Chapter 28

Why does it work?

Bloch: "Space is the field of linear operators."
Heisenberg: "Nonsense, space is blue and birds fly through it."
—Felix Bloch, *Heisenberg and the early days of quantum mechanics*

(R. Artuso, H.H. Rugh and P. Cvitanović)

As we shall see, the trace formulas and spectral determinants work well, sometimes very well. The question is: Why? And it still is. The heuristic manipulations of chapter 21 were naive and reckless, as we are facing infinite-dimensional vector spaces and singular integral kernels.

We now outline the key ingredients of proofs that put the trace and determinant formulas on solid footing. This requires taking a closer look at the evolution operators from a mathematical point of view, since up to now we have talked about eigenvalues without any reference to what kind of a function space the corresponding eigenfunctions belong to. We shall restrict our considerations to the spectral properties of the Perron-Frobenius operator for maps, as proofs for more general evolution operators follow along the same lines. What we refer to as a "the set of eigenvalues" acquires meaning only within a precisely specified functional setting: this sets the stage for a discussion of the analyticity properties of spectral determinants. In example 28.1 we compute explicitly the eigenspectrum for the three analytically tractable piecewise linear examples. In sect. 28.3 we review the basic facts of the classical Fredholm theory of integral equations. The program is sketched in sect. 28.4, motivated by an explicit study of eigenspectrum of the Bernoulli shift map, and in sect. 28.5 generalized to piecewise real-analytic hyperbolic maps acting on appropriate densities. We show on a very simple example that the spectrum is quite sensitive to the regularity properties of the functions considered.

For expanding and hyperbolic finite-subshift maps analyticity leads to a very strong result; not only do the determinants have better analyticity properties than

the trace formulas, but the spectral determinants are singled out as entire functions in the complex s plane. remark 28.1

The goal of this chapter is not to provide an exhaustive review of the rigorous theory of the Perron-Frobenius operators and their spectral determinants, but rather to give you a feeling for how our heuristic considerations can be put on a firm basis. The mathematics underpinning the theory is both hard and profound.

If you are primarily interested in applications of the periodic orbit theory, you should skip this chapter on the first reading.

fast track:
chapter 16, p. 293

28.1 Linear maps: exact spectra

We start gently; in example 28.1 we work out the *exact* eigenvalues and eigenfunctions of the Perron-Frobenius operator for the simplest example of unstable, expanding dynamics, a linear 1-dimensional map with one unstable fixed point. Ref. [28.6] shows that this can be carried over to d-dimensions. Not only that, but in example 28.5 we compute the exact spectrum for the simplest example of a dynamical system with an *infinity* of unstable periodic orbits, the Bernoulli shift.

Example 28.1 ***The simplest eigenspectrum - a single fixed point:*** *In order to get some feeling for the determinants defined so formally in sect. 22.2, let us work out a trivial example: a repeller with only one expanding linear branch*

$$f(x) = \Lambda x\,, \qquad |\Lambda| > 1\,,$$

and only one fixed point $x_q = 0$. The action of the Perron-Frobenius operator (19.10) is

$$\mathcal{L}\phi(y) = \int dx\,\delta(y - \Lambda x)\,\phi(x) = \frac{1}{|\Lambda|}\phi(y/\Lambda)\,. \tag{28.1}$$

From this one immediately gets that the monomials y^k are eigenfunctions:

$$\mathcal{L}y^k = \frac{1}{|\Lambda|\Lambda^k}y^k\,, \quad k = 0, 1, 2, \dots \tag{28.2}$$

What are these eigenfunctions? Think of eigenfunctions of the Schrödinger equation: k labels the kth eigenfunction x^k in the same spirit in which the number of nodes labels the kth quantum-mechanical eigenfunction. A quantum-mechanical amplitude with more nodes has more variability, hence a higher kinetic energy. Analogously, for a Perron-Frobenius operator, a higher k eigenvalue $1/|\Lambda|\Lambda^k$ is getting exponentially smaller because densities that vary more rapidly decay more rapidly under the expanding action of the map.

Example 28.2 ***The trace formula for a single fixed point:*** *The eigenvalues Λ^{-k-1} fall off exponentially with k, so the trace of $\mathcal{L}$ is a convergent sum*

$$\operatorname{tr}\mathcal{L} = \frac{1}{|\Lambda|}\sum_{k=0}^{\infty}\Lambda^{-k} = \frac{1}{|\Lambda|(1-\Lambda^{-1})} = \frac{1}{|f(0)'-1|}\,,$$

in agreement with (21.6). A similar result follows for powers of $\mathcal{L}$, yielding the single-fixed point version of the trace formula for maps (21.9):

$$\sum_{k=0}^{\infty}\frac{z e^{s_k}}{1-z e^{s_k}} = \sum_{r=1}^{\infty}\frac{z^r}{|1-\Lambda^r|}\,, \qquad e^{s_k} = \frac{1}{|\Lambda|\Lambda^k}\,. \tag{28.3}$$

The left hand side of (28.3) is a meromorphic function, with the leading zero at $z = |\Lambda|$. So what?

Example 28.3 ***Meromorphic functions and exponential convergence:*** *As an illustration of how exponential convergence of a truncated series is related to analytic properties of functions, consider, as the simplest possible example of a meromorphic function, the ratio*

$$h(z) = \frac{z-a}{z-b}$$

with a, b real and positive and $a < b$. Within the spectral radius *$|z| < b$ the function h can be represented by the power series*

$$h(z) = \sum_{k=0}^{\infty}\sigma_k z^k\,,$$

where $\sigma_0 = a/b$, and the higher order coefficients are given by $\sigma_j = (a-b)/b^{j+1}$. Consider now the truncation of order N of the power series

$$h_N(z) = \sum_{k=0}^{N}\sigma_k z^k = \frac{a}{b} + \frac{z(a-b)(1-z^N/b^N)}{b^2(1-z/b)}\,.$$

Let $\hat{z}_N$ be the solution of the truncated series $h_N(\hat{z}_N) = 0$. To estimate the distance between a and $\hat{z}_N$ it is sufficient to calculate $h_N(a)$. It is of order $(a/b)^{N+1}$, so finite order estimates converge exponentially to the asymptotic value.

This example shows that: (1) an estimate of the leading pole (the leading eigenvalue of $\mathcal{L}$) from a finite truncation of a trace formula converges exponentially, and (2) the non-leading eigenvalues of $\mathcal{L}$ lie outside of the radius of convergence of the trace formula and cannot be computed by means of such cycle expansion. However, as we shall now see, the whole spectrum is reachable at no extra effort, by computing it from a determinant rather than a trace.

Example 28.4 ***The spectral determinant for a single fixed point:*** *The spectral determinant (22.3) follows from the trace formulas of example 28.2:*

$$\det(1-z\mathcal{L}) = \prod_{k=0}^{\infty}\left(1-\frac{z}{|\Lambda|\Lambda^k}\right) = \sum_{n=0}^{\infty}(-t)^n Q_n\,, \quad t = \frac{z}{|\Lambda|}\,, \tag{28.4}$$

where the cummulants Q_n are given explicitly by the Euler formula

exercise 28.3

$$Q_n = \frac{1}{1-\Lambda^{-1}}\,\frac{\Lambda^{-1}}{1-\Lambda^{-2}}\cdots\frac{\Lambda^{-n+1}}{1-\Lambda^{-n}}\,. \tag{28.5}$$

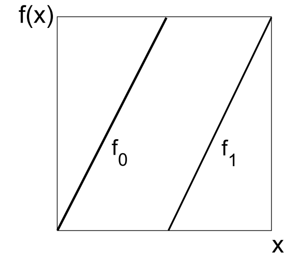

Figure 28.1: The Bernoulli shift map.

The main lesson to glean from this simple example is that the cummulants Q_n decay asymptotically *faster* than exponentially, as $\Lambda^{-n(n-1)/2}$. For example, if we approximate series such as (28.4) by the first 10 terms, the error in the estimate of the leading zero is $\approx 1/\Lambda^{50}$!

So far all is well for a rather boring example, a dynamical system with a single repelling fixed point. What about chaos? Systems where the number of unstable cycles increases exponentially with their length? We now turn to the simplest example of a dynamical system with an infinity of unstable periodic orbits.

Example 28.5 ***Eigenfunction of Bernoulli shift map.*** *(continued from example 14.6) The* Bernoulli shift map *figure 28.1*

$$f(x) = \begin{cases} f_0(x) = 2x\,, & x \in I_0 = [0, 1/2) \\ f_1(x) = 2x-1\,, & x \in I_1 = (1/2, 1] \end{cases} \tag{28.6}$$

models the 50-50% probability of a coin toss. The associated Perron-Frobenius operator (19.9) assembles $\rho(y)$ from its two preimages

$$\mathcal{L}\rho(y) = \frac{1}{2}\rho\left(\frac{y}{2}\right) + \frac{1}{2}\rho\left(\frac{y+1}{2}\right)\,. \tag{28.7}$$

For this simple example the eigenfunctions can be written down explicitly: they coincide, up to constant prefactors, with the Bernoulli polynomials $B_n(x)$. These polynomials are generated by the Taylor expansion of the exponential generating function

$$\mathcal{G}(x,t) = \frac{t e^{xt}}{e^t-1} = \sum_{k=0}^{\infty}B_k(x)\frac{t^k}{k!}\,, \quad B_0(x) = 1\,,\ B_1(x) = x - \frac{1}{2}\,,\ldots$$

The Perron-Frobenius operator (28.7) acts on the exponential generating function $\mathcal{G}$ as

$$\mathcal{L}\mathcal{G}(x,t) = \frac{1}{2}\left(\frac{t e^{xt/2}}{e^t-1} + \frac{t e^{t/2}e^{xt/2}}{e^t-1}\right) = \frac{t}{2}\frac{e^{xt/2}}{e^{t/2}-1} = \sum_{k=1}^{\infty}B_k(x)\frac{(t/2)^k}{k!}\,,$$

hence each $B_k(x)$ is an eigenfunction of $\mathcal{L}$ with eigenvalue $1/2^k$.

The full operator has two components corresponding to the two branches. For the n times iterated operator we have a full binary shift, and for each of the 2^n branches the above calculations carry over, yielding the same trace $(2^n-1)^{-1}$ for every cycle on length n. Without further ado we substitute everything back and obtain the determinant,

$$\det(1-z\mathcal{L}) = \exp\left(-\sum_{n=1}\frac{z^n}{n}\frac{2^n}{2^n-1}\right) = \prod_{k=0}\left(1-\frac{z}{2^k}\right)\,, \tag{28.8}$$

verifying that the Bernoulli polynomials are eigenfunctions with eigenvalues $1, 1/2, \ldots, 1/2^n, \ldots$.

The Bernoulli map spectrum looks reminiscent of the single fixed-point spectrum (28.2), with the difference that the leading eigenvalue here is 1, rather than $1/|\Lambda|$. The difference is significant: the single fixed-point map is a repeller, with escape rate (1.7) given by the $\mathcal{L}$ leading eigenvalue $\gamma = \ln|\Lambda|$, while there is no escape in the case of the Bernoulli map. As already noted in discussion of the relation (22.18), for bounded systems the local expansion rate (here $\ln|\Lambda| = \ln 2$) is balanced by the entropy (here $\ln 2$, the log of the number of preimages F_s), yielding zero escape rate.

section 22.4

So far we have demonstrated that our periodic orbit formulas are correct for two piecewise linear maps in 1 dimension, one with a single fixed point, and one with a full binary shift chaotic dynamics. For a single fixed point, eigenfunctions are monomials in x. For the chaotic example, they are orthogonal polynomials on the unit interval. What about higher dimensions? We check our formulas on a 2-dimensional hyperbolic map next.

Example 28.6 ***The simplest of 2-dimensional maps - a single hyperbolic fixed point:*** *We start by considering a very simple linear hyperbolic map with a single hyperbolic fixed point,*

$$f(x) = (f_1(x_1,x_2), f_2(x_1,x_2)) = (\Lambda_s x_1, \Lambda_u x_2)\,, \quad 0 < |\Lambda_s| < 1\,, \quad |\Lambda_u| > 1\,.$$

The Perron-Frobenius operator (19.10) acts on the 2-dimensional density functions as

$$\mathcal{L}\rho(x_1,x_2) = \frac{1}{|\Lambda_s\Lambda_u|}\,\rho(x_1/\Lambda_s, x_2/\Lambda_u) \tag{28.9}$$

What are good eigenfunctions? Cribbing the 1-dimensional eigenfunctions for the stable, contracting x_1 direction from example 28.1 is not a good idea, as under the iteration of $\mathcal{L}$ the high terms in a Taylor expansion of $\rho(x_1,x_2)$ in the x_1 variable would get multiplied by exponentially exploding eigenvalues $1/\Lambda_s^k$. This makes sense, as in the contracting directions hyperbolic dynamics crunches up initial densities, instead of smoothing them. So we guess instead that the eigenfunctions are of form

$$\varphi_{k_1k_2}(x_1,x_2) = x_2^{k_2}/x_1^{k_1+1}\,, \qquad k_1, k_2 = 0,1,2,\ldots\,, \tag{28.10}$$

a mixture of the Laurent series in the contraction x_1 direction, and the Taylor series in the expanding direction, the x_2 variable. The action of Perron-Frobenius operator on this set of basis functions

$$\mathcal{L}\varphi_{k_1k_2}(x_1,x_2) = \frac{\sigma}{|\Lambda_u|}\frac{\Lambda_s^{k_1}}{\Lambda_u^{k_2}}\,\varphi_{k_1k_2}(x_1,x_2)\,, \qquad \sigma = \Lambda_s/|\Lambda_s|$$

is smoothing, with the higher k_1, k_2 eigenvectors decaying exponentially faster, by $\Lambda_s^{k_1}/\Lambda_u^{k_2+1}$ factor in the eigenvalue. One verifies by an explicit calculation (undoing the geometric series expansions to lead to (22.8)) that the trace of $\mathcal{L}$ indeed equals $1/|\det(\mathbf{1}-M)| = 1/|(1-\Lambda_u)(1-\Lambda_s)|$, from which it follows that all our trace and spectral determinant formulas apply. The argument applies to any hyperbolic map linearized around the fixed point of form $f(x_1....,x_d) = (\Lambda_1 x_1, \Lambda_2 x_2, \ldots, \Lambda_d x_d)$.

So far we have checked the trace and spectral determinant formulas derived heuristically in chapters 21 and 22, but only for the case of 1-dimensional and 2-dimensional linear maps. But for infinite-dimensional vector spaces this game is fraught with dangers, and we have already been mislead by piecewise linear examples into spectral confusions: contrast the spectra of example 19.1 and example 20.4 with the spectrum computed in example 21.2.

We show next that the above results do carry over to a sizable class of piecewise analytic expanding maps.

28.2 Evolution operator in a matrix representation

The standard, and for numerical purposes sometimes very effective way to look at operators is through their matrix representations. Evolution operators are moving density functions defined over some state space, and as in general we can implement this only numerically, the temptation is to discretize the state space as in sect. 19.3. The problem with such state space discretization approaches that they sometimes yield plainly wrong spectra (compare example 20.4 with the result of example 21.2), so we have to think through carefully what is it that we *really* measure.

An expanding map $f(x)$ takes an initial smooth density $\phi_n(x)$, defined on a subinterval, stretches it out and overlays it over a larger interval, resulting in a new, smoother density $\phi_{n+1}(x)$. Repetition of this process smoothes the initial density, so it is natural to represent densities $\phi_n(x)$ by their Taylor series. Expanding

$$\phi_n(y) = \sum_{k=0}^{\infty}\phi_n^{(k)}(0)\frac{y^k}{k!}\,, \qquad \phi_{n+1}(y)_k = \sum_{\ell=0}^{\infty}\phi_{n+1}^{(\ell)}(0)\frac{y^\ell}{\ell!}\,,$$

$$\phi_{n+1}^{(\ell)}(0) = \int dx\;\delta^{(\ell)}(y-f(x))\phi_n(x)\Big|_{y=0}\,, \qquad x = f^{-1}(0)\,,$$

and substitute the two Taylor series into (19.6):

$$\phi_{n+1}(y) = (\mathcal{L}\phi_n)(y) = \int_{\mathcal{M}} dx\,\delta(y-f(x))\,\phi_n(x)\,.$$

The matrix elements follow by evaluating the integral

$$\mathbf{L}_{\ell k} = \frac{\partial^\ell}{\partial y^\ell}\int dx\,\mathcal{L}(y,x)\frac{x^k}{k!}\bigg|_{y=0}\,. \tag{28.11}$$

we obtain a matrix representation of the evolution operator

$$\int dx\,\mathcal{L}(y,x)\frac{x^k}{k!} = \sum_{k'}\frac{y^{k'}}{k'!}\mathbf{L}_{k'k}\,, \qquad k,k' = 0,1,2,\ldots$$

which maps the x^k component of the density of trajectories $\phi_n(x)$ into the $y^{k'}$ component of the density $\phi_{n+1}(y)$ one time step later, with $y = f(x)$.

We already have some practice with evaluating derivatives $\delta^{(\ell)}(y) = \frac{\partial^\ell}{\partial y^\ell}\delta(y)$ from sect. 19.2. This yields a representation of the evolution operator centered on the fixed point, evaluated recursively in terms of derivatives of the map f:

$$\begin{aligned}\mathbf{L}_{\ell k} &= \int dx\, \delta^{(\ell)}(x - f(x)) \frac{x^k}{k!}\bigg|_{x=f(x)} \\ &= \frac{1}{|f'|}\left(\frac{d}{dx}\frac{1}{f'(x)}\right)^\ell \frac{x^k}{k!}\bigg|_{x=f(x)} .\end{aligned} \tag{28.12}$$

The matrix elements vanish for $\ell < k$, so $\mathbf{L}$ is a lower triangular matrix. The diagonal and the successive off-diagonal matrix elements are easily evaluated iteratively by computer algebra

$$\mathbf{L}_{kk} = \frac{1}{|\Lambda|\Lambda^k}\,, \quad \mathbf{L}_{k+1,k} = -\frac{(k+2)!f''}{2k!|\Lambda|\Lambda^{k+2}}\,, \quad \cdots .$$

For chaotic systems the map is expanding, $|\Lambda| > 1$. Hence the diagonal terms drop off exponentially, as $1/|\Lambda|^{k+1}$, the terms below the diagonal fall off even faster, and truncating $\mathbf{L}$ to a finite matrix introduces only exponentially small errors.

The trace formula (28.3) takes now a matrix form

$$\mathrm{tr}\,\frac{z\mathcal{L}}{1 - z\mathcal{L}} = \mathrm{tr}\,\frac{z\mathbf{L}}{1 - z\mathbf{L}}\,. \tag{28.13}$$

In order to illustrate how this works, we work out a few examples.

In example 28.7 we show that these results carry over to any analytic single-branch 1-dimensional repeller. Further examples motivate the steps that lead to a proof that spectral determinants for general analytic 1-dimensional expanding maps, and - in sect. 28.5, for 1-dimensional hyperbolic mappings - are also entire functions.

Example 28.7 ***Perron-Frobenius operator in a matrix representation:*** *As in example 28.1, we start with a map with a single fixed point, but this time with a nonlinear piecewise analytic map f with a nonlinear inverse $F = f^{-1}$, sign of the derivative $\sigma = \sigma(F') = F'/|F'|$, and the Perron-Frobenius operator acting on densities analytic in an open domain enclosing the fixed point $x = w_q$,*

$$\mathcal{L}\phi(y) = \int dx\, \delta(y - f(x))\, \phi(x) = \sigma\, F'(y)\, \phi(F(y))\,.$$

Assume that F is a contraction of the unit disk in the complex plane, i.e.,

$$|F(z)| < \theta < 1 \quad \text{and} \quad |F'(z)| < C < \infty \quad \text{for} \quad |z| < 1\,, \tag{28.14}$$

and expand ϕ in a polynomial basis with the Cauchy integral formula

$$\phi(z) = \sum_{n=0}^{\infty} z^n \phi_n = \oint \frac{dw}{2\pi i}\, \frac{\phi(w)}{w - z}\,, \quad \phi_n = \oint \frac{dw}{2\pi i}\, \frac{\phi(w)}{w^{n+1}}$$

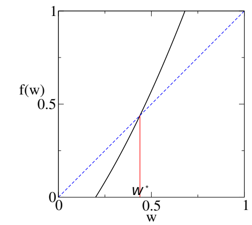

Figure 28.2: A nonlinear one-branch repeller with a single fixed point w_q.

Combining this with (28.22), we see that in this basis Perron-Frobenius operator $\mathcal{L}$ is represented by the matrix

$$\mathcal{L}\phi(w) = \sum_{m,n} w^m \mathbf{L}_{mn}\phi_n\,, \quad \mathbf{L}_{mn} = \oint \frac{dw}{2\pi i}\, \frac{\sigma\, F'(w)(F(w))^n}{w^{m+1}}\,. \tag{28.15}$$

Taking the trace and summing we get:

$$\mathrm{tr}\,\mathcal{L} = \sum_{n\geq 0} \mathbf{L}_{nn} = \oint \frac{dw}{2\pi i}\, \frac{\sigma\, F'(w)}{w - F(w)}\,.$$

This integral has but one simple pole at the unique fixed point $w^* = F(w^*) = f(w^*)$. Hence exercise 28.6

$$\mathrm{tr}\,\mathcal{L} = \frac{\sigma\, F'(w^*)}{1 - F'(w^*)} = \frac{1}{|f'(w^*) - 1|}\,.$$

This super-exponential decay of cummulants Q_k ensures that for a repeller consisting of a single repelling point the spectral determinant (28.4) is *entire* in the complex z plane.

In retrospect, the matrix representation method for solving the density evolution problems is eminently sensible — after all, that is the way one solves a close relative to classical density evolution equations, the Schrödinger equation. *When* available, matrix representations for $\mathcal{L}$ enable us to compute many more orders of cumulant expansions of spectral determinants and many more eigenvalues of evolution operators than the cycle expensions approach.

Now, if the spectral determinant is entire, formulas such as (22.28) imply that the dynamical zeta function is a meromorphic function. The practical import of this observation is that it guarantees that finite order estimates of zeroes of dynamical zeta functions and spectral determinants converge exponentially, or - in cases such as (28.4) - super-exponentially to the exact values, and so the cycle expansions to be discussed in chapter 23 represent a *true perturbative* approach to chaotic dynamics.

Before turning to specifics we summarize a few facts about classical theory of integral equations, something you might prefer to skip on first reading. The purpose of this exercise is to understand that the Fredholm theory, a theory that works so well for the Hilbert spaces of quantum mechanics does not necessarily work for deterministic dynamics - the ergodic theory is much harder.

fast track:
sect. 28.4, p. 522

28.3 Classical Fredholm theory

He who would valiant be 'gainst all disaster
Let him in constancy follow the Master.
—John Bunyan, *Pilgrim's Progress*

The Perron-Frobenius operator

$$\mathcal{L}\phi(x) = \int dy\, \delta(x - f(y))\, \phi(y)$$

has the same appearance as a classical Fredholm integral operator

$$\mathcal{K}\varphi(x) = \int_{\mathcal{M}} dy\, \mathcal{K}(x,y)\varphi(y)\,, \tag{28.16}$$

and one is tempted to resort to classical Fredholm theory in order to establish analyticity properties of spectral determinants. This path to enlightenment is blocked by the singular nature of the kernel, which is a distribution, whereas the standard theory of integral equations usually concerns itself with regular kernels $\mathcal{K}(x,y) \in L^2(\mathcal{M}^2)$. Here we briefly recall some steps of Fredholm theory, before working out the example of example 28.5.

The general form of Fredholm integral equations of the second kind is

$$\varphi(x) = \int_{\mathcal{M}} dy\, \mathcal{K}(x,y)\varphi(y) + \xi(x) \tag{28.17}$$

where $\xi(x)$ is a given function in $L^2(\mathcal{M})$ and the kernel $\mathcal{K}(x,y) \in L^2(\mathcal{M}^2)$ (Hilbert-Schmidt condition). The natural object to study is then the linear integral operator (28.16), acting on the Hilbert space $L^2(\mathcal{M})$: the fundamental property that follows from the $L^2(Q)$ nature of the kernel is that such an operator is *compact*, that is close to a finite rank operator.A compact operator has the property that for every $\delta > 0$ only a *finite* number of linearly independent eigenvectors exist corresponding to eigenvalues whose absolute value exceeds δ, so we immediately realize (figure 28.5) that much work is needed to bring Perron-Frobenius operators into this picture.

We rewrite (28.17) in the form

$$\mathcal{T}\varphi = \xi\,, \qquad \mathcal{T} = 1\!\!1 - \mathcal{K}\,. \tag{28.18}$$

The Fredholm alternative is now applied to this situation as follows: the equation $\mathcal{T}\varphi = \xi$ has a unique solution for every $\xi \in L^2(\mathcal{M})$ **or** there exists a non-zero solution of $\mathcal{T}\varphi_0 = 0$, with an eigenvector of $\mathcal{K}$ corresponding to the eigenvalue 1. The theory remains the same if instead of $\mathcal{T}$ we consider the operator $\mathcal{T}_\lambda = 1\!\!1 - \lambda\mathcal{K}$ with $\lambda \neq 0$. As $\mathcal{K}$ is a compact operator there is at most a denumerable set of λ for which the second part of the Fredholm alternative holds: apart from this set the inverse operator $(1\!\!1 - \lambda\mathcal{T})^{-1}$ exists and is bounded (in the operator sense). When λ is sufficiently small we may look for a perturbative expression for such an inverse, as a geometric series

$$(1\!\!1 - \lambda\mathcal{K})^{-1} = 1\!\!1 + \lambda\mathcal{K} + \lambda^2\mathcal{K}^2 + \cdots = 1\!\!1 + \lambda\mathcal{W}\,, \tag{28.19}$$

where $\mathcal{K}^n$ is a compact integral operator with kernel

$$\mathcal{K}^n(x,y) = \int_{\mathcal{M}^{n-1}} dz_1 \ldots dz_{n-1}\, \mathcal{K}(x,z_1)\cdots\mathcal{K}(z_{n-1},y)\,,$$

and $\mathcal{W}$ is also compact, as it is given by the convergent sum of compact operators. The problem with (28.19) is that the series has a finite radius of convergence, while apart from a denumerable set of λ's the inverse operator is well defined. A fundamental result in the theory of integral equations consists in rewriting the resolving kernel $\mathcal{W}$ as a ratio of two *analytic* functions of λ

$$\mathcal{W}(x,y) = \frac{\mathcal{D}(x,y;\lambda)}{D(\lambda)}.$$

If we introduce the notation

$$\mathcal{K}\begin{pmatrix} x_1 \ldots x_n \\ y_1 \ldots y_n \end{pmatrix} = \begin{vmatrix} \mathcal{K}(x_1,y_1) & \ldots & \mathcal{K}(x_1,y_n) \\ \ldots & \ldots & \ldots \\ \mathcal{K}(x_n,y_1) & \ldots & \mathcal{K}(x_n,y_n) \end{vmatrix}$$

we may write the explicit expressions

$$\begin{aligned} D(\lambda) &= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} \int_{\mathcal{M}^n} dz_1 \ldots dz_n\, \mathcal{K}\begin{pmatrix} z_1 \ldots z_n \\ z_1 \ldots z_n \end{pmatrix} \\ &= \exp\left(-\sum_{m=1}^{\infty} \frac{\lambda^m}{m} \mathrm{tr}\, \mathcal{K}^m\right) \\ \mathcal{D}(x,y;\lambda) &= \mathcal{K}\begin{pmatrix} x \\ y \end{pmatrix} + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_{\mathcal{M}^n} dz_1 \ldots dz_n\, \mathcal{K}\begin{pmatrix} x & z_1 & \ldots & z_n \\ y & z_1 & \ldots & z_n \end{pmatrix} \end{aligned} \tag{28.20}$$

The quantity $D(\lambda)$ is known as the Fredholm determinant (see (22.19)):it is an entire analytic function of λ, and $D(\lambda) = 0$ if and only if $1/\lambda$ is an eigenvalue of $\mathcal{K}$.

Worth emphasizing again: the Fredholm theory is based on the compactness of the integral operator, i.e., on the functional properties (summability) of its kernel. As the Perron-Frobenius operator is not compact, there is a bit of wishful thinking involved here.

28.4 Analyticity of spectral determinants

> They savored the strange warm glow of being much more ignorant than ordinary people, who were only ignorant of ordinary things.
> —Terry Pratchett

Spaces of functions integrable L^1, or square-integrable L^2 on interval $[0, 1]$ are mapped into themselves by the Perron-Frobenius operator, and in both cases the constant function $\phi_0 \equiv 1$ is an eigenfunction with eigenvalue 1. If we focus our attention on L^1 we also have a family of L^1 eigenfunctions,

$$\phi_\theta(y) = \sum_{k\neq 0} \exp(2\pi iky)\frac{1}{|k|^\theta} \tag{28.21}$$

with complex eigenvalue $2^{-\theta}$, parameterized by complex θ with Re $\theta > 0$. By varying θ one realizes that such eigenvalues fill out the entire unit disk. Such *essential spectrum*, the case $k = 0$ of figure 28.5, hides all fine details of the spectrum.

What's going on? Spaces L^1 and L^2 contain arbitrarily ugly functions, allowing any singularity as long as it is (square) integrable - and there is no way that expanding dynamics can smooth a kinky function with a non-differentiable singularity, let's say a discontinuous step, and that is why the eigenspectrum is dense rather than discrete. Mathematicians love to wallow in this kind of muck, but there is no way to prepare a nowhere differentiable L^1 initial density in a laboratory. The only thing we can prepare and measure are piecewise smooth (real-analytic) density functions.

For a bounded linear operator $\mathcal{A}$ on a Banach space $\mathbf{\Omega}$, the spectral radius is the smallest positive number ρ_{spec} such that the spectrum is inside the disk of radius ρ_{spec}, while the essential spectral radius is the smallest positive number ρ_{ess} such that outside the disk of radius ρ_{ess} the spectrum consists only of isolated eigenvalues of finite multiplicity (see figure 28.5). exercise 28.5

We may shrink the essential spectrum by letting the Perron-Frobenius operator act on a space of smoother functions, exactly as in the one-branch repeller case of sect. 28.1. We thus consider a smaller space, $\mathbb{C}^{k+\alpha}$, the space of k times differentiable functions whose k'th derivatives are Hölder continuous with an exponent $0 < \alpha \leq 1$: the expansion property guarantees that such a space is mapped into itself by the Perron-Frobenius operator. In the strip $0 <$ Re $\theta < k + \alpha$ most ϕ_θ will cease to be eigenfunctions in the space $\mathbb{C}^{k+\alpha}$; the function ϕ_n survives only for integer valued $\theta = n$. In this way we arrive at a finite set of *isolated* eigenvalues $1, 2^{-1}, \cdots, 2^{-k}$, and an essential spectral radius $\rho_{ess} = 2^{-(k+\alpha)}$.

We follow a simpler path and restrict the function space even further, namely to a space of analytic functions, i.e., functions for which the Taylor expansion is convergent at each point of the interval $[0, 1]$. With this choice things turn out easy and elegant. To be more specific, let ϕ be a holomorphic and bounded function on

the disk $D = B(0, R)$ of radius $R > 0$ centered at the origin. Our Perron-Frobenius operator preserves the space of such functions provided $(1 + R)/2 < R$ so all we need is to choose $R > 1$. If F_s, $s \in \{0, 1\}$, denotes the s inverse branch of the Bernoulli shift (28.6), the corresponding action of the Perron-Frobenius operator is given by $\mathcal{L}_s h(y) = \sigma\, F_s'(y)\, h \circ F_s(y)$, using the Cauchy integral formula along the ∂D boundary contour:

$$\mathcal{L}_s h(y) = \sigma \oint_{\partial D} \frac{dw}{2\pi i} \frac{h(w)F_s'(y)}{w - F_s(y)} . \tag{28.22}$$

For reasons that will be made clear later we have introduced a sign $\sigma = \pm 1$ of the given real branch $|F'(y)| = \sigma\, F'(y)$. For both branches of the Bernoulli shift $s = 1$, but in general one is not allowed to take absolute values as this could destroy analyticity. In the above formula one may also replace the domain D by *any domain* containing $[0, 1]$ such that the inverse branches maps the closure of D into the interior of D. Why? simply because the kernel remains non-singular under this condition, i.e., $w - F(y) \neq 0$ whenever $w \in \partial D$ and $y \in \mathrm{Cl}\, D$. The problem is now reduced to the standard theory for Fredholm determinants, sect. 28.3. The integral kernel is no longer singular, traces and determinants are well-defined, and we can evaluate the trace of $\mathcal{L}_F$ by means of the Cauchy contour integral formula:

$$\mathrm{tr}\, \mathcal{L}_F = \oint \frac{dw}{2\pi i} \frac{\sigma F'(w)}{w - F(w)} .$$

Elementary complex analysis shows that since F maps the closure of D into its own interior, F has a unique (real-valued) fixed point x^* with a multiplier strictly smaller than one in absolute value. Residue calculus therefore yields exercise 28.6

$$\mathrm{tr}\, \mathcal{L}_F = \frac{\sigma F'(x^*)}{1 - F'(x^*)} = \frac{1}{|f'(x^*) - 1|} ,$$

justifying our previous *ad hoc* calculations of traces using Dirac delta functions.

Example 28.8 ***Perron-Frobenius operator in a matrix representation:*** *As in example 28.1, we start with a map with a single fixed point, but this time with a nonlinear piecewise analytic map f with a nonlinear inverse $F = f^{-1}$, sign of the derivative $\sigma = \sigma(F') = F'/|F'|$*

$$\mathcal{L}\phi(z) = \int dx\, \delta(z - f(x))\, \phi(x) = \sigma\, F'(z)\, \phi(F(z)) .$$

Assume that F is a contraction of the unit disk, i.e.,

$$|F(z)| < \theta < 1 \quad \text{and} \quad |F'(z)| < C < \infty \quad \text{for} \quad |z| < 1 , \tag{28.23}$$

and expand ϕ in a polynomial basis by means of the Cauchy formula

$$\phi(z) = \sum_{n\geq 0} z^n \phi_n = \oint \frac{dw}{2\pi i} \frac{\phi(w)}{w - z} , \quad \phi_n = \oint \frac{dw}{2\pi i} \frac{\phi(w)}{w^{n+1}}$$

Combining this with (28.22), we see that in this basis $\mathcal{L}$ is represented by the matrix

$$\mathcal{L}\phi(w) = \sum_{m,n} w^m \mathbf{L}_{mn} \phi_n , \quad \mathbf{L}_{mn} = \oint \frac{dw}{2\pi i} \frac{\sigma\, F'(w)(F(w))^n}{w^{m+1}} . \tag{28.24}$$

Taking the trace and summing we get:

$$\mathrm{tr}\ \mathcal{L} = \sum_{n\geq 0} \mathbf{L}_{nn} = \oint \frac{dw}{2\pi i}\ \frac{\sigma\ F'(w)}{w - F(w)}\,.$$

This integral has but one simple pole at the unique fixed point $w^* = F(w^*) = f(w^*)$. *Hence*

$$\mathrm{tr}\ \mathcal{L} = \frac{\sigma\ F'(w^*)}{1 - F'(w^*)} = \frac{1}{|f'(w^*) - 1|}\,.$$

We worked out a very specific example, yet our conclusions can be generalized, provided a number of restrictive requirements are met by the dynamical system under investigation: exercise 28.6

1) the evolution operator is *multiplicative* along the flow,
2) the symbolic dynamics is a *finite subshift*,
3) all cycle eigenvalues are *hyperbolic* (exponentially bounded in magnitude away from 1),
4) the map (or the flow) is *real analytic*, i.e., it has a piecewise analytic continuation to a complex extension of the state space.

These assumptions are romantic expectations not satisfied by the dynamical systems that we actually desire to understand. Still, they are not devoid of physical interest; for example, nice repellers like our 3-disk game of pinball do satisfy the above requirements.

Properties 1 and 2 enable us to represent the evolution operator as a finite matrix in an appropriate basis; properties 3 and 4 enable us to bound the size of the matrix elements and control the eigenvalues. To see what can go wrong, consider the following examples:

Property 1 is violated for flows in 3 or more dimensions by the following weighted evolution operator

$$\mathcal{L}^t(y, x) = |\Lambda^t(x)|^\beta \delta\big(y - f^t(x)\big)\ ,$$

where $\Lambda^t(x)$ is an eigenvalue of the Jacobian matrix transverse to the flow. Semiclassical quantum mechanics suggest operators of this form with $\beta = 1/2$.The problem with such operators arises from the fact that when considering the Jacobian matrices $J_{ab} = J_a J_b$ for two successive trajectory segments a and b, the corresponding eigenvalues are in general *not* multiplicative, $\Lambda_{ab} \neq \Lambda_a \Lambda_b$ (unless a, b are iterates of the same prime cycle p, so $J_a J_b = J_p^{r_a + r_b}$). Consequently, this evolution operator is not multiplicative along the trajectory. The theorems require that the evolution be represented as a matrix in an appropriate polynomial basis, and thus cannot be applied to non-multiplicative kernels, i.e., kernels that do not satisfy the semi-group property $\mathcal{L}^{t'}\mathcal{L}^t = \mathcal{L}^{t'+t}$.

Property 2 is violated by the 1-dimensional tent map (see figure 28.3 (a))

$$f(x) = \alpha(1\ -\ |1 - 2x|)\ ,\quad 1/2 < \alpha < 1\ .$$

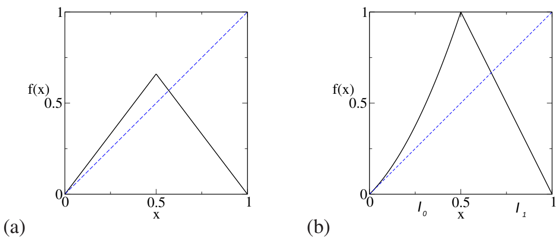

Figure 28.3: (a) A (hyperbolic) tent map without a finite Markov partition. (b) A Markov map with a marginal fixed point.

All cycle eigenvalues are hyperbolic, but in general the critical point $x_c = 1/2$ is not a pre-periodic point, so there is no finite Markov partition and the symbolic dynamics does not have a finite grammar (see sect. 15.4 for definitions). In practice, this means that while the leading eigenvalue of $\mathcal{L}$ might be computable, the rest of the spectrum is very hard to control; as the parameter α is varied, the non-leading zeros of the spectral determinant move wildly about.

Property 3 is violated by the map (see figure 28.3 (b))

$$f(x) = \begin{cases} x + 2x^2 & ,\quad x \in I_0 = [0, \frac{1}{2}] \\ 2 - 2x & ,\quad x \in I_1 = [\frac{1}{2}, 1] \end{cases}\ .$$

Here the interval $[0, 1]$ has a Markov partition into two subintervals I_0 and I_1, and f is monotone on each. However, the fixed point at $x = 0$ has marginal stability $\Lambda_0 = 1$, and violates condition 3. This type of map is called "intermittent" and necessitates much extra work. The problem is that the dynamics in the neighborhood of a marginal fixed point is very slow, with correlations decaying as power laws rather than exponentially. We will discuss such flows in chapter 29.

Property 4 is required as the heuristic approach of chapter 21 faces two major hurdles:

1. The trace (21.7) is not well defined because the integral kernel is singular.

2. The existence and properties of eigenvalues are by no means clear.

Actually, property 4 is quite restrictive, but we need it in the present approach, so that the Banach space of analytic functions in a disk is preserved by the Perron-Frobenius operator.

In attempting to generalize the results, we encounter several problems. First, in higher dimensions life is not as simple. Multi-dimensional residue calculus is at our disposal but in general requires that we find poly-domains (direct product of domains in each coordinate) and this need not be the case. Second, and perhaps somewhat surprisingly, the 'counting of periodic orbits' presents a difficult problem. For example, instead of the Bernoulli shift consider the doubling map (14.19) of the circle, $x \mapsto 2x \bmod 1$, $x \in R/Z$. Compared to the shift on the interval $[0, 1]$ the only difference is that the endpoints 0 and 1 are now glued together. Because these endpoints are fixed points of the map, the number of cycles of length n decreases by 1. The determinant becomes:

$$\det(1 - z\mathcal{L}) = \exp\left(-\sum_{n=1} \frac{z^n}{n} \frac{2^n - 1}{2^n - 1}\right) = 1 - z. \tag{28.25}$$

The value $z = 1$ still comes from the constant eigenfunction, but the Bernoulli polynomials no longer contribute to the spectrum (as they are not periodic). Proofs of these facts, however, are difficult if one sticks to the space of analytic functions.

Third, our Cauchy formulas *a priori* work only when considering purely expanding maps. When stable and unstable directions co-exist we have to resort to stranger function spaces, as shown in the next section.

28.5 Hyperbolic maps

I can give you a definion of a Banach space, but I do not know what that means.
—Federico Bonnetto, *Banach space*

(H.H. Rugh)

Proceeding to hyperbolic systems, one faces the following paradox: If f is an area-preserving hyperbolic and real-analytic map of, for example, a 2-dimensional torus then the Perron-Frobenius operator is unitary on the space of L^2 functions, and its spectrum is confined to the unit circle. On the other hand, when we compute determinants we find eigenvalues scattered around inside the unit disk. Thinking back to the Bernoulli shift example 28.5 one would like to imagine these eigenvalues as popping up from the L^2 spectrum by shrinking the function space. Shrinking the space, however, can only make the spectrum smaller so this is obviously not what happens. Instead one needs to introduce a 'mixed' function space where in the unstable direction one resorts to analytic functions, as before, but in the stable direction one instead considers a 'dual space' of distributions on analytic functions. Such a space is neither included in nor includes L^2 and we have thus resolved the paradox. However, it still remains to be seen how traces and determinants are calculated.

The linear hyperbolic fixed point example 28.6 is somewhat misleading, as we have made explicit use of a map that acts independently along the stable and unstable directions. For a more general hyperbolic map, there is no way to implement such direct product structure, and the whole argument falls apart. Her comes an idea; use the analyticity of the map to rewrite the Perron-Frobenius operator acting as follows (where σ denotes the sign of the derivative in the unstable direction):

$$\mathcal{L}h(z_1, z_2) = \oint\oint \frac{\sigma\ h(w_1, w_2)}{(z_1 - f_1(w_1, w_2)(f_2(w_1, w_2) - z_2)} \frac{dw_1}{2\pi i} \frac{dw_2}{2\pi i}. \tag{28.26}$$

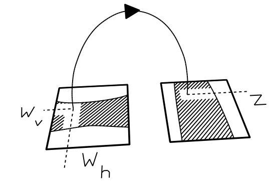

Figure 28.4: For an analytic hyperbolic map, specifying the contracting coordinate w_h at the initial rectangle and the expanding coordinate z_v at the image rectangle defines a unique trajectory between the two rectangles. In particular, w_v and z_h (not shown) are uniquely specified.

Here the function ϕ should belong to a space of functions analytic respectively *outside* a disk and *inside* a disk in the first and the second coordinates; with the additional property that the function decays to zero as the first coordinate tends to infinity. The contour integrals are along the boundaries of these disks. It is an exercise in multi-dimensional residue calculus to verify that for the above linear example this expression reduces to (28.9). Such operators form the building blocks in the calculation of traces and determinants. One can prove the following:

Theorem: *The spectral determinant for* 2*-dimensional hyperbolic analytic maps is entire.* remark 28.8

The proof, apart from the Markov property that is the same as for the purely expanding case, relies heavily on the analyticity of the map in the explicit construction of the function space. The idea is to view the hyperbolicity as a cross product of a contracting map in forward time and another contracting map in backward time. In this case the Markov property introduced above has to be elaborated a bit. Instead of dividing the state space into intervals, one divides it into rectangles. The rectangles should be viewed as a direct product of intervals (say horizontal and vertical), such that the forward map is contracting in, for example, the horizontal direction, while the inverse map is contracting in the vertical direction. For Axiom A systems (see remark 28.8) one may choose coordinate axes close to the stable/unstable manifolds of the map. With the state space divided into N rectangles $\{\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_N\}$, $\mathcal{M}_i = I_i^h \times I_i^v$ one needs a complex extension $D_i^h \times D_i^v$, with which the hyperbolicity condition (which simultaneously guarantees the Markov property) can be formulated as follows:

Analytic hyperbolic property: Either $f(\mathcal{M}_i) \cap \mathrm{Int}(\mathcal{M}_j) = \emptyset$, or for each pair $w_h \in \mathrm{Cl}(D_i^h)$, $z_v \in \mathrm{Cl}(D_j^v)$ there exist unique analytic functions of w_h, z_v: $w_v = w_v(w_h, z_v) \in \mathrm{Int}(D_i^v)$, $z_h = z_h(w_h, z_v) \in \mathrm{Int}(D_j^h)$, such that $f(w_h, w_v) = (z_h, z_v)$. Furthermore, if $w_h \in I_i^h$ and $z_v \in I_j^v$, then $w_v \in I_i^v$ and $z_h \in I_j^h$ (see figure 28.4).

In plain English, this means for the iterated map that one replaces the coordinates z_h, z_v at time n by the contracting pair z_h, w_v, where w_v is the contracting coordinate at time $n + 1$ for the 'partial' inverse map.

In two dimensions the operator in (28.26) acts on functions analytic outside D_i^h in the horizontal direction (and tending to zero at infinity) and inside D_i^v in the vertical direction. The contour integrals are precisely along the boundaries of these domains.

A map f satisfying the above condition is called *analytic hyperbolic* and the theorem states that the associated spectral determinant is entire, and that the trace formula (21.7) is correct.

Examples of analytic hyperbolic maps are provided by small analytic perturbations of the cat map, the 3-disk repeller, and the 2-dimensional baker's map.

28.6 Physics of eigenvalues and eigenfunctions

By now we appreciate that any honest attempt to look at the spectral properties of the Perron-Frobenius operator involves hard mathematics, but the reward is of this effort is that we are able to control the analyticity properties of dynamical zeta functions and spectral determinants, and thus substantiate the claim that these objects provide a powerful and well-founded theory.

Often (see chapter 20) physically important part of the spectrum is just the leading eigenvalue, which gives us the escape rate from a repeller, or, for a general evolution operator, formulas for expectation values of observables and their higher moments. Also the eigenfunction associated to the leading eigenvalue has a physical interpretation (see chapter 19): it is the density of the natural measures, with singular measures ruled out by the proper choice of the function space. This conclusion is in accord with the generalized Perron-Frobenius theorem for evolution operators. In a finite dimensional setting, the statement is: remark 28.7

- **Perron-Frobenius theorem:** Let $\mathbf{L}_{ij}$ be a non-negative matrix, such that some finite n exists for which any initial state has reached any other state, $(L^n)_{ij} > 0 \ \forall i, j$: then
  1. The maximal modulus eigenvalue is non-degenerate, real, and positive,
  2. The corresponding eigenvector (defined up to a constant) has non-negative coordinates.

We may ask what physical information is contained in eigenvalues beyond the leading one: suppose that we have a probability conserving system (so that the dominant eigenvalue is 1), for which the essential spectral radius satisfies $0 < \rho_{ess} < \theta < 1$ on some Banach space $\mathcal{B}$. Denote by $\mathbf{P}$ the projection corresponding to the part of the spectrum inside a disk of radius θ. We denote by $\lambda_1, \lambda_2 \ldots, \lambda_M$ the eigenvalues outside of this disk, ordered by the size of their absolute value, with $\lambda_1 = 1$. Then we have the following decomposition

$$\mathcal{L}\varphi = \sum_{i=1}^{M} \lambda_i \psi_i L_i \psi_i^* \varphi + \mathbf{P}\mathcal{L}\varphi \tag{28.27}$$

when L_i are (finite) matrices in Jordan canomical form ($L_0 = 0$ is a $[1\times1]$ matrix, as λ_0 is simple, due to the Perron-Frobenius theorem), whereas ψ_i is a row vector

whose elements form a basis on the eigenspace corresponding to λ_i, and ψ_i^* is a column vector of elements of $\mathcal{B}^*$ (the dual space of linear functionals over $\mathcal{B}$) spanning the eigenspace of $\mathcal{L}^*$ corresponding to λ_i. For iterates of the Perron-Frobenius operator, (28.27) becomes

$$\mathcal{L}^n\varphi = \sum_{i=1}^{M} \lambda_i^n \psi_i L_i^n \psi_i^* \varphi + \mathbf{P}\mathcal{L}^n\varphi\,. \tag{28.28}$$

If we now consider, for example, correlation between initial φ evolved n steps and final ξ,

$$\langle\xi|\mathcal{L}^n|\varphi\rangle = \int_{\mathcal{M}} dy\, \xi(y) \left(\mathcal{L}^n\varphi\right)(y) = \int_{\mathcal{M}} dw\, (\xi \circ f^n)(w)\varphi(w)\,, \tag{28.29}$$

it follows that

$$\langle\xi|\mathcal{L}^n|\varphi\rangle = \lambda_1^n \omega_1(\xi,\varphi) + \sum_{i=2}^{L} \lambda_i^n \omega_i^{(n)}(\xi,\varphi) + O(\theta^n)\,, \tag{28.30}$$

where

$$\omega_i^{(n)}(\xi,\varphi) = \int_{\mathcal{M}} dy\, \xi(y)\psi_i L_i^n \psi_i^* \varphi\,.$$

The eigenvalues beyond the leading one provide two pieces of information: they rule the convergence of expressions containing high powers of the evolution operator to leading order (the λ_1 contribution). Moreover if $\omega_1(\xi,\varphi) = 0$ then exercise 28.7 (28.29) defines a correlation function: as each term in (28.30) vanishes exponentially in the $n \to \infty$ limit, the eigenvalues $\lambda_2, \ldots, \lambda_M$ determine the exponential decay of correlations for our dynamical system. The prefactors ω depend on the choice of functions, whereas the exponential decay rates (given by logarithms of λ_i) do not: the correlation spectrum is thus a *universal* property of the dynamics (once we fix the overall functional space on which the Perron-Frobenius operator acts).

Example 28.9 ***Bernoulli shift eigenfunctions:*** *Let us revisit the Bernoulli shift example (28.6) on the space of analytic functions on a disk: apart from the origin we have only simple eigenvalues $\lambda_k = 2^{-k},\ k = 0, 1, \ldots$. The eigenvalue $\lambda_0 = 1$ corresponds to probability conservation: the corresponding eigenfunction $B_0(x) = 1$ indicates that the natural measure has a constant density over the unit interval. If we now take any analytic function $\eta(x)$ with zero average (with respect to the Lebesgue measure), it follows that $\omega_1(\eta,\eta) = 0$, and from (28.30) the asymptotic decay of the correlation function is (unless also $\omega_1(\eta,\eta) = 0$)*

$$C_{\eta,\eta}(n) \sim \exp(-n\log 2)\,. \tag{28.31}$$

Thus, $-\log\lambda_1$ gives the exponential decay rate of correlations (with a prefactor that depends on the choice of the function). Actually the Bernoulli shift case may be treated exactly, as for analytic functions we can employ the Euler-MacLaurin summation formula

$$\eta(z) = \int_0^1 dw\, \eta(w) + \sum_{m=1}^{\infty} \frac{\eta^{(m-1)}(1) - \eta^{(m-1)}(0)}{m!} B_m(z)\,. \tag{28.32}$$

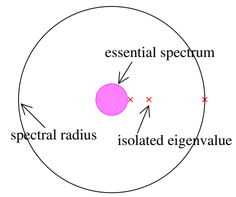

Figure 28.5: Spectrum of the Perron-Frobenius operator acting on the space of $\mathbb{C}^{k+\alpha}$ Hölder-continuous functions: only k isolated eigenvalues remain between the spectral radius, and the essential spectral radius which bounds the "essential," continuous spectrum.

As we are considering functions with zero average, we have from (28.29) and the fact that Bernoulli polynomials are eigenvectors of the Perron-Frobenius operator that

$$C_{\eta,\eta}(n) = \sum_{m=1}^{\infty} \frac{(2^{-m})^n(\eta^{(m)}(1) - \eta^{(m)}(0))}{m!} \int_0^1 dz\, \eta(z) B_m(z)\,.$$

The decomposition (28.32) is also useful in realizing that the linear functionals ψ_i^* are singular objects: if we write it as

$$\eta(z) = \sum_{m=0}^{\infty} B_m(z)\, \psi_m^*[\eta]\,,$$

we see that these functionals are of the form

$$\psi_i^*[\varepsilon] = \int_0^1 dw\, \Psi_i(w) \varepsilon(w)\,,$$

where

$$\Psi_i(w) = \frac{(-1)^{i-1}}{i!} \left(\delta^{(i-1)}(w-1) - \delta^{(i-1)}(w) \right), \tag{28.33}$$

when $i \geq 1$ and $\Psi_0(w) = 1$. This representation is only meaningful when the function ε is analytic in neighborhoods of $w, w-1$.

28.7 Troubles ahead

The above discussion confirms that for a series of examples of increasing generality formal manipulations with traces and determinants are justified: the Perron-Frobenius operator has isolated eigenvalues, the trace formulas are explicitly verified, and the spectral determinant is an entire function whose zeroes yield the eigenvalues. Real life is harder, as we may appreciate through the following considerations:

- Our discussion tacitly assumed something that is physically entirely reasonable: our evolution operator is acting on the space of analytic functions, i.e., we are allowed to represent the initial density $\rho(x)$ by its Taylor expansions

in the neighborhoods of periodic points. This is however far from being the only possible choice: mathematicians often work with the function space $\mathbb{C}^{k+\alpha}$, i.e., the space of k times differentiable functions whose k'th derivatives are Hölder continuous with an exponent $0 < \alpha \leq 1$: then every y^η with $\mathrm{Re}\,\eta > k$ is an eigenfunction of the Perron-Frobenius operator and we have exercise 28.1

$$\mathcal{L} y^\eta = \frac{1}{|\Lambda|\Lambda^\eta} y^\eta\,, \qquad \eta \in \mathbb{C}\,.$$

This spectrum differs markedly from the analytic case: only a small number of isolated eigenvalues remain, enclosed between the spectral radius and a smaller disk of radius $1/|\Lambda|^{k+1}$, see figure 28.5. In literature the radius of this disk is called the *essential spectral radius*.

In sect. 28.4 we discussed this point further, with the aid of a less trivial 1-dimensional example. The physical point of view is complementary to the standard setting of ergodic theory, where many chaotic properties of a dynamical system are encoded by the presence of a *continuous* spectrum, used to prove asymptotic decay of correlations in the space of L^2 square-integrable functions. exercise 28.2

- A deceptively innocent assumption is hidden beneath much that was discussed so far: that (28.1) maps a given function space into itself. The *expanding* property of the map guarantees that: if $f(x)$ is smooth in a domain D then $f(x/\Lambda)$ is smooth on a *larger* domain, provided $|\Lambda| > 1$. For higher-dimensional hyperbolic flows this is not the case, and, as we saw in sect. 28.5, extensions of the results obtained for expanding 1-dimensional maps are highly nontrivial.

- It is not at all clear that the above analysis of a simple one-branch, one fixed point repeller can be extended to dynamical systems with Cantor sets of periodic points: we showed this in sect. 28.4.

Résumé

Examples of analytic eigenfunctions for 1-dimensional maps are seductive, and make the problem of evaluating ergodic averages appear easy; just integrate over the desired observable weighted by the natural measure, right? No, generic natural measure sits on a fractal set and is singular everywhere. The point of this book is that you never need to construct the natural measure, cycle expansions will do that job.

A theory of evaluation of dynamical averages by means of trace formulas and spectral determinants requires a deep understanding of their analyticity and convergence. We worked here through a series of examples:

1. exact spectrum (but for a single fixed point of a linear map)
2. exact spectrum for a locally analytic map, matrix representation

3. rigorous proof of existence of discrete spectrum for 2-dimensional hyperbolic maps

In the case of especially well-behaved "Axiom *A*" systems, where both the symbolic dynamics and hyperbolicity are under control, it is possible to treat traces and determinants in a rigorous fashion, and strong results about the analyticity properties of dynamical zeta functions and spectral determinants outlined above follow.

Most systems of interest are *not* of the "axiom A" category; they are neither purely hyperbolic nor (as we have seen in chapters 14 and 15) do they have finite grammar. The importance of symbolic dynamics is generally grossly under appreciated; the crucial ingredient for nice analyticity properties of zeta functions is the existence of a finite grammar (coupled with uniform hyperbolicity).

The dynamical systems which are *really* interesting - for example, smooth bounded Hamiltonian potentials - are presumably never fully chaotic, and the central question remains: How do we attack this problem in a systematic and controllable fashion?

Theorem: Conjecture 3 with technical hypothesis is true in a lot of cases.

— M. Shub

Commentary

Remark 28.1 Surveys of rigorous theory. We recommend the references listed in remark 1.1 for an introduction to the mathematical literature on this subject. For a physicist, Driebe's monograph [1.19] might be the most accessible introduction into mathematics discussed briefly in this chapter. There are a number of reviews of the mathematical approach to dynamical zeta functions and spectral determinants, with pointers to the original references, such as refs. [28.1, 28.2]. An alternative approach to spectral properties of the Perron-Frobenius operator is given in ref. [28.3].

Ergodic theory, as presented by Sinai [28.14] and others, tempts one to describe the densities on which the evolution operator acts in terms of either integrable or square-integrable functions. For our purposes, as we have already seen, this space is not suitable. An introduction to ergodic theory is given by Sinai, Kornfeld and Fomin [28.15]; more advanced old-fashioned presentations are Walters [A39.15] and Denker, Grillenberger and Sigmund [28.16]; and a more formal one is given by Peterson [28.17].

Remark 28.2 Fredholm theory. Our brief summary of Fredholm theory is based on the exposition of ref. [28.4]. A technical introduction of the theory from an operator point of view is given in ref. [28.5]. The theory is presented in a more general form in ref. [28.6].

Remark 28.3 Bernoulli shift. For a more in-depth discussion, consult chapter 3 of ref. [1.19]. The extension of Fredholm theory to the case or Bernoulli shift on $\mathbb{C}^{k+\alpha}$ (in which the Perron-Frobenius operator is *not* compact – technically it is only *quasi-compact*. That is, the essential spectral radius is strictly smaller than the spectral radius) has been given by Ruelle [28.7]: a concise and readable statement of the results is contained in ref. [28.8]. We see from (28.31) that for the Bernoulli shift the exponential decay rate of correlations coincides with the Lyapunov exponent: while such an identity holds for a number of systems, it is by no means a general result, and there exist explicit counterexamples.

Remark 28.4 Hyperbolic dynamics. When dealing with hyperbolic systems one might try to reduce to the expanding case by projecting the dynamics along the unstable directions. As mentioned in the text this can be quite involved technically, as such unstable foliations are not characterized by strong smoothness properties. For such an approach, see ref. [28.3].

Remark 28.5 Spectral determinants for smooth flows. The theorem on page 526 also applies to hyperbolic analytic maps in d dimensions and smooth hyperbolic analytic flows in $(d + 1)$ dimensions, provided that the flow can be reduced to a piecewise analytic map by a suspension on a Poincaré section, complemented by an analytic "ceiling" function (3.5) that accounts for a variation in the section return times. For example, if we take as the ceiling function $g(x) = e^{sT(x)}$, where $T(x)$ is the next Poincaré section time for a trajectory staring at x, we reproduce the flow spectral determinant (22.26). Proofs are beyond the scope of this chapter.

Remark 28.6 Explicit diagonalization. For 1-dimensional repellers a diagonalization of an explicit truncated $\mathbf{L}_{mn}$ matrix evaluated in a judiciously chosen basis may yield many more eigenvalues than a cycle expansion (see refs. [A1.28, 28.11]). The reasons why one persists in using periodic orbit theory are partially aesthetic and partially pragmatic. The explicit calculation of $\mathbf{L}_{mn}$ demands an explicit choice of a basis and is thus non-invariant, in contrast to cycle expansions which utilize only the invariant information of the flow. In addition, we usually do not know how to construct $\mathbf{L}_{mn}$ for a realistic high-dimensional flow, such as the hyperbolic 3-disk game of pinball flow of sect. 1.3, whereas periodic orbit theory is true in higher dimensions and straightforward to apply.

Remark 28.7 Perron-Frobenius theorem. A proof of the Perron-Frobenius theorem may be found in ref. [A39.15]. For positive transfer operators, this theorem has been generalized by Ruelle [A39.13].

Remark 28.8 Axiom A systems. The proofs in sect. 28.5 follow the thesis work of H.H. Rugh [A1.22, 28.18, A1.60]. For a mathematical introduction to the subject, consult the excellent review by V. Baladi [28.1]. It would take us too far afield to give and explain the definition of Axiom A systems (see refs. [A1.7, A1.70]). Axiom A implies, however, the existence of a Markov partition of the state space from which the properties 2 and 3 assumed on page 515 follow.

Remark 28.9 Left eigenfunctions. We shall never use an explicit form of left eigenfunctions, corresponding to highly singular kernels like (28.33). Many details have been elaborated in a number of papers, such as ref. [28.20], with a daring physical interpretation.

Remark 28.10 Ulam's idea. The approximation of Perron-Frobenius operator defined by (19.11) has been shown to reproduce the spectrum for expanding maps, once finer and finer Markov partitions are used [28.21]. The subtle point of choosing a state space partitioning for a "generic case" is discussed in ref. [28.22].

Exercises

28.1. **What space does $\mathcal{L}$ act on?** Show that (28.2) is a complete basis on the space of analytic functions on a disk (and thus that we found the *complete* set of eigenvalues).

28.2. **What space does $\mathcal{L}$ act on?** What can be said about the spectrum of (28.1) on $L^1[0,1]$? Compare the result with figure 28.5.

28.3. **Euler formula.** Derive the Euler formula (28.5), $|u| < 1$:

$$\begin{aligned}\prod_{k=0}^{\infty}(1+tu^k) &= 1+\frac{t}{1-u}+\frac{t^2u}{(1-u)(1-u^2)}\\ &\quad +\frac{t^3u^3}{(1-u)(1-u^2)(1-u^3)}\cdots\\ &= \sum_{k=0}^{\infty} t^k \frac{u^{\frac{k(k-1)}{2}}}{(1-u)\cdots(1-u^k)}\,.\end{aligned}$$

28.4. **2-dimensional product expansion.** We conjecture that the expansion corresponding to exercise 28.3 is in the 2-dimensional case given by

$$\begin{aligned}&\prod_{k=0}^{\infty}(1+tu^k)^{k+1}\\ &= \sum_{k=0}^{\infty}\frac{F_k(u)}{(1-u)^2(1-u^2)^2\cdots(1-u^k)^2}t^k\\ &= 1+\frac{1}{(1-u)^2}t+\frac{2u}{(1-u)^2(1-u^2)^2}t^2\\ &\quad +\frac{u^2(1+4u+u^2)}{(1-u)^2(1-u^2)^2(1-u^3)^2}t^3+\cdots\end{aligned}$$

$F_k(u)$ is a polynomial in u, and the coefficients fall off asymptotically as $C_n \approx u^{n^{3/2}}$. Verify; if you have a proof to all orders, e-mail it to the authors. (See also solution 28.3).

28.5. **Bernoulli shift on L spaces.** Check that the family (28.21) belongs to $L^1([0,1])$. What can be said about the essential spectral radius on $L^2([0,1])$? A useful reference is ref. [28.24].

28.6. **Cauchy integrals.** Rework all complex analysis steps used in the Bernoulli shift example on analytic functions on a disk.

28.7. **Escape rate.** Consider the escape rate from a strange repeller: find a choice of trial functions ξ and φ such that (28.29) gives the fraction on particles surviving after n iterations, if their initial density distribution is $\rho_0(x)$. Discuss the behavior of such an expression in the long time limit.

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