Epilogue

> Nowadays, whatever the truth of the matter may be (and we will probably never know), the simplest solution is no longer emotionally satisfying. Everything we know about the world militates against it. The concepts of indeterminacy and chaos have filtered down to us from the higher sciences to confirm our nagging suspicions.
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> —L. Sante, "Review of 'American Tabloid' by James Ellroy," *New York Review of Books* (May 11, 1995)

A motion on a strange attractor can be approximated by shadowing long orbits by sequences of nearby shorter periodic orbits. This notion has here been made precise by approximating orbits by prime cycles, and evaluating associated curvatures. A curvature measures the deviation of a long cycle from its approximation by shorter cycles; the smoothness of the dynamical system implies exponential fall-off for (almost) all curvatures. We propose that the theoretical and experimental non–wandering sets be expressed in terms of the symbol sequences of short cycles (a topological characterization of the spatial layout of the non–wandering set) and their eigenvalues (metric structure)

Cycles as the skeleton of chaos

We wind down this all-too-long treatise by asking: why cycle?

We tend to think of a dynamical system as a smooth system whose evolution can be followed by integrating a set of differential equations. Traditionally one used integrable motions as zeroth-order approximations to physical systems, and accounted for weak nonlinearities perturbatively. However, when the evolution is actually followed through to asymptotic times, one discovers that the strongly nonlinear systems show an amazingly rich structure which is not at all apparent in their formulation in terms of differential equations. In particular, the periodic orbits are important because they form the *skeleton* onto which all trajectories trapped for long times cling. This was already appreciated century ago by H. Poincaré, who, describing in *Les méthodes nouvelles de la méchanique céleste* his discovery of homoclinic tangles, mused that "the complexity of this figure will be striking, and I shall not even try to draw it." Today such drawings are cheap and

plentiful; but Poincaré went a step further and, noting that hidden in this apparent chaos is a rigid skeleton, a tree of *cycles* (periodic orbits) of increasing lengths and self-similar structure, suggested that the cycles should be the key to chaotic dynamics.

The zeroth-order approximations to harshly chaotic dynamics are very different from those for the nearly integrable systems: a good starting approximation here is the stretching and kneading of a baker's map, rather than the winding of a harmonic oscillator.

For low dimensional deterministic dynamical systems description in terms of cycles has many virtues:

1. cycle symbol sequences are *topological* invariants: they give the spatial layout of a non–wandering set
2. cycle eigenvalues are *metric* invariants: they give the scale of each piece of a non–wandering set
3. cycles are *dense* on the asymptotic non–wandering set
4. cycles are ordered *hierarchically*: short cycles give good approximations to a non–wandering set, longer cycles only refinements. Errors due to neglecting long cycles can be bounded, and typically fall off exponentially or super-exponentially with the cutoff cycle length
5. cycles are *structurally robust*: for smooth flows eigenvalues of short cycles vary slowly with smooth parameter changes
6. asymptotic averages (such as correlations, escape rates, quantum mechanical eigenstates and other "thermodynamic" averages) can be efficiently computed from short cycles by means of *cycle expansions*

Points 1, 2: That the cycle topology and eigenvalues are invariant properties of dynamical systems follows from elementary considerations. If the same dynamics is given by a map f in one set of coordinates, and a map g in the next, then f and g (or any other good representation) are related by a reparametrization and a coordinate transformation $f = h^{-1} \circ g \circ h$. As both f and g are arbitrary representations of the dynamical system, the explicit form of the conjugacy h is of no interest, only the properties invariant under any transformation h are of general import. The most obvious invariant properties are topological; a fixed point must be a fixed point in any representation, a trajectory which exactly returns to the initial point (a cycle) must do so in any representation. Furthermore, a good representation should not mutilate the data; h must be a smooth transformation which maps nearby periodic points of f into nearby periodic points of g. This smoothness guarantees that the cycles are not only topological invariants, but that their linearized neighborhoods are also metrically invariant. In particular, the cycle eigenvalues (eigenvalues of the Jacobian matrixs $df^n(x)/dx$ of periodic orbits $f^n(x) = x$) are invariant.

Point 5: An important virtue of cycles is their *structural robustness*. Many quantities customarily associated with dynamical systems depend on the notion of "structural stability," i.e., robustness of non–wandering set to small parameter variations.

Still, the sufficiently short unstable cycles are structurally robust in the sense that they are only slightly distorted by such parameter changes, and averages computed using them as a skeleton are insensitive to small deformations of the non–wandering set. In contrast, lack of structural stability wreaks havoc with long time averages such as Lyapunov exponents, for which there is no guarantee that they converge to the correct asymptotic value in any finite time numerical computation.

The main recent theoretical advance is **point 4:** we now know how to control the errors due to neglecting longer cycles. As we seen above, even though the number of invariants is infinite (unlike, for example, the number of Casimir invariants for a compact Lie group) the dynamics can be well approximated to any finite accuracy by a small finite set of invariants. The origin of this convergence is geometrical, as we shall see in appendix I.1.2, and for smooth flows the convergence of cycle expansions can even be super-exponential.

The cycle expansions such as (20.7) outperform the pedestrian methods such as extrapolations from the finite cover sums (22.2) for a number of reasons. The cycle expansion is a better averaging procedure than the naive box counting algorithms because the strange attractor is here pieced together in a topologically invariant way from neighborhoods ("space average") rather than explored by a long ergodic trajectory ("time average"). The cycle expansion is co-ordinate and reparametrization invariant - a finite nth level sum (22.2) is not. Cycles are of finite period but infinite duration, so the cycle eigenvalues are already evaluated in the $n \to \infty$ limit, but for the sum (22.2) the limit has to be estimated by numerical extrapolations. And, crucially, the higher terms in the cycle expansion (20.7) are deviations of longer prime cycles from their approximations by shorter cycles. Such combinations vanish exactly in piecewise linear approximations and fall off exponentially for smooth dynamical flows.

In the above we have reviewed the general properties of the cycle expansions; those have been applied to a series of examples of low-dimensional chaos: 1-d strange attractors, the period-doubling repeller, the Hénon-type maps and the mode locking intervals for circle maps. The cycle expansions have also been applied to the irrational windings set of critical circle maps, to the Hamiltonian period-doubling repeller, to a Hamiltonian three-disk game of pinball, to the three-disk quantum scattering resonances and to the extraction of correlation exponents, Feasibility of analysis of experimental non–wandering set in terms of cycles is discussed in ref. [20.1].

Homework assignment

> "Lo! thy dread empire Chaos is restor'd, Light dies before thy uncreating word; Thy hand, great Anarch, lets the curtain fall, And universal darkness buries all."
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> —Alexander Pope, *The Dunciad*

We conclude cautiously with a homework assignment posed May 22, 1990 (the original due date was May 22, 2000, but alas...):

1. **Topology** Develop optimal sequences ("continued fraction approximants") of finite subshift approximations to generic dynamical systems. Apply to (a) the Hénon map, (b) the Lorenz flow and (c) the Hamiltonian standard map.
2. **Non-hyperbolicity** Incorporate power–law (marginal stability orbits,"intermittency") corrections into cycle expansions. Apply to long-time tails in the Hamiltonian diffusion problem.
3. **Phenomenology** Carry through a convincing analysis of a genuine experimentally extracted data set in terms of periodic orbits.
4. **Invariants** Prove that the scaling functions, or the cycles, or the spectrum of a transfer operator are the maximal set of invariants of an (physically interesting) dynamically generated non–wandering set.
5. **Field theory** Develop a periodic orbit theory of systems with many unstable degrees of freedom. Apply to (a) coupled lattices, (b) cellular automata, (c) neural networks.
6. **Tunneling** Add complex time orbits to quantum mechanical cycle expansions (WKB theory for chaotic systems).
7. **Unitarity** Evaluate corrections to the Gutzwiller semiclassical periodic orbit sums. (a) Show that the zeros (energy eigenvalues) of the appropriate Selberg products are real. (b) Find physically realistic systems for which the "semiclassical" periodic orbit expansions yield the exact quantization.
8. **Atomic spectra** Compute the helium spectrum from periodic orbit expansions (already accomplished by Wintgen and Tanner!).
9. **Symmetries** Include fermions, gauge fields into the periodic orbit theory.
10. **Quantum field theory** Develop quantum theory of systems with infinitely many classically unstable degrees of freedom. Apply to (a) quark confinement (b) early universe (c) the brain.

Conclusion

Good-bye. I am leaving because I am bored.
—George Saunders' dying words

Nadie puede escribir un libro. Para Que un libro sea verdaderamente, Se requieren la aurora y el poniente Siglos, armas y el mar que une y separa.
—Jorge Luis Borges El Hacedor, *Ariosto y los arabes*

The buttler did it.