

6. FROM GHOULIES TO GHOSTIES

A physical photon is massless and has only transverse degrees of freedom; still, in relativistic calculations it is convenient to pretend that the photon is a vector particle. Decoupling of the extra degree of freedom is guaranteed by Ward identities. We shall use the requirement of the decoupling of the extra degrees of freedom as the guiding principle for constructing the QCD action. In retrospect it will be clear that this diagrammatic derivation corresponds step by step to the textbook local gauge invariance arguments. Still, this kind of derivation has its charms - it shows rather explicitly how the ghosts eat up the unphysical gluon degrees of freedom, and how the Ward identities guarantee their decoupling.

A. Massless vector particles

A massive vector particle is characterized by its mass M and its polarization $\epsilon_\mu^\lambda(k)$. There are $\lambda = 1, 2, \dots, d-1$ independent polarizations; in the rest frame $k^\mu = (M, \vec{0})$, so a vector particle can point in $d-1$ directions. Another way to see this is to observe that k^μ , the direction of propagation of a free spinning particle, reduces the symmetry from $SO(1, d-1)$ to $SO(d-1)$, the rotations in the transverse spacetime directions.

In the rest frame a vector particle points in a direction $\vec{\epsilon}$. The choice of the coordinates is quite arbitrary; one can choose any $d-1$ independent basis vectors \hat{e}_λ (circular polarizations, for example) and express the polarization in this basis

$$\epsilon_i = \sum_\lambda \epsilon_i^\lambda \hat{e}_\lambda \quad \lambda, i = 1, 2, \dots, d-1 \quad .$$

To describe the polarizations covariantly, we add a fake d -th polarization ϵ_0^λ and set it equal to zero by the transversality condition

$$k^\mu \epsilon_\mu^\lambda(k) = 0 \quad , \quad \begin{array}{l} \lambda = 1, 2, \dots, d-1; \text{ polarization} \\ \mu = 1, 2, \dots, d; \text{ Minkowski} \end{array} \quad (6.1)$$

This reduces to $\epsilon_0^\lambda = 0$ in the rest frame. Being explicitly covariant, the transversality condition also describes the $d-1$

vector polarizations in any frame.

The momentum of a physical massive particle satisfies the mass-shell condition:

$$k^2 - M^2 = 0 \quad . \quad (6.2)$$

If the particle is massless

$$k^2 = 0 \quad (6.3)$$

it is not possible to bring it to a rest frame. The best we can do is to align it along the lightcone: $k^\mu = (E, 0, 0, \dots, E)$. A physical massless spinning particle is always whizzing along a spatial direction $\vec{k} = (0, 0, \dots, E)$, and the symmetry is reduced from $SO(1, d-1)$ to $SO(d-2)$, the rotations in the transverse space directions. Hence a massless vector particle has $d-2$ polarizations. The trouble is that there is no nice way of imposing the masslessness condition on the polarizations. We can, however, see that there is one degree of freedom less than in the massive case, because we can freely vary the polarizations along the longitudinal direction

$$\varepsilon_\mu(k) \rightarrow \varepsilon_\mu(k) + k_\mu \omega(k) \quad , \quad (6.4)$$

($\omega(k)$ arbitrary function) without violating the transversality condition (6.1). (Remember that $k^2 = 0$). For somewhat obscure historical reasons, this kind of transformation is called a gauge transformation[†].

Under the gauge transformation (6.4) the transition amplitudes pick up extra contributions from the longitudinal bits, or "gaugeons". We denote gaugeons diagrammatically by

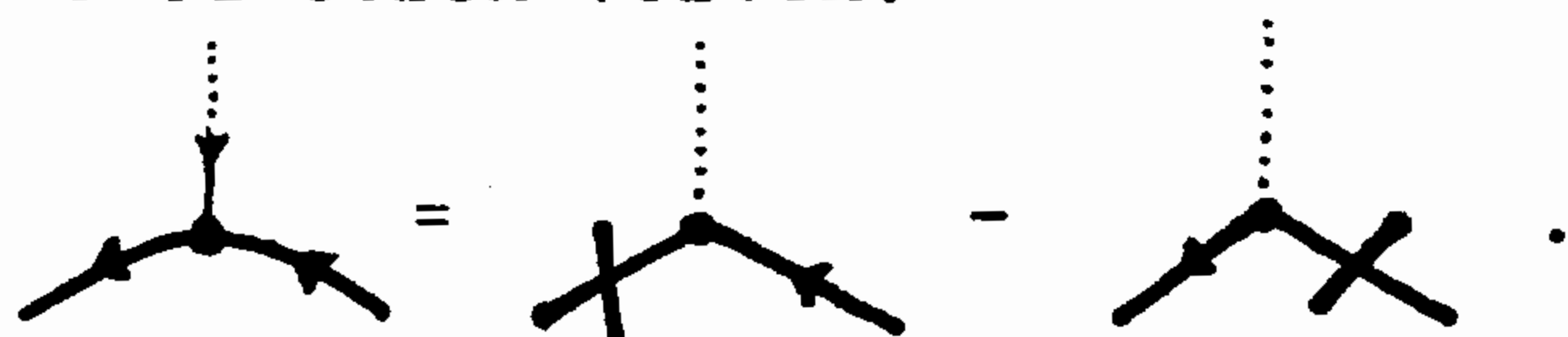
$$\begin{array}{c} \cdots \longrightarrow \bullet \\ \mu = \frac{-i}{k^2} k^\mu \end{array} \quad . \quad (6.5)$$

[†]The term "gauge symmetry" was introduced by James Joyce in *Ulysses* (p.490 of the Modern Library 1934 edition). Bloom is standing at the entrance of a whorehouse "feeling his occiput dubiously with the unparalleled embarrassment of a harassed pedlar gauging the symmetry of her peeled pears".

(The diagrammatic rules are summarized in appendix D.) At first glance, gaugeons seem like bad news because they change the transition amplitudes. However, the only thing that matters are the physical S-matrix elements (5.23), and they are unaffected by the gaugeons. In QED this follows from the trivial momentum-conservation identity

$$\not{k} = (\not{p} + \not{k} - m) - (\not{p} - m) . \quad (6.6)$$

Diagrammatically (cf. appendix D) this is the Ward identity for the bare electron vertex:




$$\text{Diagram} = \text{Diagram} - \text{Diagram} . \quad (6.7)$$

The slashed lines indicate factors of $(\not{p} - m)$. They vanish on the mass-shell by the Dirac equation

$$(\not{p} - m)u(p) = 0 . \quad (6.8)$$

It is easy to show (next exercise) that all QED diagrams with gaugeons lead to mass-shell vanishing contributions. The QCD Ward identities are not so trivial - their derivation will be the main subject of this and the next chapter.

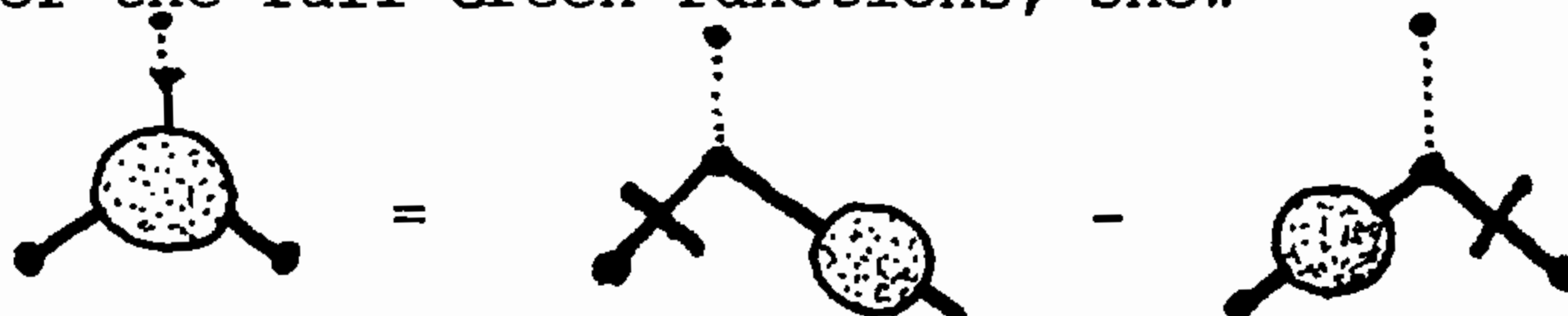
Exercise 6.A.1 Derive by iterating (6.6) the QED Ward identity



$$(\not{p}' - \not{p})_{\mu} \Gamma^{\mu}(p, p') = e[S^{-1}(p') - S^{-1}(p)] . \quad (6.9)$$

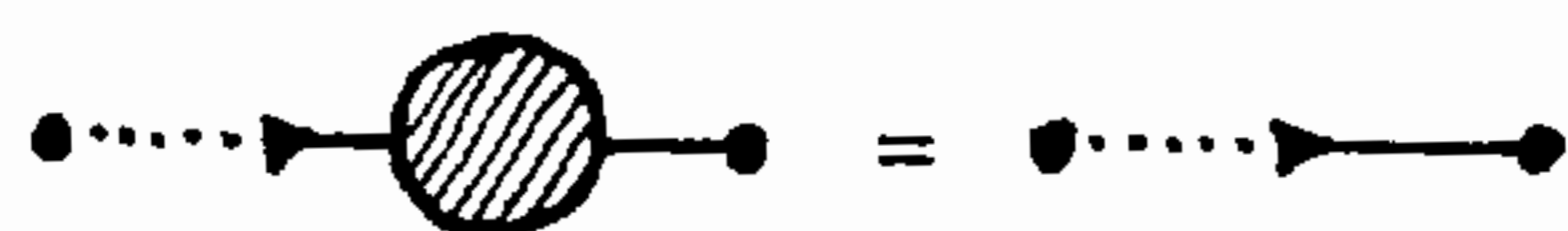
Hints:

1. For the full Green functions, show



Rewrite this for connected Green functions.

2. Show that



3. Finally, use the result of exercise 2.H.1 for the 1PI Green function.

B. Photon propagator

We have shown that QED gaugeons are innocuous; they do not affect the physical predictions of QED. One could even claim that the gaugeons are actually good news, as the gauge invariance (6.3) gives us great flexibility in defining the bare photon propagator $\langle A_\mu(x)A_\nu(y) \rangle$. Whatever your favorite way of deriving propagators may be (I like random walks of the preceding chapter), the end result for the vector particles must be

$$D_{\mu\nu}(k) = \frac{-i}{k^2} \sum_{\lambda}^{\text{polar.}} \epsilon_{\mu}^{\lambda}(k) \epsilon_{\lambda\nu}(k) . \quad (6.10)$$

The polarization tensors ϵ_{μ}^{λ} are Clebsch-Gordan coefficients which project the physical $d-1$ (or $d-2$) transverse polarizations out of the space of d -dimensional vectors. Explicit construction of Clebsch-Gordan coefficients is a tedious and unrewarding business. Fortunately we do not need them: we need only their sum in (6.10).

For massive vector particles this is easy to evaluate. We write all rank-two tensors available and fix the constants by the mass-shell conditions (6.1) and (6.2):

$$\begin{aligned} \sum_{\lambda} \epsilon_{\mu}^{\lambda}(k) \epsilon_{\lambda\nu}(k) &= Ag_{\mu\nu} + Bk_{\mu}k_{\nu} \\ &= g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{M^2} . \end{aligned} \quad (6.11)$$

For massless vector particles there is no such unique choice. One's first impulse is to replace (6.11) by

$$g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2} .$$

However, any gauge-transformed polarization (6.4) should lead to an equally good propagator, so we are lead to propagators of general form

$$\text{wavy} = D_{\mu\nu}(k) = -\frac{i}{k^2} (g_{\mu\nu} + k_{\mu}f_{\nu} + f_{\mu}k_{\nu}) , \quad (6.12)$$

where $f_{\nu}(k)$ is an arbitrary function. The most popular gauge choices of this type are listed in appendix C; which one is the most convenient depends on the application. More perverse gauges

can be thought up, and are[†]. Each gauge choice generates its gaugeons - and if the theory is to make any sense, we must insist on their decoupling from physical processes. This is the principle from which we shall presently construct the QCD action.

More precisely, the sacred principle is the gauge invariance, which in the language of Feynman diagrams comes in two guises:

(a) external gauge invariance, or invariance under transformation (6.4):

$$\epsilon_{\mu} \rightarrow \epsilon_{\mu} + \delta\omega k_{\mu} \quad . \quad (6.13)$$

(b) internal gauge invariance, or invariance under variation of gauge-fixing parameters:

$$D_{\mu\nu} \rightarrow D_{\mu\nu} + k_{\mu} \delta f_{\nu} + \delta f_{\mu} k_{\nu} \quad . \quad (6.14)$$

Exercise 6.B.1 Gauge fixing. Any not too pathological function f in the propagator (6.12) will do, as it must decouple anyway. One usually fixes $f_{\nu}(k)$ by some physically motivated condition. For interactions of nearly static particles, Coulomb gauge is the natural choice. For highly relativistic situations the covariant, planar or lightcone gauges might be convenient, and so on. The gain is of purely computational nature - the physical results must be the same in all gauges. The Coulomb gauge condition

$$\sum_{i=1}^3 \partial_i A^i(x) = 0 \quad (6.15)$$

is a typical example. This condition introduces a spacetime direction $n^{\mu} = (1, 0, 0, 0)$, so the most general form of f is

$$f^{\mu} = Bk^{\mu} + Cn^{\mu} \quad .$$

The coefficients B and C are fixed by substituting f into the gauge condition on the propagator:

$$\begin{aligned} 0 &= \langle \vec{k} \cdot \vec{A} A^{\mu} \rangle = (k^{\nu} - (n \cdot k) n^{\nu}) \langle A^{\nu} A^{\mu} \rangle \\ &= (k^{\nu} - (n \cdot k) n^{\nu}) D^{\nu\mu} \quad . \end{aligned}$$

Here the three-vector \vec{k} is expressed covariantly by $(0, \vec{k}) = k^{\mu} - (n \cdot k) n^{\mu}$. Compute the propagators listed in appendix C by this method. Observe that it is sufficient to do one calculation; once the axial gauge propagator is known, the others are obtained by special choices of the vector n^{μ} .

[†] useful in some contexts.

Exercise 6.B.2 Physical polarizations. In (6.1) we have insisted on the transversality of the physical polarizations. This seems to be in conflict with imposing a noncovariant gauge condition such as (6.15). (a) Straighten out this confusion. (b) Communicate the resolution to the author.

C. Colored quarks

We start the construction of Quantum Chromodynamics by attempting a simple generalization of QED: we replace the electron by a set of quarks[†] of n different "colors", and the photon by N gluons. A free quark or gluon propagates without changing color, so the spacetime propagators are the same as in QED, while the color factors are simply Kronecker deltas. However, a quark can change color by emitting a gluon, and the QED coupling constant e generalizes to quark-antiquark-gluon ($q\bar{q}G$) coupling matrices T_i

$$\begin{array}{c} \mu, i \\ \text{wavy line} \\ \swarrow \quad \searrow \\ a \quad \quad b \end{array} = ig(T_i)_{ab}^b (\gamma^\mu)_{\alpha\beta}$$

$a, b = 1, 2, \dots, n$ quark colors

$i, j = 1, 2, \dots, N$ gluon colors .

In QED the strength of radiative corrections is measured by the fine structure constant $\alpha = e^2 / (4\pi)$. In QCD the corresponding quantity (color weight for 1-quark loop correction to the gluon propagator) is $\text{Tr}(T_i T_j)$. If T_i is a hermitian matrix, this can be diagonalized

$$\text{tr}(T_i T_j) = a_i \delta_{ij} , \quad a_i \geq 0 ; \quad (\text{no sum on } i) . \tag{6.16}$$

The a_i is the "fine structure constant" with which the i -th color gluon couples. If T_i are not hermitian, we might be in trouble, because some a_i could be negative (that is like taking imaginary e in QED). Henceforth we shall always take coupling matrices T_i to be hermitian.

Thinking exercise 6.C.1: What could go wrong if $q\bar{q}G$ couplings were not hermitian?

[†] Quarks have also been introduced by James Joyce: "tree quarks for Muster Mark", *Finnegans Wake*, II.iv.

D. Compton scattering

We assume that the gluons are massless vector particles, just like photons. They should be transverse, and the gaugeons introduced by the longitudinal polarizations (6.4) must not contribute to the S-matrix.

Let us check this by considering the simplest conceivable process: the Compton scattering in the lowest order. The contributing (QED-like) Feynman diagrams are (the rules are summarized in appendix D)

$$\begin{aligned}
 \mathcal{M} = & \text{Diagram 1} + \text{Diagram 2} \\
 & = (ig)^2 \bar{u}(\bar{p}', s') \left[\not{\epsilon}'^j (T_j)_b^c \frac{i}{\not{p} + \not{k} - m} (T_i)_a^b \not{\epsilon} + \not{\epsilon} (T_i)_b^c \frac{i}{\not{p} - \not{k} - m} (T_j)_c^b \not{\epsilon}'^j \right] u(p, s) \quad , \\
 & \hspace{15em} (6.17)
 \end{aligned}$$

(from now on we shall suppress the polarization and spinor wave functions ϵ_μ, u, \bar{u}).

The gaugeon insertions from (6.4) lead to extra contributions to the S-matrix:

$$\delta \mathcal{M} = \text{Diagram 3} + \text{Diagram 4}$$

The bare Ward identity (6.7) yields

$$\begin{aligned}
 \delta \mathcal{M} = & \text{Diagram 5} - \text{Diagram 6} \\
 & - \text{Diagram 7} + \text{Diagram 8} \hspace{10em} (6.18)
 \end{aligned}$$

The first two terms vanish on the mass-shell. The last two terms differ only in the color factors and yield

$$- (T_i T_j - T_j T_i)_a^b (-i\gamma^\mu) \quad . \hspace{10em} (6.19)$$

In QED $T_i \rightarrow e$, and this vanishes, ensuring the gauge invariance of the Compton scattering. What happens in QCD?

E. Color algebra

So far we have put no restrictions on the color couplings other than that T_i be hermitian. A gluon can change an initial quark of any color into a final quark of any color, so there are $i = 1, 2, \dots, n^2$ gluon colors, and there should be n^2 linearly independent coupling matrices T_i . In other words, T_i form a complete basis for expanding hermitian matrices[†]

$$M_a^b = \sum_{i=1}^{n^2} m_i (T_i)_a^b, \quad \text{real } m_i. \quad (6.20)$$

The color factor $i(T_i T_j - T_j T_i)$ in (6.19) is also a hermitian matrix, so it can be expanded in the T_i basis (repeated indices summed over)

$$T_i T_j - T_j T_i = i C_{ij}^k T_k \quad (6.21)$$

with real constants C_{ij}^k . This is a Lie algebra, and C_{ij}^k are called structure constants. It is convenient to choose the generators T_i in such a way that the Killing-Cartan metric (6.16) is diagonal. We take all $a_i > 0$ (if any a_i were vanishing, the corresponding gluons would not couple at all). If $a_i \neq a_j$, the corresponding gluons couple with different strengths, and the generators T_i can be divided into mutually commuting subsectors (the Lie algebra is semi-simple). The interesting case is the simple Lie algebras, for which all gluons couple with the same strength. (6.16) becomes a normalization convention for Lie algebra generators

$$\text{tr}(T_i T_j) = a \delta_{ij} \quad i = 1, 2, \dots, N \leq n^2. \quad (6.22)$$

Physically a is the (unrenormalized) "fine structure constant". With this normalization convention, the structure constants $C_{ij}^k = C_{ijk}$ are fully antisymmetric

[†]This is the completeness relation for $U(n)$ generators. In general the color group can be any subgroup of $U(n)$, in which case (6.20) should be replaced by the appropriate completeness relation.

$$C_{ijk} = -C_{jik} = -C_{ikj} \quad . \quad (6.23)$$

So much for the color algebra. The important result is that (6.19) has no reason to vanish, so the QCD gaugeons do not (yet) decouple.

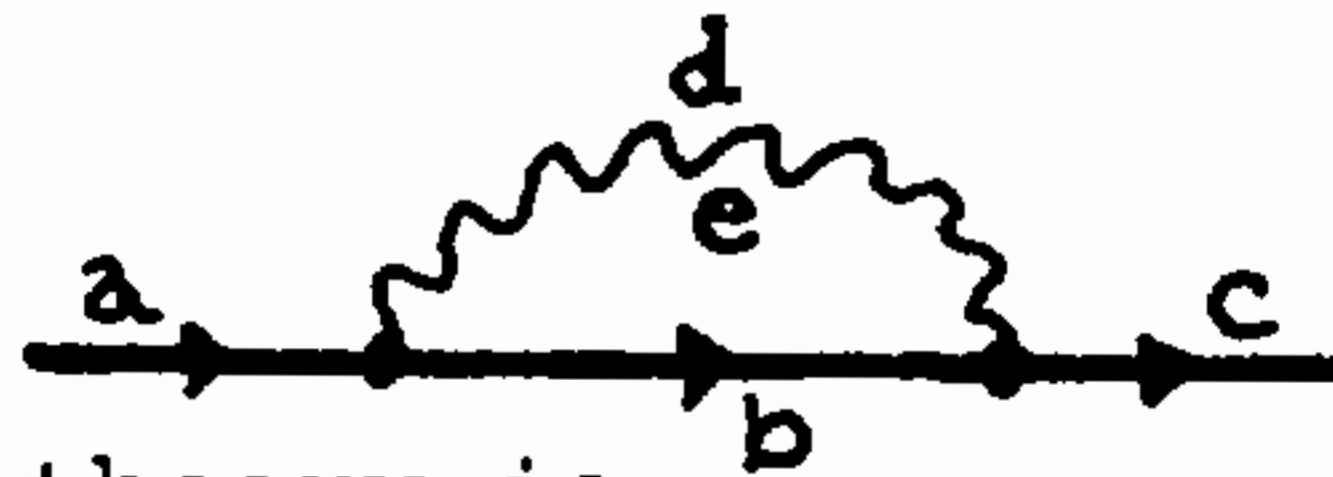
Exercise 6.E.1 Evaluation of color weights. Instead of labeling the gluon colors by $i=1,2,\dots,N$, it is often more convenient to label them by the colors (a,b) , $a,b=1,2,\dots,n$ of the corresponding quark-antiquark pairs. It is very easy to construct generators T_b^a explicitly; for example, $U(n)$ is generated by

$$\left(T_a^b\right)_c^d = \delta_c^b \delta_a^d \quad , \quad (6.24)$$

and $SU(n)$ (the Lie algebra of all traceless hermitian matrices) by

$$\left(T_a^b\right)_c^d = \delta_c^b \delta_a^d - \frac{1}{n} \delta_a^b \delta_c^d \quad . \quad (6.25)$$

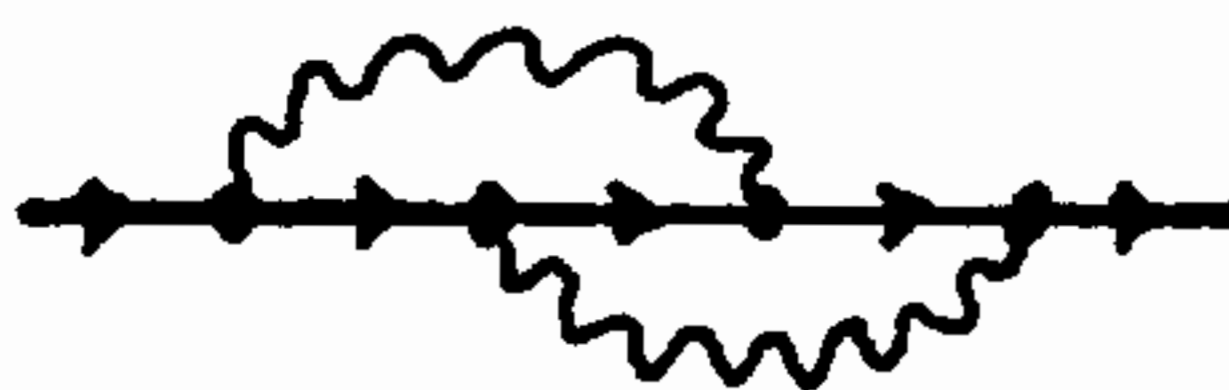
These explicit expressions for the generators enable us to compute the color weights associated with various QCD-graphs. For example, the color weight for the graph



in $U(n)$ gauge theory is

$$\left(T_d^e\right)_b^a \left(T_e^d\right)_c^b = \delta_b^e \delta_d^a \delta_c^d \delta_e^b = \delta_c^a \delta_b^b = 1 \times \text{Tr } \underline{1} = n \quad .$$

Color weights have a very simple physical interpretation. The momentum space integral is the same for any choice of the external and internal quark and gluon colorings, and each coloring contributes the same amount. The color weight is the number of distinct colorings. In the example above the color weight is n , because the internal quark line can be colored in n ways. What is the $SU(n)$ color weight for the above diagram? Compute the $U(n)$ and $SU(n)$ color weights for



F. Three-gluon vertex

We are in trouble; gaugeons do contribute to the Compton scattering. That is not acceptable, as they are unphysical. We shall now show that the theory can be repaired by introducing a 3-gluon vertex. The physical reason why 3-gluon couplings are needed is that gluons are charged (they carry quark-antiquark colors). A 3-gluon coupling is also suggested by the form of

the uncancelled term in (6.19). The Lie algebra (6.21) relates this to emission of a single gluon with coupling T_i , followed by splitting into two gluons with coupling strength iC_{ijk} . We can cancel the extra terms in (6.18) by adding such diagram:

$$\begin{array}{c}
 \text{---} \\
 \diagup \\
 \text{---} \\
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 \end{array}
 +
 \begin{array}{c}
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 \end{array}
 +
 \begin{array}{c}
 \text{---} \\
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 \text{---} \\
 \diagdown \\
 \text{---}
 \end{array}
 = 0 \tag{6.26}$$

The three terms have the same momentum space structure (diagrammatics is explained in appendix D), so this is simply a diagrammatic statement of the Lie algebra.

Now we have to invent a 3-gluon vertex which will, upon a gaugeon insertion, yield the desired term

$$\begin{array}{c}
 \text{---} \\
 \diagup \\
 \text{---} \\
 \diagdown \\
 \text{---}
 \end{array}
 =
 \begin{array}{c}
 \text{---} \\
 \diagup \\
 \text{---} \\
 \diagdown \\
 \text{---}
 \end{array}
 + (\text{terms vanishing on the mass-shell}) . \tag{6.27}$$

This is reminiscent of the bare quark vertex Ward identity (6.7). That identity is simply a statement of momentum conservation. For vectors, the momentum conservation can be diagrammatically stated as

$$\begin{array}{c}
 \text{---} \\
 \diagup \\
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 \diagdown \\
 \text{---}
 \end{array}
 +
 \begin{array}{c}
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 \diagup \\
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 \diagdown \\
 \text{---}
 \end{array}
 +
 \begin{array}{c}
 \text{---} \\
 \diagup \\
 \text{---} \\
 \diagdown \\
 \text{---}
 \end{array}
 = 0$$

$$i(-iC_{ijk})(k_1^\rho + k_2^\rho + k_3^\rho) = 0 . \tag{6.28}$$

To get something that has a hope of becoming a 3-gluon vertex, we need two more Minkowski indices: the only candidates are $g^{\mu\nu}$ and $k^\mu k^\nu$. $k^\mu k^\nu$ is no good (see exercise 6.G.1), so we try multiplying by $g^{\mu\nu}$:

$$\begin{array}{c}
 \text{---} \\
 \diagup \\
 \text{---} \\
 \diagdown \\
 \text{---}
 \end{array}
 =
 \begin{array}{c}
 \text{---} \\
 \diagup \\
 \text{---} \\
 \diagdown \\
 \text{---}
 \end{array}
 +
 \begin{array}{c}
 \text{---} \\
 \diagup \\
 \text{---} \\
 \diagdown \\
 \text{---}
 \end{array}
 \tag{6.29}$$

$$(-iC_{ijk})ig^{\mu\nu}k_2^\rho = (-iC_{ijk})ig^{\mu\nu}(k_1^\rho + k_3^\rho) .$$

Contracting with k_2^ρ gives

$$\text{Diagram with slash} = \text{Diagram with arrow} + \text{Diagram with arrow} \quad (6.30)$$

where

$$\text{Diagram with slash} = k^2 g^{\mu\nu} \quad (6.31)$$

As this is very reminiscent of (6.27), we are tempted to define a three-gluon vertex by

$$\text{Diagram with slash} = \text{Diagram with arrow}$$

This cannot be right. Gluons are bosons, and the vertex must be symmetric. So we symmetrize our guess:

$$\text{Diagram with slash} = \text{Diagram with arrow} + \text{Diagram with arrow} + \text{Diagram with arrow} - \text{Diagram with arrow} - \text{Diagram with arrow} - \text{Diagram with arrow} \quad (6.32)$$

$$\gamma_{\mu_1 \mu_2 \mu_3}^{ijk}(k_1, k_2, k_3) = (-iC_{ijk}) \left(ig^{\mu_2 \mu_3} (k_3 - k_2)^{\mu_1} + ig^{\mu_1 \mu_2} (k_2 - k_1)^{\mu_3} + ig^{\mu_1 \mu_3} (k_1 - k_3)^{\mu_2} \right) .$$

Does this satisfy the condition (6.27)? A simple computation yields

$$k_\mu \gamma^{\mu\nu\sigma}(k, k_2, k_3) = (-iC_{ijk}) i \left[g^{\mu_2 \mu_3} (k_2^2 - k_3^2) - (k_2^\mu k_2^\mu k_3^\nu - k_3^\mu k_3^\mu k_2^\nu) \right] . \quad (6.33)$$

This looks right, at least in the Feynman gauge. For gluons in the arbitrary gauge (6.12) we use the identity

$$\begin{aligned} & -\frac{i}{k^2} (g^{\mu\sigma} + f^\mu k^\sigma + k^\mu f^\sigma) (g_\sigma^\nu k^2 - k_\sigma^\nu k^\nu) \\ & = \frac{i}{k^2} (g^{\mu\nu} k^2 - k^\mu k^\nu) \end{aligned} \quad (6.34)$$

where

$$h^\nu = k^\nu - k_\mu (k^\mu f^\nu - f^\mu k^\nu) \quad (6.35)$$

$$k_\mu h^\mu = k^2 ; \quad \cdots \blacktriangleleft \cdots = - \cdots / \cdots \quad (6.36)$$

to rewrite (6.33) as the bare 3-gluon vertex Ward identity:

$$\text{Diagrammatic equation (6.37)} \quad (6.37)$$

Here $\cdots \blacktriangleleft$ stands for h^ν , and the wiggly lines are gluon propagators (see appendix D for diagrammatics). This identity and the three-gluon vertex (6.32) are the main results of this section.

With the three-gluon vertex, the Compton scattering is given by

$$\mathcal{M} = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \quad (6.38)$$

rather than by (6.17). One can easily check that the gaugeons now decouple. The Ward identity (6.37) generates 3 extra terms beyond the desired (6.27), but they all vanish on the mass-shell by the transversality condition (6.1) and the mass-shell condition (6.3).

Exercise 6.F.1 A three-gluon vertex has three Minkowski indices. Show that they cannot all be carried by the momentum vectors: $\gamma^{\mu\nu\sigma} \neq ck_1^\mu k_2^\nu k_3^\sigma$. Hint: the color factor is antisymmetric.

Exercise 6.F.2 Scalar QED vertices. For scalar charged particles the only available vectors are p_μ and p'_μ and the photon vertex is given by

$$\text{Diagrammatic equation (6.39)} \quad (6.39)$$

The propagator is the usual scalar propagator

$$\text{Diagrammatic equation (6.40)} \quad (6.40)$$

Show that gaugeons do not decouple if (a) we add a $(p-p')^\mu$ part to the vertex (6.39), or

(b) the Compton amplitude is given by diagrams in (6.17).
Save the day by devising a 2-photon vertex



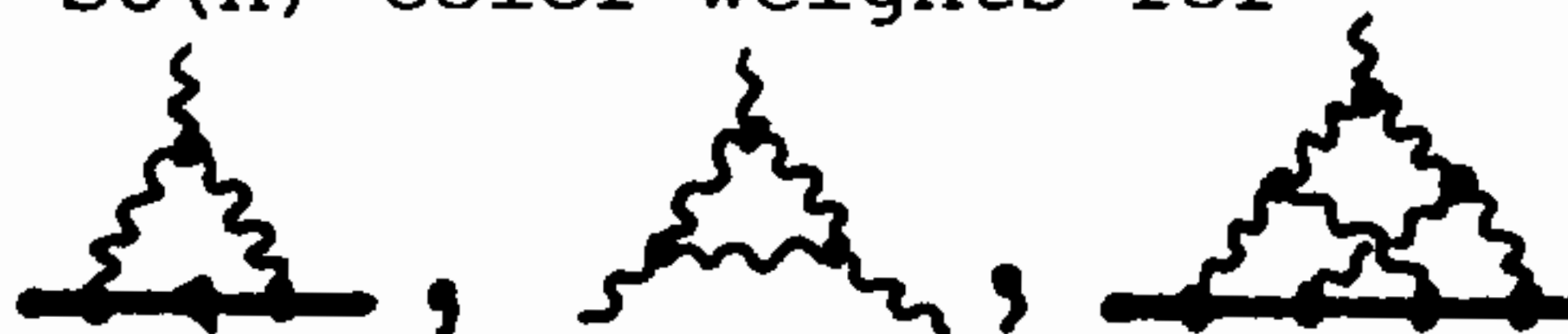
which ensures the decoupling of gaugeons.

Exercise 6.F.3 Derive the bare three-gluon vertex Ward identity (6.34).
Check the gaugeon decoupling in (6.35).

Exercise 6.F.4 (Continuation of exercise 6.D.1). Show that for SU(n)
the color weight for the gluon self-energy diagram is



What is it for U(n)? U(n) is non-semisimple - how does that
manifest itself? Is color weight reducible to (6.22)? Compute
also U(n), SU(n) color weights for



Hints: Lie algebra (6.21) together with normalization (6.22)
implies that

$$(-iC_{ijk}) = \frac{1}{a} \text{tr}(T_i T_j T_k - T_k T_j T_i) .$$

Use this to eliminate the 3-gluon color factors. Resulting
color weights can be evaluated by (6.24) and (6.25).

"Birdtracks" are a convenient method for evaluating color
weights. In this formalism the gluon projection operators (6.24)
and (6.25) are replaced by diagrams:

$$U(n): \quad (T_a^b)_c^d = \frac{1}{a} \begin{array}{c} b \quad c \\ \diagdown \quad \diagup \\ a \quad d \end{array} = \begin{array}{c} b \quad c \\ \curvearrowright \\ a \quad d \end{array}$$

$$SU(n): \quad (T_a^b)_c^d = \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} - \frac{1}{n} \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array}$$

The number of quark colors, the normalization (6.16), and the
structure constants are given by

$$\bigcirc = \delta_a^a = \text{Tr } \mathbf{1} = n$$

$$\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} = a \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array}$$

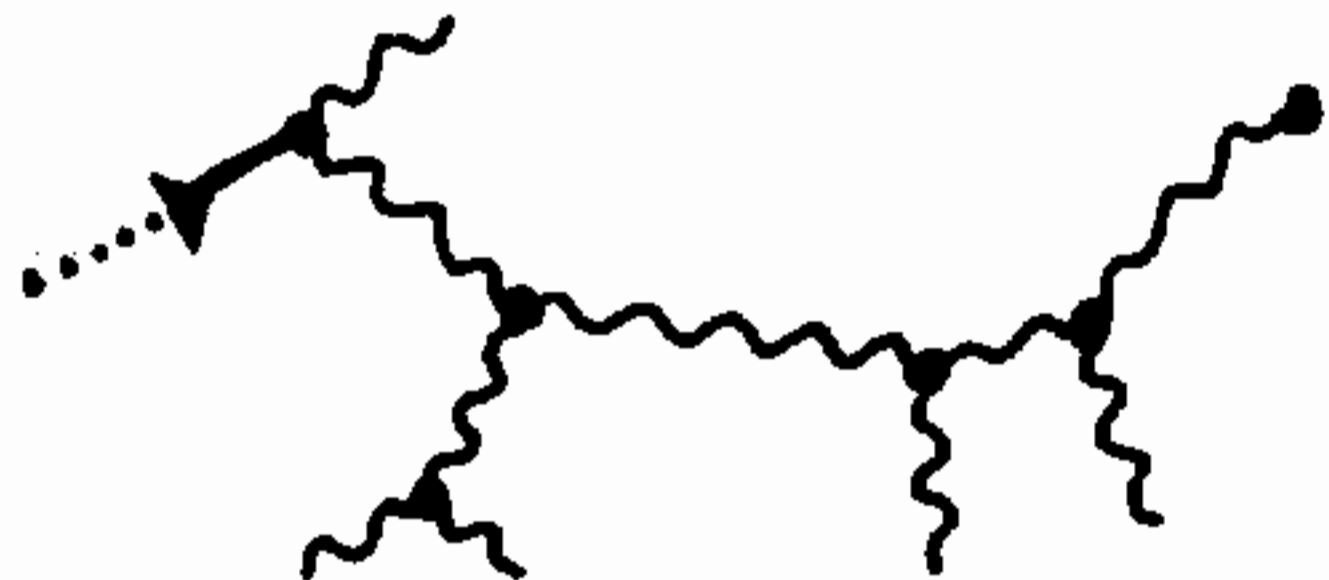
$$\begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} = \frac{1}{a} \left\{ \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} - \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right\}$$

For example, the above gluon self-energy is evaluated by sub-
stituting the diagrammatic gluon projection operators in

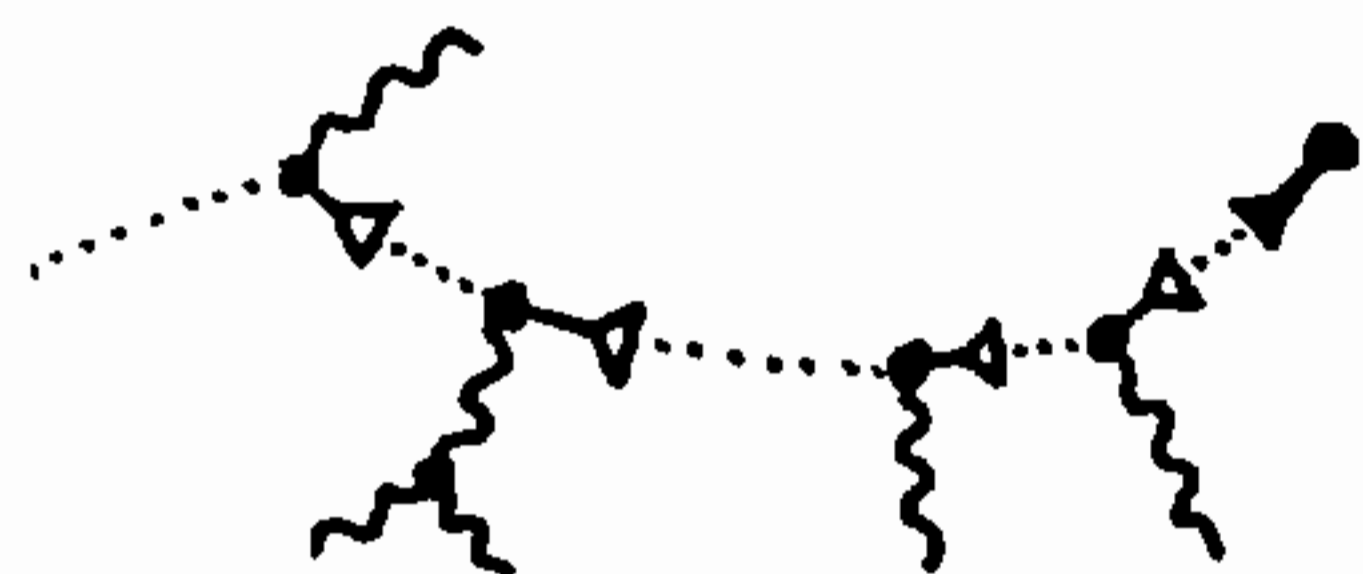
$$\begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} = \frac{2}{a} \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} = \frac{2}{a} \left(\begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} - \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right) .$$

G. Ghosts

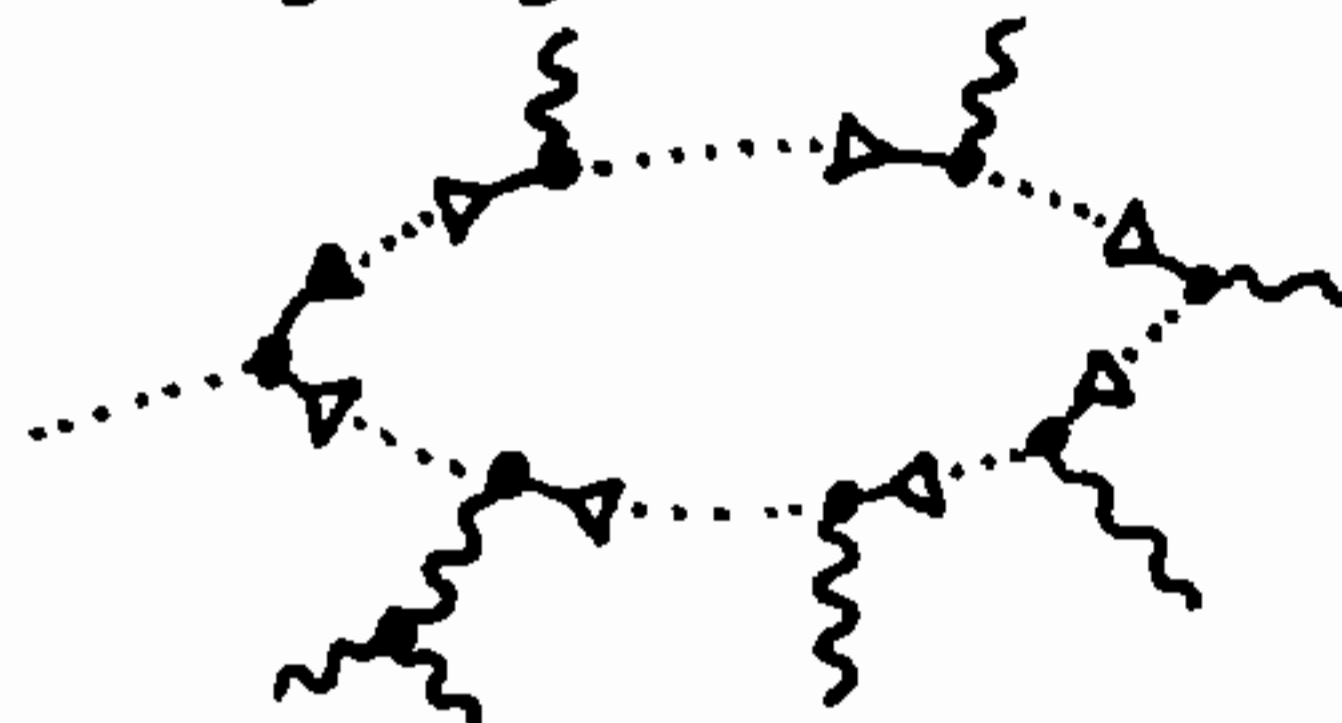
So far so good - we have repaired the Compton scattering by introducing a 3-gluon vertex. However, the 3-gluon bare Ward identity is rather complicated; beyond the terms analogous to the spinor Ward identity (6.7) there are two extra terms with $k^\mu h^\nu$ numerators. If a diagram has a number of 3-gluon vertices



a k^μ insertion will (after repeated applications of the Ward identity (6.37)) yield contributions like



If such a gaugeon line ends up on a quark line, it will (by applications of the fermion Ward identity (6.7)) eventually yield mass-shell vanishing contributions. But if it loops onto itself, we are stuck with gaugeon contributions of the type



which have no reason to vanish. The problem is that the physical gluon has only $d-2$ degrees of freedom, but with our Feynman rules, all d components contribute; there are too many degrees of freedom circulating along the loops.

This disease has a drastic cure. We introduce a new particle, called a ghost, whose sole purpose is to (in the manner of ghoulies) eat up the longitudinal degrees of freedom. It couples to gluons just like the gaugeon

$$\begin{array}{c} | \\ \bullet \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \end{array} = \begin{array}{c} | \\ \bullet \\ \swarrow \quad \searrow \\ \text{---} \quad \text{---} \end{array} , \tag{6.41}$$

(cf. appendix D), but each ghost loop carries a minus sign and thus cancels the corresponding gaugeon loop. As we have seen in chapter 4, such particles must obey Fermi statistics. The arrow on the ghost line keeps track of the $h^\mu(k)$ factors in (6.41). We shall prove in the next chapter that ghosts indeed cure the

gaugeon loop problem. For that we shall also need the bare ghost vertex Ward identity, which, as always, is simply a statement of the momentum conservation, this time combined with the identity (6.36):

$$C_{ijk} \left[-k_{\mu} h^{\mu} (k+k') - k'_{\mu} h^{\mu} (k+k') \right] = - C_{ijk} (k+k')^2 . \quad (6.42)$$

(Note that because of the color factor the "vertex" on the right-hand side is antisymmetric - this is another indication that the ghosts must be treated as fermions.)

In order to verify the correctness of the ghost prescription we shall have to go through some algebra. However, the physics of ghosts should already be clear; gaugeons are unphysical degrees of freedom, and the ghosts are here to cancel them. Neither "particle" has any physical meaning by itself.

Exercise 6.G.1 Show that for axial gauges $n^{\mu} D_{\mu\nu} = 0$, so that the gaugeons decouple

$$\text{[Diagram: ghost-gluon vertex with slash]} = 0 . \quad (6.43)$$

This means that the axial gauges are "ghost-free"; the Ward identities will turn out to be no more complicated than the QED ones. This is the reason that the axial gauges are often used in general diagrammatic gauge invariance arguments. Computationally they are horrid.

H. Four-gluon vertex

The next thing we have to check is the gauge invariance of the gluon-gluon Compton scattering:

$$\mathcal{M} = \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]} \quad (6.44)$$

Inserting a gaugeon, using the gluon Ward identity (6.37), and discarding the mass-shell vanishing contributions, we end up with

$$\delta \mathcal{M} = - \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]} \quad (6.45)$$

Replacing each 3-gluon vertex by (6.32) yields lots of terms

$$\delta\mathcal{M} = - \left(\text{diagram 1} + 5 \text{ terms} \right) + \left(\text{diagram 2} + 5 \text{ terms} \right) + \left(\text{diagram 3} + 5 \text{ terms} \right) \quad (6.46)$$

There is no reason for this to vanish. To rescue the theory we have to devise a 4-gluon vertex for which a gaugeon insertion

$$\text{diagram 4} \quad (6.47)$$

precisely cancels (6.46). We do this by rewriting (6.46) in a form that resembles a gaugeon insertion into a 4-vertex. The tools that we have at our disposal are the momentum conservation (6.28) and the Lie algebra commutator (6.21), which, for 3-gluon couplings, is the Jacobi relation

$$C_{ijm} C_{mkl} - C_{jml} C_{kim} = C_{jkm} C_{mli} \quad (6.48)$$

We can use the Jacobi identity to combine the (6.46) terms with the same Minkowski structure. For example,

$$\begin{aligned} & \text{diagram 1} - \text{diagram 2} = \text{diagram 3} \\ & ig^{v\rho} ik_4^\mu \left((-iC_{ijm})(-iC_{mkl}) - (-iC_{imk})(-iC_{jml}) \right) (-i) \\ & = ig^{v\rho} ik_4^\mu (-C_{iml})(-iC_{mjk})(-i) \quad (6.49) \end{aligned}$$

This reduces the number of terms in (6.46) to twelve:

$$\delta\mathcal{M} = - \left(\text{diagram 3} + \text{diagram 4} + (10 \text{ terms}) \right) \quad (6.50)$$

By the momentum conservation (6.28) these add up to six terms

$$\delta\mathcal{M} = \text{diagram 3} + (5 \text{ terms}) \quad (6.51)$$

Now the gaugeon contribution is of the desired form (6.47). If

we define the four-gluon vertex by

$$\text{Four-gluon vertex} = - \text{Diagram 1} - \text{Diagram 2} - \text{Diagram 3} - \text{Diagram 4} - \text{Diagram 5} - \text{Diagram 6} - \text{Diagram 7} \quad (6.52)$$

the gluon-gluon scattering amplitude

$$M = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} \quad (6.53)$$

is gauge invariant.

Definition of the four-gluon vertex (6.52), together with the bare four-gluon vertex Ward identity

$$\text{Ward identity diagram} = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \quad (6.54)$$

are the main results of this section. (The Ward identity follows from (6.45)).

So far we have succeeded in making the quark-gluon and the gluon-gluon tree level scattering amplitudes gauge invariant, but at what a price: three new kinds of vertices and even ghosts. This looks like a story without end; next one might need a 5-gluon vertex to fix up the five-leg Green functions, etc. Indeed, in theories like gravity, one would find 5-graviton vertex, 6-graviton vertex,..... . For QCD the buck stops here - we shall prove that in the next chapter. To carry out the proof, we shall also need the following invariance condition for the four-gluon vertex:

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = 0 \quad (6.55)$$

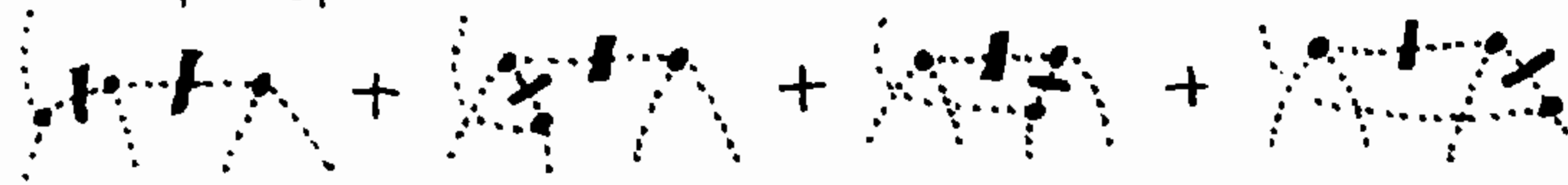
This is simply a statement that $C_{ijk} C_{klm}$ is an invariant tensor (exercise 6.H.2).

Exercise 6.H.1 (Continuation of exercise 6.F.4). Compute the SU(n) color weight for the diagram



Hint: the 4-gluon vertex (6.52) is really composed of pairs of 3-vertices, so group-theoretically there are no 4-vertices.

Exercise 6.H.2 Prove the invariance of the 4-gluon vertex, (6.55).
 Hint: note that nothing in (6.55) depends on the momentum.
 Substituting (6.52) you will observe that each Minkowski
 factor $g_{\mu\nu}g_{\sigma\rho}$ is multiplied by color factor



Prove (by using the Jacobi identity (6.48)) that this vanishes.

I. QCD action

As explained in chapter 2, Feynman rules can be compactly summarized by the action functional (2.13). Carrying this out for the QCD Feynman rules is a straightforward but somewhat tedious continuation of exercises 2.E.2 and 2.D.1. The compact indices are replaced by the full set of explicit indices:

$$\phi_i \rightarrow \left(A_\mu^i(k), \bar{\omega}^i(k), \omega^i(k), \bar{q}_\alpha^a(k), q_{a\alpha}(k) \right), \quad (6.56)$$

where A is the gluon field, ω and $\bar{\omega}$ is the ghost and antighost fields, and q and \bar{q} the quark and antiquark fields.

The result is known to everybody:

$$\begin{aligned}
 S[\phi] &= i \int dx \mathcal{L} \\
 \mathcal{L} &= \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{fix}} + \mathcal{L}_{\text{ghost}} + \mathcal{L}_{\text{quark}} \\
 \mathcal{L}_{\text{YM}} &= -\frac{1}{4} \left(F_{\mu\nu}^i \right)^2 \\
 F_{\mu\nu}^i &= \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + gC_{ijk} A_\mu^j A_\nu^k \\
 \mathcal{L}_{\text{fix}} &= -\frac{1}{2a} \left(\partial^\mu A_\mu^i \right)^2 \quad (\text{covariant gauges}) \\
 \mathcal{L}_{\text{ghost}} &= \bar{\omega}^i \partial^\mu D_\mu^{ij} \omega^j \quad (\text{covariant gauges}) \\
 D_\mu^{ij} &= \delta^{ij} \partial_\mu + gC_{ikj} A_\mu^k \\
 \mathcal{L}_{\text{quark}} &= i \bar{q}_a \not{D}_b^a q^b - m \bar{q}_a q^a \\
 D_{b\mu}^a &= \delta_b^a \partial_\mu - ig(T_i)_{ba}^a A_\mu^i \quad (6.57)
 \end{aligned}$$

Checking the equivalence between the above action and our Feynman rules is dullness embodied (though nothing compared to doing the same for the supergravity actions). The only non-

trivial step is the inversion of the gluon propagator (this is needed for the quadratic part of the action (2.13)). The general case is unilluminating and we relegate it to the exercise 7.H.1; the problem can be understood by looking just at the covariant gauges. The covariant propagator (appendix C) can be decomposed into the transverse and longitudinal parts

$$ik^2 D^{\mu\nu} = (g^{\mu\nu} - k^\mu k^\nu / k^2) + ak^\mu k^\nu / k^2 \quad . \quad (6.58)$$

The inverse is simply

$$-ik^{-2} (D^{-1})^{\mu\nu} = (g^{\mu\nu} - k^\mu k^\nu / k^2) + \frac{1}{a} k^\mu k^\nu / k^2 \quad . \quad (6.59)$$

However, if $a = 0$ the propagator is purely transverse, and it cannot be inverted. There is nothing wrong in using $a = 0$ (Landau) gauge in evaluating Feynman diagrams, but non-invertibility is a problem for the path integral formulation: a zero eigenvalue for the propagator (6.58) means that the path integral (3.7) has no gaussian damping factor for integrations over longitudinal fields $A_\mu^i \propto \omega^i(k) k_\mu$. These troublesome directions are just our old gaugeons in a new guise, and the cure is gauge fixing.

Multiplying (6.59) by the momentum conservation delta function and $A_\mu^i(k) A_\nu^i(k')$, summing over color and Minkowski indices, integrating over momenta and Fourier transforming, we obtain the quadratic part of \mathcal{L}_{YM}

$$S_{\text{transverse}} = -\frac{i}{4} \int dx \left(\partial_\mu A_\nu^i - \partial_\nu A_\mu^i \right)^2 \quad (6.60)$$

and the gauge fixing term \mathcal{L}_{fix} in (6.57). The remainder of (6.57) is obtained in the same way.

Exercise 6.I.1 Inverting gluon propagators. Under the gauge transformation (6.4) the polarization sum (6.10) transforms into

$$\epsilon^\mu \cdot \epsilon^\nu \rightarrow \epsilon^\mu \cdot \epsilon^\nu + k^\mu (\omega \cdot \epsilon^\nu) + (\epsilon^\mu \cdot \omega) k^\nu + (\omega \cdot \omega) k^\mu k^\nu .$$

Define functions $h_T^\mu(k)$, $B(k)$ by

$$\epsilon^\mu \cdot \omega = -h_T^\mu / k^2 \quad , \quad \omega \cdot \omega = B / k^2 \quad .$$

With transverse ϵ_μ , equation (6.1), the gluon propagator (6.10) can be written as

$$ik^2 D^{\mu\nu}(k) = (g^{\mu\nu} - k^\mu k^\nu / k^2) - (h_T^\mu k^\nu + k^\mu h_T^\nu) / k^2 + B k^\mu k^\nu / k^2$$

$$k_\mu h_T^\mu = 0 \quad . \quad (6.61)$$

This is nothing but a rewrite of (6.12) in terms of the transverse, mixed and longitudinal parts, convenient for inversion. Check that h_T is the transverse part of the ghost vertex (6.35), $h = k + h_T$. Show that the inverse propagator is given by

$$(ik^2 D^{-1})_{\mu\nu} = (g_{\mu\nu} - k_\mu k_\nu / k^2) + h_\mu h_\nu / (B - h_T^2 / k^2) . \quad (6.62)$$

Show that the gauge fixing terms in the action are given by

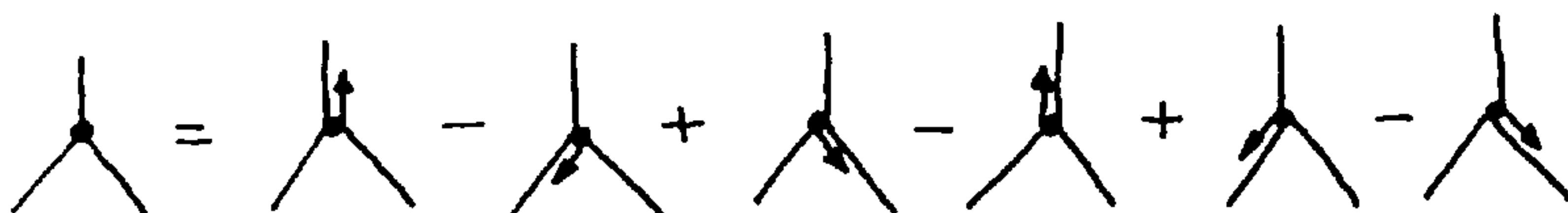
$$\begin{aligned} \text{Covariant:} \quad \mathcal{L}_{\text{fix}} &= -\frac{1}{2a} (\partial_\mu A^\mu)^2 \\ \text{Axial} \quad : \quad \mathcal{L}_{\text{fix}} &= -\frac{1}{2a} (n_\mu A^\mu)^2 \\ \text{Planar} \quad : \quad \mathcal{L}_{\text{fix}} &= -\frac{1}{2an^2} (n_\mu A^\mu) \partial^2 (n_\nu A^\nu) \\ \text{Coulomb} \quad : \quad \mathcal{L}_{\text{fix}} &= -\frac{1}{2a} [(n^2 \partial_\mu - n \cdot \partial n_\mu) A_a^\mu]^2 . \end{aligned} \quad (6.63)$$

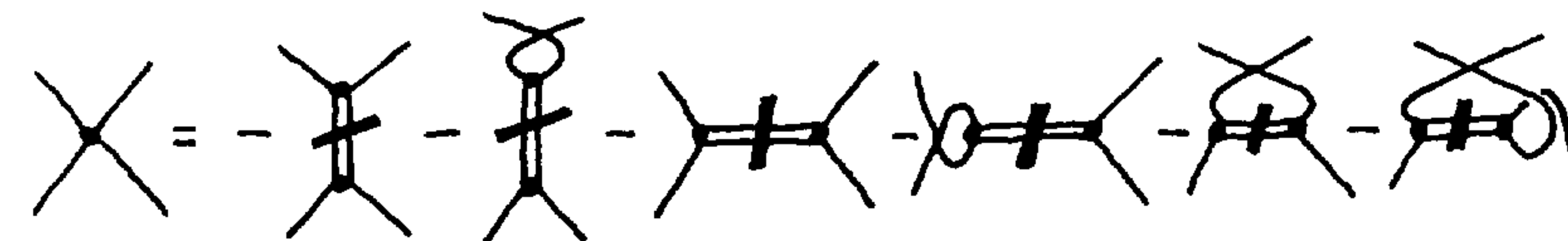
Corresponding propagators are given in appendix C. Note that most of the popular gauges (Landau, axial, etc.) correspond to the singular $a \rightarrow 0$ and/or $n^2 \rightarrow 0$ limits. Do you feel uncomfortable? Reflect briefly upon whether you are really enjoying this.

Exercise 6.I.2 Construct (6.57) from our Feynman rules, or verify the Feynman rules from (6.57), whichever is more to your taste. Note that the ghost propagators and vertices differ in the two formulations. Is that a problem?

J. Summary:

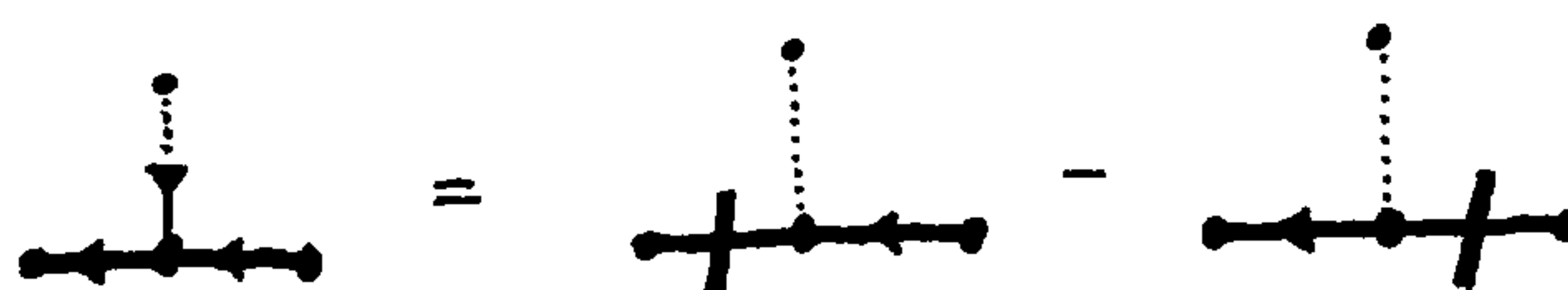
bare vertices:

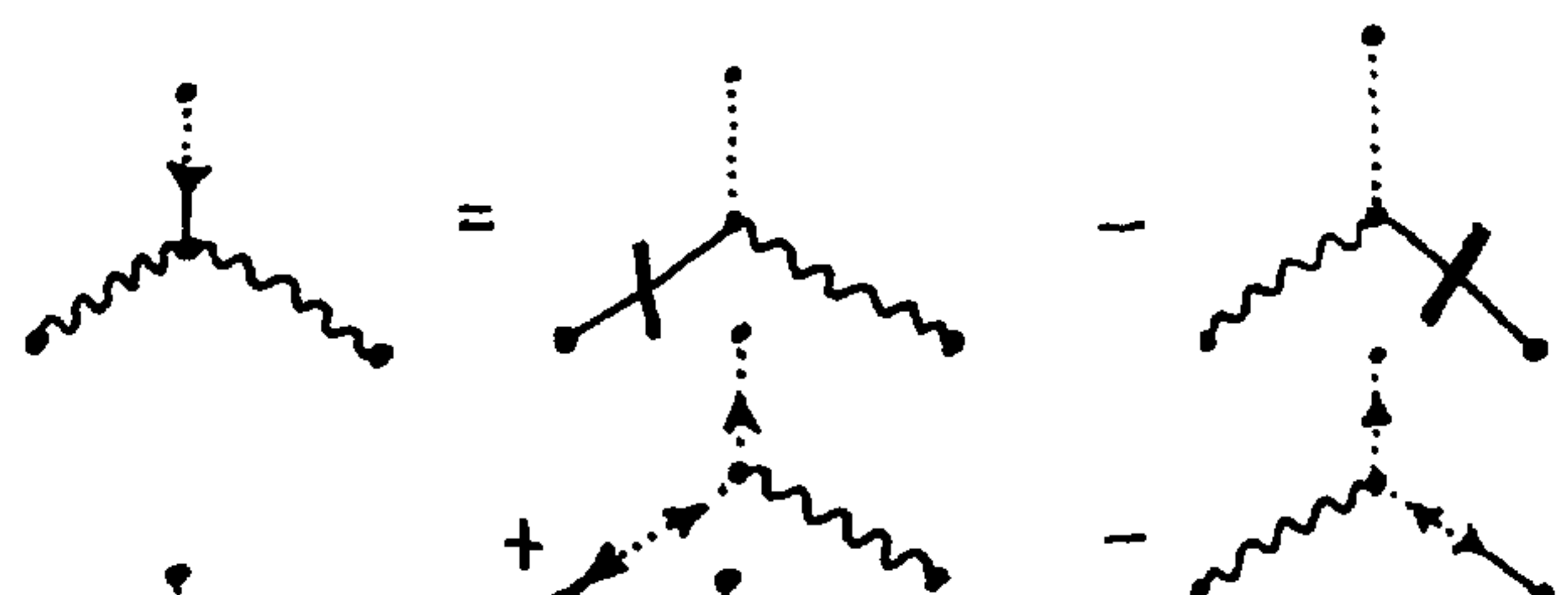
3-gluon  (6.64a)

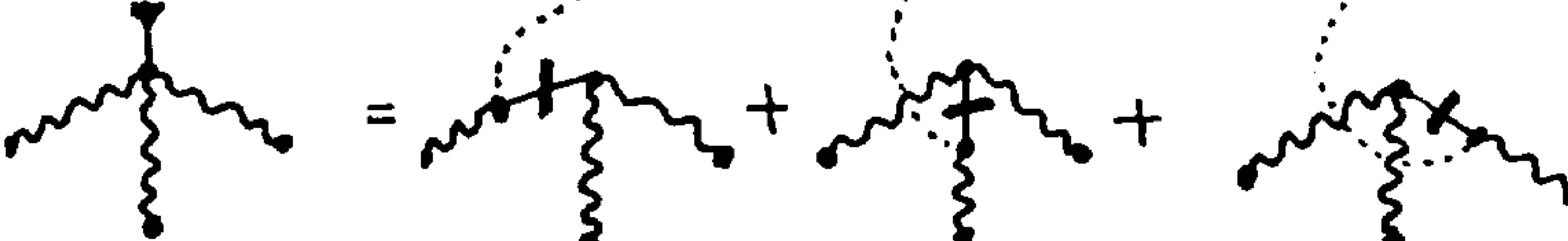
4-gluon  (6.64b)

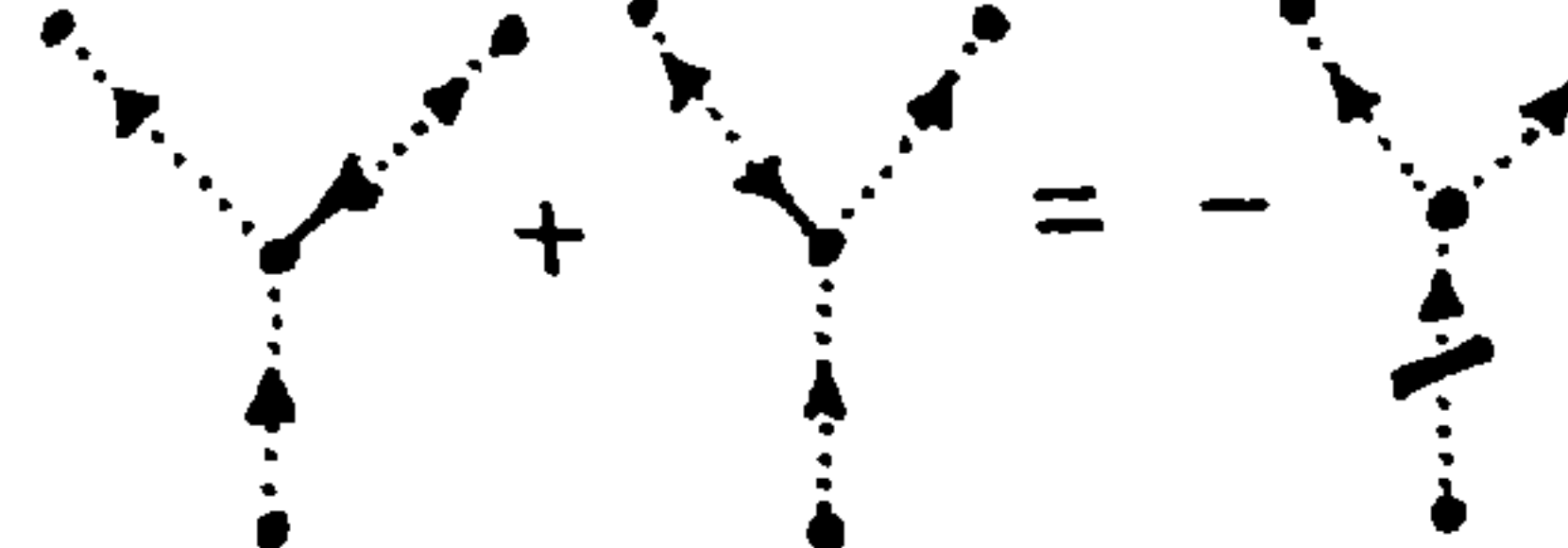
ghost-gluon  (6.64c)

bare vertex Ward identities:

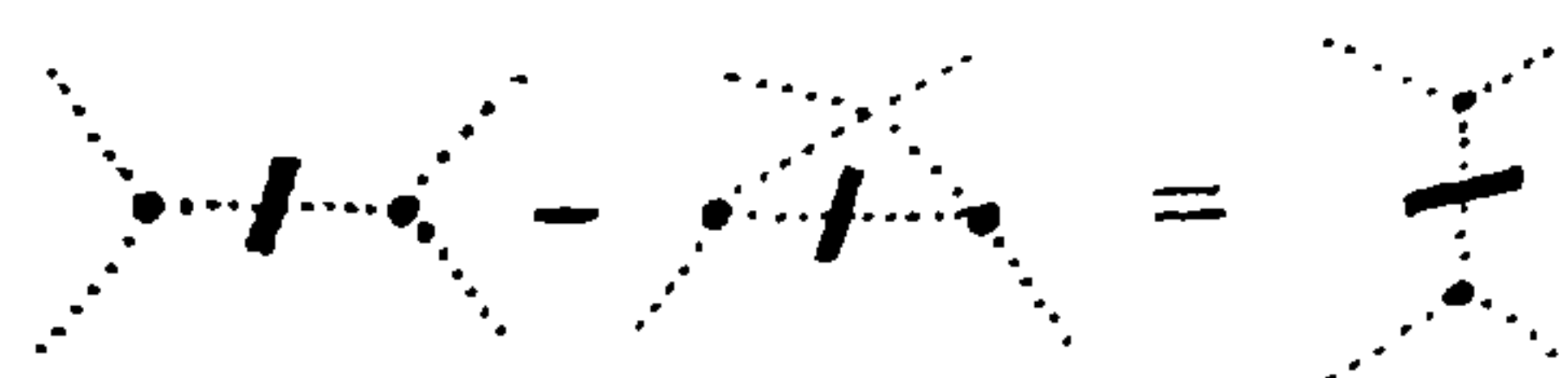
quark-gluon  (6.65a)

3-gluon  (6.65b)

4-gluon  (6.65c)

ghost-gluon  (6.65d)

invariance conditions:

Jacobi identities  (6.66a)

(and similarly with Minkowski factors)

4-gluon  (6.66b)

The diagrammatic rules are explained in appendix D.