# Appendix D

# Symbolic dynamics techniques

THE KNEADING THEORY for unimodal mappings is developed in sect. D.1. The prime factorization for dynamical itineraries of sect. D.2 illustrates the sense in which prime cycles are "prime" - the product structure of zeta functions is a consequence of the unique factorization property of symbol sequences.

## D.1 Topological zeta functions for infinite subshifts

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The transition graph methods outlined in chapter 11 are well suited for symbolic dynamics of finite subshift type. A sequence of well defined rules leads to the answer, the topological zeta function, which turns out to be a polynomial. For infinite subshifts one would have to go through an infinite sequence of graph constructions and it is of course very difficult to make any asymptotic statements about the outcome. Luckily, for some simple systems the goal can be reached by much simpler means. This is the case for unimodal maps.

We will restrict our attention to the topological zeta function for unimodal maps with one external parameter $f_\Lambda(x) = \Lambda g(x)$. As usual, symbolic dynamics is introduced by mapping a time series $\ldots x_{i-1}x_ix_{i+1}\ldots$ onto a sequence of symbols $\ldots s_{i-1}s_is_{i+1}\ldots$ where

$$
\begin{aligned}
s_i &= 0 \quad x_i < x_c \\
s_i &= C \quad x_i = x_c \\
s_i &= 1 \quad x_i > x_c
\end{aligned}
\tag{D.1}
$$

and $x_c$ is the critical point of the map (i.e., maximum of $g$). In addition to the usual binary alphabet we have added a symbol $C$ for the critical point. The kneading

| $I(C)$ | $\zeta_{top}^{-1}(z)/(1-z)$ |
|---|---|
| $1C$ | |
| $101C$ | |
| $1011101C$ | |
| $H^\infty(1)$ | $\prod_{n=0}^{\infty}(1-z^{2^n})$ |
| $10111C$ | |
| $1011111C$ | |
| $101^\infty$ | $(1-2z^2)/(1+z)$ |
| $10111111C$ | |
| $101111C$ | |
| $1011C$ | |
| $101101C$ | |
| $10C$ | $(1-z-z^2)$ |
| $10010C$ | |
| $100101C$ | |

| $I(C)$ | $\zeta_{top}^{-1}(z)/(1-z)$ |
|---|---|
| $1001C$ | |
| $100111C$ | |
| $10011C$ | |
| $100110C$ | |
| $100C$ | |
| $100010C$ | |
| $10001C$ | |
| $100011C$ | |
| $1000C$ | |
| $100001C$ | |
| $10000C$ | |
| $100000C$ | |
| $10^\infty$ | $(1-2z)/(1-z)$ |

**Table D.1:** All ordered kneading sequences up to length seven, as well as some longer kneading sequences. Harmonic extension $H^\infty(1)$ is defined below.

sequence $K_\Lambda$ is the itinerary of the critical point (11.13). The crucial observation is that no periodic orbit can have a topological coordinate (see sect. D.1.1) beyond that of the kneading sequence. The kneading sequence thus inserts a border in the list of periodic orbits (ordered according to maximal topological coordinate), cycles up to this limit are allowed, all beyond are pruned. All unimodal maps (obeying some further constraints) with the same kneading sequence thus have the same set of periodic orbitsand the same topological zeta function. The topological coordinate of the kneading sequence increases with increasing $\Lambda$.

The kneading sequence can be of one of three types

1. It maps to the critical point again, after $n$ iterations. If so, we adopt the convention to terminate the kneading sequence with a $C$, and refer to the kneading sequence as finite.
2. Preperiodic, i.e., it is infinite but with a periodic tail.
3. Aperiodic.

As an archetype unimodal map we will choose the *tent map*

$$x \mapsto f(x) = \begin{cases} \Lambda x & x \in [0, 1/2] \\ \Lambda(1-x) & x \in (1/2, 1] \end{cases}, \tag{D.2}$$

where the parameter $\Lambda \in (1, 2]$. The topological entropy is $h = \log \Lambda$. This follows from the fact any trajectory of the map is bounded, the escape rate is strictly zero, and so the dynamical zeta function

$$1/\zeta(z) = \prod_p \left(1 - \frac{z^{n_p}}{|\Lambda_p|}\right) = \prod_p \left(1 - \left(\frac{z}{\Lambda}\right)^{n_p}\right) = 1/\zeta_{\text{top}}(z/\Lambda)$$

has its leading zero at $z = 1$.

The set of periodic points of the tent map is countable. A consequence of this fact is that the set of parameter values for which the kneading sequence (11.13) is periodic or preperiodic are countable and thus of measure zero and consequently *the kneading sequence is aperiodic for almost all* $\Lambda$. For general unimodal maps the corresponding statement is that the kneading sequence is aperiodic for almost all topological entropies.

For a given periodic kneading sequence of period $n$, $\underline{K_\Lambda} = PC = s_1 s_2 \ldots s_{n-1} C$ there is a simple expansion for the topological zeta function. Then the expanded zeta function is a polynomial of degree $n$

$$1/\zeta_{\rm top}(z) = \prod_p (1 - z_p^n) = (1-z)\sum_{i=0}^{n-1} a_i z^i \,, \qquad a_i = \prod_{j=1}^{i} (-1)^{s_j} \tag{D.3}$$

and $a_0 = 1$.

Aperiodic and preperiodic kneading sequences are accounted for by simply replacing $n$ by $\infty$.

**Example**. Consider as an example the kneading sequence $K_\Lambda = 10C$. From (D.3) we get the topological zeta function $1/\zeta_{top}(z) = (1-z)(1-z-z^2)$, see table D.1. This can also be realized by redefining the alphabet. The only forbidden subsequence is 100. All allowed periodic orbits, except $\overline{0}$, can can be built from a alphabet with letters $\underline{10}$ and $\underline{1}$. We write this alphabet as $\{\underline{10}, \underline{1}; \overline{0}\}$, yielding the topological zeta function $1/\zeta_{top}(z) = (1-z)(1-z-z^2)$. The leading zero is the inverse golden mean $z_0 = (\sqrt{5}-1)/2$.

**Example**. As another example we consider the preperiodic kneading sequence $K_\Lambda = 101^\infty$. From (D.3) we get the topological zeta function $1/\zeta_{top}(z) = (1-z)(1-2z^2)/(1+z)$, see table D.1. This can again be realized by redefining the alphabet. There are now an infinite number of forbidden subsequences, namely $101^{2n}0$ where $n \geq 0$. These pruning rules are respected by the alphabet $\{\underline{01^{2n+1}}; \overline{1}, \overline{0}\}$, yielding the topological zeta function above. The pole in the zeta function $\zeta^{-1}_{top}(z)$ is a consequence of the infinite alphabet.

An important consequence of (D.3) is that the sequence $\{a_i\}$ has a periodic tail if and only if the kneading sequence has one (however, their period may differ by a factor of two). We know already that the kneading sequence is aperiodic for almost all $\Lambda$.

The analytic structure of the function represented by the infinite series $\sum a_i z_i$ with unity as radius of convergence, depends on whether the tail of $\{a_i\}$ is periodic or not. If the period of the tail is $N$ we can write

$$1/\zeta_{top}(z) = p(z) + q(z)(1 + z^N + z^{2N} \ldots) = p(z) + \frac{q(z)}{1-z^N} \,,$$

for some polynomials $p(z)$ and $q(z)$. The result is a set of poles spread out along the unit circle. This applies to the preperiodic case. An aperiodic sequence of coefficients would formally correspond to infinite $N$ and it is natural to assume that the singularities will fill the unit circle. There is indeed a theorem ensuring that this is the case [12.58], provided the $a_i$'s can only take on a finite number of values. The unit circle becomes a *natural boundary*, already apparent in a finite polynomial approximations to the topological zeta function, as in figure 15.2. A function with a natural boundary lacks an analytic continuation outside it.

To conclude: The topological zeta function $1/\zeta_{top}$ for unimodal maps has the unit circle as a natural boundary for almost all topological entropies and for the tent map (D.2), for almost all $\Lambda$.

Let us now focus on the relation between the analytic structure of the topological zeta function and the number of periodic orbits, or rather (15.8), the number $N_n$ of fixed points of $f^n(x)$. The trace formula is (see sect. 15.4)

$$N_n = \mathrm{tr}\, T^n = \frac{1}{2\pi i}\oint_{\gamma_r} dz\, z^{-n}\frac{d}{dz}\log\zeta_{top}^{-1}$$

where $\gamma_r$ is a (circular) contour encircling the origin $z = 0$ in clockwise direction. Residue calculus turns this into a sum over zeros $z_0$ and poles $z_p$ of $\zeta_{top}^{-1}$

$$N_n = \sum_{z_0:r<|z_0|<R} z_0^{-n} - \sum_{z_p:r<|z_p|<R} z_0^{-n} + \frac{1}{2\pi i}\oint_{\gamma_R} dz\, z^{-n}\frac{d}{dz}\log\zeta_{top}^{-1}$$

and a contribution from a large circle $\gamma_R$. For meromorphic topological zeta functions one may let $R \to \infty$ with vanishing contribution from $\gamma_R$, and $N_n$ will be a sum of exponentials.

The leading zero is associated with the topological entropy, as discussed in chapter 15.

We have also seen that for preperiodic kneading there will be poles on the unit circle.

To appreciate the role of natural boundaries we will consider a (very) special example. Cascades of period doublings is a central concept for the description of unimodal maps. This motivates a close study of the function

$$\Xi(z) = \prod_{n=0}^{\infty}(1 - z^{2^n}) \quad . \tag{D.4}$$

This function will appear again when we derive (D.3).

The expansion of $\Xi(z)$ begins as $\Xi(z) = 1 - z - z^2 + z^3 - z^4 + z^5 \ldots$. The radius of convergence is obviously unity. The simple rule governing the expansion will effectively prohibit any periodicity among the coefficients making the unit circle a natural boundary.

It is easy to see that $\Xi(z) = 0$ if $z = \exp(2\pi m/2^n)$ for any integer $m$ and $n$. (Strictly speaking we mean that $\Xi(z) \to 0$ when $z \to \exp(2\pi m/2^n)$ from inside). Consequently, zeros are dense on the unit circle. One can also show that singular points are dense on the unit circle, for instance $|\Xi(z)| \to \infty$ when $z \to \exp(2\pi m/3^n)$ for any integer $m$ and $n$.

As an example, the topological zeta function at the accumulation point of the first Feigenbaum cascade is $\zeta_{top}^{-1}(z) = (1-z)\Xi(z)$. Then $N_n = 2^{l+1}$ if $n = 2^l$, otherwise $N_n = 0$. The growth rate in the number of cycles is anything but exponential. It is clear that $N_n$ cannot be a sum of exponentials, the contour $\gamma_R$ cannot be pushed away to infinity, $R$ is restricted to $R \leq 1$ and $N_n$ is entirely determined by $\int_{\gamma_R}$ which picks up its contribution from the natural boundary.

We have so far studied the analytic structure for some special cases and we know that the unit circle is a natural boundary for almost all $\Lambda$. But how does it look out there in the complex plane for some typical parameter values? To explore that we will imagine a journey from the origin $z = 0$ out towards the unit circle. While traveling we let the parameter $\Lambda$ change slowly. The trip will have a distinct science fiction flavor. The first zero we encounter is the one connected to the topological entropy. Obviously it moves smoothly and slowly. When we move outward to the unit circle we encounter zeros in increasing densities. The closer to the unit circle they are, the wilder and stranger they move. They move from and back to the horizon, where they are created and destroyed through bizarre bifurcations. For some special values of the parameter the unit circle suddenly gets transparent and and we get (infinitely) short glimpses of another world beyond the horizon.

We end this section by deriving eqs (D.5) and (D.6). The impenetrable prose is hopefully explained by the accompanying tables.

We know one thing from chapter 11, namely for that finite kneading sequence of length $n$ the topological polynomial is of degree $n$. The graph contains a node which is connected to itself only via the symbol 0. This implies that a factor $(1-z)$ may be factored out and $\zeta_{top}(z) = (1-z)\sum_{i=0}^{n-1} a_i z^i$. The problem is to find the coefficients $a_i$.

The ordered list of (finite) kneading sequences table D.1 and the ordered list of periodic orbits (on maximal form) are intimately related. In table D.2 we indicate how they are nested during a period doubling cascade. Every finite kneading sequence $PC$ is bracketed by two periodic orbits, $\overline{P1}$ and $\overline{P0}$. We have $\overline{P1} < PC < \overline{P0}$ if $P$ contains an odd number of 1's, and $\overline{P0} < PC < \overline{P1}$ otherwise. From now on we will assume that $P$ contains an odd number of 1's. The other case can be worked out in complete analogy. The first and second harmonic of $PC$ are displayed in table D.2. The periodic orbit $\overline{P1}$ (and the corresponding infinite kneading sequence) is sometimes referred to as the antiharmonic extension of $PC$ (denoted $A^\infty(P)$) and the accumulation point of the cascade is called the harmonic extension of $PC$ [11.8] (denoted $H^\infty(P)$).

| periodic orbits | finite kneading sequences |
|---|---|
| $\overline{P1} = A^\infty(P)$ | |
| | $PC$ |
| $\overline{P0}$ | |
| | $P0PC$ |
| $\overline{P0P1}$ | |
| | $P0P1P0PC$ |
| $\downarrow$ | $\downarrow$ |
| $H^\infty(P)$ | $H^\infty(P)$ |

**Table D.2:** Relation between periodic orbits and finite kneading sequences in a harmonic cascade. The string $P$ is assumed to contain an odd number of 1's.

A central result is the fact that a period doubling cascade of $PC$ is not interfered by any other sequence. Another way to express this is that a kneading sequence $PC$ and its harmonic are adjacent in the list of kneading sequences to any order.

| $I(C)$ | | | $\zeta_{top}^{-1}(z)/(1-z)$ |
|---|---|---|---|
| $P_1$ | $=$ | $100C$ | $1-z-z^2-z^3$ |
| $H^\infty(P_1)$ | $=$ | $10001001100\ldots$ | $1-z-z^2-z^3-z^4+z^5+z^6+z^7-z^8\ldots$ |
| $P'$ | $=$ | $10001C$ | $1-z-z^2-z^3-z^4+z^5$ |
| $A^\infty(P_2)$ | $=$ | $1000110001\ldots$ | $1-z-z^2-z^3-z^4+z^5-z^6-z^7-z^8\ldots$ |
| $P_2$ | $=$ | $1000C$ | $1-z-z^2-z^3-z^4$ |

**Table D.3:** Example of a step in the iterative construction of the list of kneading sequences $PC$.

Table D.3 illustrates another central result in the combinatorics of kneading sequences. We suppose that $P_1C$ and $P_2C$ are neighbors in the list of order 5 (meaning that the shortest finite kneading sequence $P'C$ between $P_1C$ and $P_2C$ is longer than 5.) The important result is that $P'$ (of length $n' = 6$) has to coincide with the first $n' - 1$ letters of both $H^\infty(P_1)$ and $A^\infty(P_2)$. This is exemplified in the left column of table D.3. This fact makes it possible to generate the list of kneading sequences in an iterative way.

The zeta function at the accumulation point $H^\infty(P_1)$ is

$$\zeta_{P_1}^{-1}(z)\Xi(z^{n_1}) \;, \tag{D.5}$$

and just before $A^\infty(P_2)$

$$\zeta_{P_2}^{-1}(z)/(1-z^{n_2}) \;. \tag{D.6}$$

A short calculation shows that this is exactly what one would obtain by applying (D.3) to the antiharmonic and harmonic extensions directly, provided that it applies to $\zeta_{P_1}^{-1}(z)$ and $\zeta_{P_2}^{-1}(z)$. This is the key observation.

Recall now the product representation of the zeta function $\zeta^{-1} = \prod_p (1 - z^{n_p})$. We will now make use of the fact that the zeta function associated with

$P'C$ is a polynomial of order $n'$. There is no periodic orbit of length shorter than $n' + 1$ between $H^\infty(P_1)$ and $A^\infty(P_2)$. It thus follows that the coefficients of this polynomial coincides with those of (D.5) and (D.6), see Table D.3. We can thus conclude that our rule can be applied directly to $P'C$.

This can be used as an induction step in proving that the rule can be applied to every finite and infinite kneading sequences.

**Remark D.1** How to prove things. The explicit relation between the kneading sequence and the coefficients of the topological zeta function is not commonly seen in the literature. The result can proven by combining some theorems of Milnor and Thurston [11.14]. That approach is hardly instructive in the present context. Our derivation was inspired by Metropolis, Stein and Stein classical paper [11.8]. For further detail, consult [15.14].

### D.1.1 Periodic orbits of unimodal maps

A *periodic point* (*cycle* point) $x_k$ belonging to a cycle of period $n$ is a real solution of

$$f^n(x_k) = f(f(\ldots f(x_k)\ldots)) = x_k, \quad k = 0, 1, 2, \ldots, n-1\,. \tag{D.7}$$

The $n$th iterate of a unimodal map has at most $2^n$ monotone segments, and therefore there will be $2^n$ or fewer periodic points of length $n$. Similarly, the backward and the forward Smale horseshoes intersect at most $2^n$ times, and therefore there will be $2^n$ or fewer periodic points of length $n$. A periodic orbit of length $n$ corresponds to an infinite repetition of a length $n = n_p$ symbol string, customarily indicated by a line over the string:

$$S_p = (s_1 s_2 s_3 \ldots s_n)^\infty = \overline{s_1 s_2 s_3 \ldots s_n}\,.$$

As all itineraries are infinite, we shall adopt convention that a finite string itinerary $S_p = s_1 s_2 s_3 \ldots s_n$ stands for infinite repetition of a finite block, and routinely omit the overline. $x_0$, its cyclic permutation
$\overline{s_k s_{k+1} \ldots s_n s_1 \ldots s_{k-1}}$ corresponds to the point $x_{k-1}$ in the same cycle. A cycle $p$ is called *prime* if its itinerary $S$ cannot be written as a repetition of a shorter block $S'$.

Each cycle $p$ is a set of $n_p$ rational-valued full tent map periodic points $\gamma$. It follows from (11.9) that if the repeating string $s_1 s_2 \ldots s_n$ contains an odd number "1"s, the string of well ordered symbols $w_1 w_2 \ldots w_{2n}$ has to be of the double length before it repeats itself. The cycle-point $\gamma$ is a geometrical sum which we can rewrite as the fraction

$$\gamma(\overline{s_1 s_2 \ldots s_n}) = \frac{2^{2n}}{2^{2n}-1} \sum_{t=1}^{2n} w_t / 2^t \tag{D.8}$$

Using this we can calculate the $\hat{\gamma}(S)$ for all short cycles. For orbits up to length 5 this is done in table 11.1.

Here we give explicit formulas for the topological coordinate of a periodic point, given its itinerary. For the purpose of what follows it is convenient to compactify the itineraries by replacing the binary alphabet $s_i = \{0, 1\}$ by the infinite alphabet

$$\{a_1, a_2, a_3, a_4, \cdots; \overline{0}\} = \{1, 10, 100, 1000, \ldots; \overline{0}\}\,. \tag{D.9}$$

In this notation the itinerary $S = a_i a_j a_k a_l \cdots$ and the corresponding topological coordinate (11.9) are related by $\gamma(S) = .1^i 0^j 1^k 0^l \cdots$. For example:

$$\begin{array}{lclcl} S & = & 111011101001000\ldots & = & a_1 a_1 a_2 a_1 a_1 a_2 a_3 a_4 \ldots \\ \gamma(S) & = & .101101001110000\ldots & = & .1^1 0^1 1^2 0^1 1^1 0^2 1^3 0^4 \ldots \end{array}$$

Cycle points whose itineraries start with $w_1 = w_2 = \ldots = w_i = 0$, $w_{i+1} = 1$ remain on the left branch of the tent map for $i$ iterations, and satisfy $\gamma(0 \ldots 0S) = \gamma(S)/2^i$.

Periodic points correspond to rational values of $\gamma$, but we have to distinguish *even* and *odd* cycles. The even (odd) cycles contain even (odd) number of $a_i$ in the repeating block, with periodic points given by

$$\gamma(a_i a_j \cdots a_k a_\ell) = \begin{cases} \frac{2^n}{2^n - 1} .1^i 0^j \cdots 1^k & \text{even} \\ \frac{1}{2^n + 1} \left(1 + 2^n \times .1^i 0^j \cdots 1^\ell\right) & \text{odd} \end{cases}\,, \tag{D.10}$$

where $n = i + j + \cdots + k + \ell$ is the cycle period. The maximal value periodic point is given by the cyclic permutation of $S$ with the largest $a_i$ as the first symbol, followed by the smallest available $a_j$ as the next symbol, and so on. For example:

$$\begin{array}{lclclclcl} \hat{\gamma}(1) & = & \gamma(a_1) & = & .10101\ldots & = & .\overline{10} & = & 2/3 \\ \hat{\gamma}(10) & = & \gamma(a_2) & = & .1^2 0^2 \ldots & = & .\overline{1100} & = & 4/5 \\ \hat{\gamma}(100) & = & \gamma(a_3) & = & .1^3 0^3 \ldots & = & .\overline{111000} & = & 8/9 \\ \hat{\gamma}(101) & = & \gamma(a_2 a_1) & = & .1^2 0^1 \ldots & = & .\overline{110} & = & 6/7 \end{array}$$

An example of a cycle where only the third symbol determines the maximal value periodic point is

$$\hat{\gamma}(1101110) = \gamma(a_2 a_1 a_2 a_1 a_1) = .\overline{11011010010010} = 100/129\,.$$

Maximal values of all cycles up to length 5 are given in table!?

## D.2 Prime factorization for dynamical itineraries

The Möbius function is not only a number-theoretic function, but can be used to manipulate ordered sets of noncommuting objects such as symbol strings. Let $\mathbf{P} = \{p_1, p_2, p_3, \cdots\}$ be an ordered set of *prime* strings, and

$$\mathcal{N} = \{n\} = \left\{ p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots p_j^{k_j} \right\} ,$$

$j \in \mathbb{N}$, $k_i \in \mathbb{Z}_+$, be the set of all strings $n$ obtained by the ordered concatenation of the "primes" $p_i$. By construction, every string $n$ has a unique prime factorization. We say that a string has a divisor $d$ if it contains $d$ as a substring, and define the string division $n/d$ as $n$ with the substring $d$ deleted. Now we can do things like this: defining $t_n := t_{p_1}^{k_1} t_{p_2}^{k_2} \cdots t_{p_j}^{k_j}$ we can write the inverse dynamical zeta function (20.2) as

$$\prod_p (1 - t_p) = \sum_n \mu(n) t_n ,$$

and, if we care (we do in the case of the Riemann zeta function), the dynamical zeta function as .

$$\prod_p \frac{1}{1 - t_p} = \sum_n t_n \qquad \text{(D.11)}$$

A striking aspect of this formula is its resemblance to the factorization of natural numbers into primes: the relation of the cycle expansion (D.11) to the product over prime cycles is analogous to the Riemann zeta (exercise 19.10) represented as a sum over natural numbers vs. its Euler product representation.

We now implement this factorization explicitly by decomposing recursively binary strings into ordered concatenations of prime strings. There are 2 strings of length 1, both prime: $p_1 = 0$, $p_2 = 1$. There are 4 strings of length 2: 00, 01, 11, 10. The first three are ordered concatenations of primes: $00 = p_1^2$, $01 = p_1 p_2$, $11 = p_2^2$; by ordered concatenations we mean that $p_1 p_2$ is legal, but $p_2 p_1$ is not. The remaining string is the only prime of length 2, $p_3 = 10$. Proceeding by discarding the strings which are concatenations of shorter primes $p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}$, with primes lexically ordered, we generate the standard list of primes, in agreement with table 15.1: 0, 1, 10, 101, 100, 1000, 1001, 1011, 10000, 10001, 10010, 10011, 10110, 10111, 100000, 100001, 100010, 100011, 100110, 100111, 101100, 101110, 101111, . . .. This factorization is illustrated in table D.4.

| factors | string |
|---|---|
| $p_1$ | 0 |
| $p_2$ | 1 |
| $p_1^2$ | 00 |
| $p_1 p_2$ | 01 |
| $p_2^2$ | 11 |
| $p_3$ | 10 |
| $p_1^3$ | 000 |
| $p_1^2 p_2$ | 001 |
| $p_1 p_2^2$ | 011 |
| $p_2^3$ | 111 |
| $p_1 p_3$ | 010 |
| $p_2 p_3$ | 110 |
| $p_4$ | 100 |
| $p_5$ | 101 |

| factors | string |
|---|---|
| $p_1^4$ | 0000 |
| $p_1^3 p_2$ | 0001 |
| $p_1^2 p_2^2$ | 0011 |
| $p_1 p_2^3$ | 0111 |
| $p_2^4$ | 1111 |
| $p_1^2 p_3$ | 0010 |
| $p_1 p_2 p_3$ | 0110 |
| $p_2^2 p_3$ | 1110 |
| $p_3^2$ | 1010 |
| $p_1 p_4$ | 0100 |
| $p_2 p_4$ | 1100 |
| $p_1 p_5$ | 0101 |
| $p_2 p_5$ | 1101 |
| $p_6$ | 1000 |
| $p_7$ | 1001 |
| $p_8$ | 1011 |

| factors | string |
|---|---|
| $p_1^5$ | 00000 |
| $p_1^4 p_2$ | 00001 |
| $p_1^3 p_2^2$ | 00011 |
| $p_1^2 p_2^3$ | 00111 |
| $p_1 p_2^4$ | 01111 |
| $p_2^5$ | 11111 |
| $p_1^3 p_3$ | 00010 |
| $p_1^2 p_2 p_3$ | 00110 |
| $p_1 p_2^2 p_3$ | 01110 |
| $p_2^3 p_3$ | 11110 |
| $p_1 p_3^2$ | 01010 |
| $p_2 p_3^2$ | 11010 |
| $p_1^2 p_4$ | 00100 |
| $p_1 p_2 p_4$ | 01100 |
| $p_2^2 p_4$ | 11100 |
| $p_3 p_4$ | 10100 |

| factors | string |
|---|---|
| $p_1^2 p_5$ | 00101 |
| $p_1 p_2 p_5$ | 01101 |
| $p_2^2 p_5$ | 11101 |
| $p_3 p_5$ | 10101 |
| $p_1 p_6$ | 01000 |
| $p_2 p_6$ | 11000 |
| $p_1 p_7$ | 01001 |
| $p_2 p_7$ | 11001 |
| $p_1 p_8$ | 01011 |
| $p_2 p_8$ | 11011 |
| $p_9$ | 10000 |
| $p_{10}$ | 10001 |
| $p_{11}$ | 10010 |
| $p_{12}$ | 10011 |
| $p_{13}$ | 10110 |
| $p_{14}$ | 10111 |

**Table D.4:** Factorization of all periodic points strings up to length 5 into ordered concatenations $p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$ of prime strings $p_1 = 0$, $p_2 = 1$, $p_3 = 10$, $p_4 = 100$, . . . , $p_{14} = 10111$.

### D.2.1 Prime factorization for spectral determinants

Following sect. D.2, the spectral determinant cycle expansions is obtained by expanding $F$ as a multinomial in prime cycle weights $t_p$

$$F = \prod_p \sum_{k=0}^{\infty} C_{p^k} t_p^k = \sum_{k_1 k_2 k_3 \cdots = 0}^{\infty} \tau_{p_1^{k_1} p_2^{k_2} p_3^{k_3} \ldots} \tag{D.12}$$

where the sum goes over all pseudocycles. In the above we have defined

$$\tau_{p_1^{k_1} p_2^{k_2} p_3^{k_3} \ldots} = \prod_{i=1}^{\infty} C_{p_i{}^{k_i}} t_{p_i}^{k_i} \ . \tag{D.13}$$

exercise 19.10

A striking aspect of the spectral determinant cycle expansion is its resemblance to the factorization of natural numbers into primes: as we already noted in sect. D.2, the relation of the cycle expansion (D.12) to the product formula (19.9) is analogous to the Riemann zeta represented as a sum over natural numbers vs. its Euler product representation.

This is somewhat unexpected, as the cycle weights factorize exactly with respect to $r$ repetitions of a prime cycle, $t_{pp\ldots p} = t_p^r$, but only approximately (*shadowing*) with respect to subdividing a string into prime substrings, $t_{p_1 p_2} \approx t_{p_1} t_{p_2}$.

The coefficients $C_{p^k}$ have a simple form only in $1-$*dimensional*, given by the Euler formula (23.5). In higher dimensions $C_{p^k}$ can be evaluated by expanding (19.9), $F(z) = \prod_p F_p$, where

$$F_p = 1 - \left(\sum_{r=1}^{\infty} \frac{t_p^r}{r d_{p,r}}\right) + \frac{1}{2}\left(\sum_{r=1}^{\infty} \frac{t_p^r}{r d_{p,r}}\right)^2 - \ldots .$$

Expanding and recollecting terms, and suppressing the $p$ cycle label for the moment, we obtain

$$\begin{aligned} F_p &= \sum_{r=1}^{\infty} C_k t^k, \quad C_k = (-)^k c_k / D_k, \\ D_k &= \prod_{r=1}^{k} d_r = \prod_{a=1}^{d} \prod_{r=1}^{k} (1 - u_a^r) \end{aligned} \tag{D.14}$$

where evaluation of $c_k$ requires a certain amount of not too luminous algebra:

$$\begin{aligned} c_0 &= 1 \\ c_1 &= 1 \\ c_2 &= \frac{1}{2}\left(\frac{d_2}{d_1} - d_1\right) = \frac{1}{2}\left(\prod_{a=1}^{d}(1+u_a) - \prod_{a=1}^{d}(1-u_a)\right) \\ c_3 &= \frac{1}{3!}\left(\frac{d_2 d_3}{d_1^2} + 2 d_1 d_2 - 3 d_3\right) \\ &= \frac{1}{6}\left(\prod_{a=1}^{d}(1 + 2u_a + 2u_a^2 + u_a^3)\right. \\ &\quad \left. + 2\prod_{a=1}^{d}(1 - u_a - u_a^2 + u_a^3) - 3\prod_{a=1}^{d}(1 - u_a^3)\right) \end{aligned}$$

*etc.*. For example, for a general 2-dimensional map we have

$$F_p = 1 - \frac{1}{D_1} t + \frac{u_1 + u_2}{D_2} t^2 - \frac{u_1 u_2 (1 + u_1)(1 + u_2) + u_1^3 + u_2^3}{D_3} t^3 + \ldots . \tag{D.15}$$

We discuss the convergence of such cycle expansions in sect. I.4.

With $\tau_{p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}}$ defined as above, the prime factorization of symbol strings is unique in the sense that *each symbol string can be written as a unique concatenation of prime strings*, up to a convention on ordering of primes. This factorization is a nontrivial example of the utility of generalized Möbius inversion, sect. D.2.

How is the factorization of sect. D.2 used in practice? Suppose we have computed (or perhaps even measured in an experiment) all prime cycles up to length

$n$, i.e., we have a list of $t_p$'s and the corresponding Jacobian matrix eigenvalues $\Lambda_{p,1}, \Lambda_{p,2}, \ldots \Lambda_{p,d}$. A cycle expansion of the Selberg product is obtained by generating all strings in order of increasing length $j$ allowed by the symbolic dynamics and constructing the multinomial

$$F = \sum_n \tau_n \tag{D.16}$$

where $n = s_1 s_2 \cdots s_j$, $s_i$ range over the alphabet, in the present case $\{0, 1\}$. Factorizing every string $n = s_1 s_2 \cdots s_j = p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}$ as in table D.4, and substituting $\tau_{p_1^{k_1} p_2^{k_2} \cdots}$ we obtain a multinomial approximation to $F$. For example, $\tau_{001001010101} = \tau_{\underline{001}\,\underline{001}\,\underline{01}\,\underline{01}\,\underline{01}} = \tau_{001^2}\tau_{01^3}$, and $\tau_{01^3}$, $\tau_{001^2}$ are known functions of the corresponding cycle eigenvalues. The zeros of $F$ can now be easily determined by standard numerical methods. The fact that as far as the symbolic dynamics is concerned, the cycle expansion of a Selberg product is simply an average over all symbolic strings makes Selberg products rather pretty.

To be more explicit, we illustrate the above by expressing binary strings as concatenations of prime factors. We start by computing $N_n$, the number of terms in the expansion (D.12) of the total cycle length $n$. Setting $C_{p^k} t_p^k = z^{n_p k}$ in (D.12), we obtain

$$\sum_{n=0}^{\infty} N_n z^n = \prod_p \sum_{k=0}^{\infty} z^{n_p k} = \frac{1}{\prod_p (1 - z^{n_p})} \,.$$

So the generating function for the number of terms in the Selberg product is the topological zeta function. For the complete binary dynamics we have $N_n = 2^n$ contributing terms of length $n$:

$$\zeta_{top} = \frac{1}{\prod_p (1 - z^{n_p})} = \frac{1}{1 - 2z} = \sum_{n=0}^{\infty} 2^n z^n$$

Hence the number of distinct terms in the expansion (D.12) is the same as the number of binary strings, and conversely, the set of binary strings of length $n$ suffices to label all terms of the total cycle length $n$ in the expansion (D.12).