that the combination t_{00011} minus the "shadow" t_0t_{0011} in (15.20) cancels exactly, and does not contribute to the topological zeta function (15.21). Are you able to construct a smaller transition graph than figure 14.6 (e)?

15.10. **Whence Möbius function?** To understand the origin of the Möbius function (15.37), consider the function

$$f(n) = \sum_{d|n} g(d) \tag{15.43}$$

where $d|n$ stands for sum over all divisors d of n. Invert recursively this infinite tower of equations and derive the *Möbius inversion formula*

$$g(n) = \sum_{d|n} \mu(n/d) f(d)\,. \tag{15.44}$$

15.11. **Counting prime binary cycles.** In order to get comfortable with Möbius inversion reproduce the results of the second column of table 15.3.

Write a program that determines the number of prime cycles of length n. You might want to have this program later on to be sure that you have missed no 3-pinball prime cycles.

15.12. **Counting subsets of cycles.** The techniques developed above can be generalized to counting subsets of cycles. Consider the simplest example of a dynamical system with a complete binary tree, a repeller map (11.4) with two straight branches, which we label 0 and 1. Every cycle weight for such map factorizes, with a factor t_0 for each 0, and factor t_1 for each 1 in its symbol string. Prove that the transition matrix traces (15.7) collapse to $tr(T^k) = (t_0+t_1)^k$, and $1/\zeta$ is simply

$$\prod_p \left(1-t_p\right) = 1-t_0-t_1 \tag{15.45}$$

Substituting (15.45) into the identity

$$\prod_p \left(1+t_p\right) = \prod_p \frac{1-t_p{}^2}{1-t_p}$$

we obtain

$$\begin{aligned} \prod_p \left(1+t_p\right) &= \frac{1-t_0^2-t_1^2}{1-t_0-t_1} \\ &= 1+t_0+t_1+\frac{2t_0t_1}{1-t_0-t_1} \end{aligned}$$

$$= 1 + t_0 + t_1 + \sum_{n=2}^{\infty}\sum_{k=1}^{n-1} 2\binom{n-2}{k-1} t_0^k t_1^{n-k}.$$

Hence for $n \geq 2$ the number of terms in the cumulant expansion with k 0's and $n-k$ 1's in their symbol sequences is $2\binom{n-2}{k-1}$.

In order to count the number of prime cycles in each such subset we denote with $M_{n,k}$ $(n = 1, 2, \ldots; k = \{0, 1\}$ for $n = 1$; $k = 1, \ldots, n-1$ for $n \geq 2)$ the number of prime n-cycles whose labels contain k zeros. Show that

$$M_{1,0} = M_{1,1} = 1, \quad n \geq 2, k = 1, \ldots, n-1$$
$$nM_{n,k} = \sum_{m \left| \frac{n}{k} \right.} \mu(m) \binom{n/m}{k/m}$$

where the sum is over all m which divide both n and k. (continued as exercise 20.7)

15.13. **Logarithmic periodicity of $\ln N_n$.** (medium difficulty) Plot $(\ln N_n, nh)$ for a system with a nontrivial finite transition graph. Do you see any periodicity? If yes, why?

15.14. **Symmetric 4-disk pinball topological zeta function.** Show that the 4-disk pinball topological zeta function (the pruning affects only the fixed points and the 2-cycles) is given by

$$\begin{aligned} 1/\zeta_{\text{top}}^{4-disk} &= (1-3z)\frac{(1-z^2)^6}{(1-z)^3(1-z^2)^3} \\ &= (1-3z)(1+z)^3 \\ &= 1 - 6z^2 - 8z^3 - 3z^4 \,. \end{aligned} \tag{15.46}$$

15.15. **Symmetric N-disk pinball topological zeta function.** Show that for an N-disk pinball, the topological zeta function is given by

$$\begin{aligned} 1/\zeta_{\text{top}}^{N-disk} &= (1-(N-1)z) \times \\ &\quad \frac{(1-z^2)^{N(N-1)/2}}{(1-z)^{N-1}(1-z^2)^{(N-1)(N-2)/2}} \\ &= (1-(N-1)z)(1+z)^{N-1} \,. \end{aligned} \tag{15.47}$$

The topological zeta function has a root $z^{-1} = N-1$, as we already know it should from (15.38) or (15.18). We shall see in sect. 21.4 that the other roots reflect the symmetry factorizations of zeta functions.

15.16. **Alphabet $\{a, b, c\}$, prune $\_ab\_$.** Write down the topological zeta function for this pruning rule.

15.17. **Alphabet {0,1}, prune n repeats of "0" $\_000\ldots00\_$.** This is equivalent to the n symbol alphabet {1, 2, . . ., n} unrestricted symbolic dynamics, with symbols corresponding to the possible 10. . .00 block lengths: 2:=10, 3:=100, . . ., n:=100. . .00. Show that the cycle expansion (15.15) becomes

$$\begin{aligned} 1/\zeta &= 1 - t_1 - t_2 \ldots - t_n - (t_{12} - t_1 t_2) \ldots \\ &\quad -(t_{1n} - t_1 t_n) \ldots . \end{aligned}$$

15.18. **Alphabet {0,1}, prune $\_1000\_$, $\_00100\_$, $\_01100\_$.** Show that the topological zeta function is given by

$$1/\zeta = (1-t_0)(1 - t_1 - t_2 - t_{23} - t_{113}) \tag{15.48}$$

with the unrestricted 4-letter alphabet {1, 2, $\underline{23}$, $\underline{113}$}. Here 2 and 3 refer to 10 and 100 respectively, as in exercise 15.17.

15.19. **Alphabet {0,1}, prune $\_1000\_$, $\_00100\_$, $\_01100\_$, $\_10011\_$.** (This grammar arises from Hénon map pruning, see remark 12.3.) The first three pruning rules were incorporated in the preceeding exercise.

(a) Show that the last pruning rule $\_10011\_$ leads (in a way similar to exercise 15.18) to the alphabet {$\underline{21^k}$, $\underline{23}$, $\underline{21^k113}$; $\overline{1}$, $\overline{0}$}, and the cycle expansion

$$1/\zeta = (1-t_0)(1-t_1-t_2-t_{23}+t_1 t_{23}-t_{2113}) \,. \tag{15.49}$$

Note that this says that 1, 23, 2, 2113 are the fundamental cycles; not all cycles up to length 7 are needed, only 2113.

(b) Show that the topological zeta function is

$$1/\zeta_{\text{top}} = (1-z)(1-z-z^2-z^5+z^6-z^7) \tag{15.50}$$

and that it yields the entropy $h = 0.522737642\ldots$.

15.20. **Alphabet {0,1}, prune only the fixed point $\overline{0}$.** This is equivalent to the *infinite* alphabet {1, 2, 3, 4, . . .} unrestricted symbolic dynamics. The prime cycles are labeled by all non-repeating sequences of integers, ordered lexically: $t_n, n > 0$; $t_{mn}, t_{mmn}, \ldots, n > m > 0$; $t_{mnr}, r > n > m > 0, \ldots$ (see sect. 24.3). Now the number of fundamental cycles is infinite as well:

$$\begin{aligned} 1/\zeta &= 1 - \sum_{n>0} t_n - \sum_{n>m>0} (t_{mn} - t_n t_m) \\ &\quad - \sum_{n>m>0} (t_{mmn} - t_m t_{mn}) \\ &\quad - \sum_{n>m>0} (t_{mnn} - t_{mn} t_n) \\ &\quad - \sum_{r>n>m>0} (t_{mnr} + t_{mrn} - t_{mn} t_r \\ &\quad - t_{mr} t_n - t_m t_{nr} + t_m t_n t_r) \cdots \end{aligned} \tag{15.51}$$

. As shown in table 24.1, this grammar plays an important role in description of fixed points of marginal stability.