Appendix A16

Finding cycles

(C. Chandre)

A16.1 Newton-Raphson method

A16.1.1 Contraction rate

Consider a $d$-dimensional map $x' = f(x)$ with an unstable fixed point $x_*$. The Newton-Raphson algorithm is obtained by iterating the following map

$$x' = g(x) = x - (J(x) - 1)^{-1} (f(x) - x).$$

The linearization of $g$ near $x_*$ leads to

$$x_* + \epsilon' = x_* + \epsilon - (J(x_*) - 1)^{-1} (f(x_*) + J(x_*)\epsilon - x_* - \epsilon) + O(\|\epsilon\|^2),$$

where $\epsilon = x - x_*$. Therefore,

$$x' - x_* = O(\|x - x_*\|^2).$$

After $n$ steps and if the initial guess $x_0$ is close to $x_*$, the error decreases exponentially

$$g^n(x_0) - x_* = O(\|x_0 - x_*\|^2^n).$$

A16.1.2 Computation of the inverse

The Newton-Raphson method for finding $n$-cycles of $d$-dimensional mappings using the multi-shooting method reduces to the following equation

$$
\begin{bmatrix}
1 & -Df(x_1) & \cdots & -Df(x_n) \\
-1 & \cdots & \cdots & \cdots \\
-1 & \cdots & \cdots & \cdots \\
\end{bmatrix}
\begin{bmatrix}
\delta_1 \\
\delta_2 \\
\vdots \\
\delta_n \\
\end{bmatrix}
= 
\begin{bmatrix}
F_1 \\
F_2 \\
\vdots \\
F_n \\
\end{bmatrix}.
$$

(A16.1)

where $Df(x_0)$ is the $[d \times d]$ Jacobian matrix of the map evaluated at the point $x_0$ and $\delta_m = x_m - x_0$ and $F_m = x_m - f(x_{m-1})$ are $d$-dimensional vectors. By some straightforward algebra, the vectors $\delta_m$ are expressed as functions of the vectors $F_m$:

$$\delta_m = -\sum_{k=1}^{m} \beta_{m,k} F_k - \beta_{m,1} (1 - \beta_{1,1})^{-1} \sum_{k=1}^{m} \beta_{m,k} F_k + \beta_{m,1} F_1.$$  

(A16.2)

for $m = 1, \ldots, n$, where $\beta_{m,k} = Df(x_{k+1})Df(x_{k}) \cdots Df(x_1)$ for $k < m$ and $\beta_{m,k} = 1$ for $k \geq m$. Therefore, finding $n$-cycles by a Newton-Raphson method with multiple shooting requires the inversing of a $[d \times d]$ matrix $1 - Df(x_0)Df(x_{n-1}) \cdots Df(x_1)$.

A16.2 Hybrid Newton-Raphson / relaxation method

Consider a $d$-dimensional map $x' = f(x)$ with an unstable fixed point $x_*$. The transformed map is the following one:

$$x' = g(x) = x + \gamma C f(x),$$

where $\gamma > 0$ and $C$ is a $d \times d$ invertible constant matrix. We note that $x_*$ is also a fixed point of $g$. Consider the stability matrix at the fixed point $x_*$

$$A_{\gamma} = \frac{dg}{dx_{\mid x=x_*}} = 1 + \gamma C(A_{\gamma} - I).$$

The matrix $C$ is constructed such that the eigenvalues of $A_{\gamma}$ are of modulus less than one. Assume that $A_{\gamma}$ is diagonalizable. In the basis of diagonalization, the matrix writes:

$$A_{\gamma} = A_{\gamma} = 1 + \gamma \tilde{C}(A_{\gamma} - I),$$

where $A_{\gamma}$ is diagonal with elements $\mu_i$. We restrict the set of matrices $\tilde{C}$ to diagonal matrices with $\tilde{C}_{ii} = \epsilon_i$ where $\epsilon_i = \pm 1$. Thus $A_{\gamma}$ is diagonal with eigenvalues $\gamma_i = 1 + \gamma \epsilon_i (\mu_i - 1)$. The choice of $\gamma$ and $\epsilon$ is such that $|\gamma_i| < 1$. It is easy to see that if $\text{Re}(\mu_i) < 1$ one has to choose $\epsilon_i = 1$, and if $\text{Re}(\mu_i) > 1$, $\epsilon_i = -1$. If it is chosen such that

$$0 < \gamma < \min_{i=1,\ldots,d} \left( \frac{2|\text{Re}(\mu_i) - 1|}{|\mu_i - 1|^2} \right),$$

all the eigenvalues of $A_{\gamma}$ have modulus less than one. The contraction rate at the fixed point for the map $g$ is then max, $|1 + \gamma \epsilon_i (\mu_i - 1)|$. If $\text{Re}(\mu_i) = 1$, it is not possible to stabilize $x_*$ by the set of matrices $\gamma C$.

From the construction of $C$, we see that $2^d$ choices of matrices are possible. For example, for 2-dimensional systems, these matrices are

$$C \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

For 2-dimensional dissipative maps, the eigenvalues satisfy $\text{Re}(\mu_1)\text{Re}(\mu_2) \leq \text{det} Df < 1$. The case $(\text{Re}(\mu_1), \text{Re}(\mu_2)) > 1$ which is stabilized by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has to be discarded. The minimal set is reduced to three matrices.
Figure A16.1: Illustration of the optimal Poincaré section. The original section \( y = 0 \) yields a large distance \( x - f(x) \) for the Newton iteration. A much better choice is \( y = 0.7 \).

**A16.2.1 Newton method with optimal section**

(F. Christiansen)

In some systems it might be hard to find a good starting guess for a fixed point. This can happen, for example, if the topology and/or the symbolic dynamics of the flow is not well understood. By changing the Poincaré section one might get a better initial guess in the sense that \( x \) and \( f(x) \) are closer together. We illustrate this in figure A16.1. The figure shows a Poincaré section, \( y = 0 \), an initial guess \( x \), the corresponding \( f(x) \) and pieces of the trajectory near these two points.

If Newton iteration does not converge for the initial guess \( x \) we might have to work very hard to find a better guess, particularly if this is in a high-dimensional system (high-dimensional in this context might mean a Hamiltonian system with 3 or more degrees of freedom). Clearly, we could easily obtain a much better guess by simply shifting the Poincaré section to \( y = 0.7 \) where the distance \( x - f(x) \) would be much smaller. Naturally, one cannot so easily determine by inspection the best section for a higher dimensional system. Rather, the way to proceed is as follows: We want to have a minimal distance between our initial guess \( x \) and its image \( f(x) \). We therefore integrate the flow looking for a minimum in the distance \( d(t) = |f^t(x) - x| \). \( d(t) \) is now a minimum with respect to variations in \( f^t(x) \), but not necessarily with respect to \( x \). We therefore integrate \( x \) either forward or backward in time. Doing this minimizes \( d \) with respect to \( x \), but now it is no longer minimal with respect to \( f^t(x) \). We therefore repeat the steps, alternating between correcting \( x \) and \( f^t(x) \). In most cases this process converges quite rapidly. The result is a trajectory for which the vector \( (f^t(x) - x) \) connecting the two end points is perpendicular to the flow at both points. We can now define a Poincaré section as the hyper-plane that goes through \( x \) and is normal to the flow at \( x \),

\[(x' - x) \cdot v(x) = 0.
\]

The image \( f(x) \) lies in the section. This section is optimal in the sense that a close return on the section is a local minimum of the distance between \( x \) and \( f^t(x) \). More important, the part of the stability matrix that describes linearization perpendicular to the flow is exactly the stability of the flow in the section when \( f(x) \) is close to \( x \). With this method, the Poincaré section changes with each Newton iteration. Should we later want to put the fixed point on a specific Poincaré section, it will only be a matter of moving along the trajectory.