Chapter 1

Overture

> If I have seen less far than other men it is because I have stood behind giants.
>
> Edoardo Specchio

Sect. 1.1 together with the appendix A offers a historical overview of the development of the theory of chaotic dynamics, mostly of scholarly and human interest only. The book proper starts in sect. 1.3. A pinball game is used to motivate and illustrate most of the concepts to be developed in this book: unstable dynamical flows, Poincaré sections, Smale horseshoes, symbolic dynamics, pruning, discrete symmetries, periodic orbits, averaging over chaotic sets, evolution operators, dynamical zeta functions, spectral determinants, cycle expansions, quantum trace formulas and zeta functions, and so on to the semiclassical quantization of helium. This chapter is a quick parcourse of the main topics covered in the book.

1.1 Death of the old Quantum Theory

> In 1913 Otto Stern and Max Theodor Felix von Laue went up for a walk up the Uetliberg. On the top they sat down and talked about physics. In particular they talked about the new atom model of Bohr. There and then they made the "Uetli Schwur": If that crazy model of Bohr turned out to be right, then they would leave physics. It did and they didn't.
>
> A. Pais, *Inward Bound: of Matter and Forces in the Physical World*

An afternoon of May 1991 Dieter Wintgen sat in an office at the Niels Bohr Institute beaming with the unparalleled glee of a boy who has just committed a major mischief. The starting words of the manuscript that he has just penned were:

The failure of the Copenhagen School to obtain a reasonable . . .

Dieter was a scruffy kind of guy, always in sandals and holed out jeans, a left winger and a mountain climber, working around the clock with his students Gregor and Klaus to complete the work that Bohr himself would have loved to see done back in 1916: a "planetary" calculation of the helium spectrum.

Never mind that "the Copenhagen School" refers not to the old quantum theory, but to something else. The old quantum theory was no theory at all; it was a set of rules bringing some order in a set of phenomena which the defied logic of classical theory. The electrons were supposed to describe planetary orbits around the nucleus; their wave aspects were yet to be discovered. The foundations seemed obscure, but Bohr's answer for the once-ionized helium to hydrogen ratio was correct to five significant figures and hard to ignore. The old quantum theory marched on, until by 1924 it reached impasse: the helium spectrum and the Zeeman effect were its death knell.

Since late 1890's it had been known that the helium spectrum consists of the orthohelium and parahelium lines. In 1915 Bohr suggested that the two kinds of helium lines might be associated with two distinct shapes of orbits (a suggestion that turned out to be wrong). In 1916 he got Kramers to work on the problem, and wrote to Rutherford: "I have used all my spare time in the last months to make a serious attempt to solve the problem of ordinary helium spectrum . . . I think really that at last I have a clue to the problem." To other colleagues he wrote that "the theory was worked out in the fall of 1916" and of having obtained a "partial agreement with the measurements." Nevertheless, the Bohr-Sommerfeld theory, while by and large successful for hydrogen, was a disaster for neutral helium. Heroic efforts of the young generation, including Kramers and Heisenberg, were of no avail.

For a while Heisenberg thought that he had the ionization potential for helium, which he had obtained by a simple perturbative scheme. He wrote enthusiastic letters to Sommerfeld and was drawn into a collaboration with Max Born to compute the spectrum of helium using Born's systematic perturbative scheme. In first approximation, they reproduced the earlier calculations. The next level of corrections turned out to be larger than the computed effect. The paper they wrote on their efforts concludes with a somber tone. The concluding paragraph of Max Born's classic "Vorlesungen über Atommechanik" from 1925 sums it up:

> (. . .) the systematic application of the principles of the quantum theory (. . .) gives results in agreement with experiment only in those cases where the motion of a single electron is considered; it fails even in the treatment of the motion of the two electrons in the helium atom.
>
> This is not surprising, for the principles used are not really consistent. (. . .) A complete systematic transformation of the classical mechanics into a discontinuous mechanics is the goal towards which the quantum theory strives.

That year Heisenberg suffered a bout of hay fever, and the old quantum theory was dead. In 1926 he gave the first quantitative explanation of the helium spectrum. He explained the distinction between the experimentally observed orthohelium and parahelium spectral lines, and predicted that orthohelium lines were triplets. He used wave mechanics, electron spin and the Pauli exclusion principle, none of which belonged to the old quantum theory, and planetary orbits of electrons were cast away for nearly half a century.

Why did Pauli and Heisenberg fail with the helium atom? It was not the fault of the old quantum mechanics, but rather it reflected their lack of understanding of the subtleties of classical mechanics. Today we know what is it that they missed in 1913-24: the role of conjugate points (Maslov indices) along classical trajectories was not accounted for, and they had no clue of what the role of periodic orbits in nonintegrable systems should be.

Since then the calculation for helium using the methods of the old quantum mechanics has been fixed. Leopold and Percival added the Maslov indices in 1980, and in 1991 Wintgen and collaborators understood the role of periodic orbits. Dieter had good reasons to gloat; while rest of us were preparing to sharpen our pencils and supercomputers in order to approach the dreaded 3-body problem, they just went ahead and did it. What it took - and much else - is described in this book. One is also free to ponder what would quantum theory look like today if all this was worked out in 1917. When asked this question, Hans Bethe responded with an exasperated look: it would be just the same, he said.

Remark **1.1** Sources. This tale, aside from a few personal recollections, is in large part lifted from Abraham Pais' accounts of the demise of the old quantum theory [1.6, 1.7], as well as Jammer's account [1.3]. In August 1994 Dieter Wintgen died in a climbing accident in the Swiss Alps.

1.2 Why this book?

> It seems sometimes that through a preoccupation with science, we acquire a firmer hold over the vicissitudes of life and meet them with greater calm, but in reality we have done no more than to find a way to escape from our sorrows.
>
> Hermann Minkowski in a letter to David Hilbert

The problem has been with us since Newton's first frustrating crack at the 3-body problem. Nature is rich in systems governed by simple deterministic laws whose asymptotic dynamics might be complex beyond belief, systems which are locally unstable (almost) everywhere but globaly recurrent. How do we describe their long term dynamics?

The answer turns out to be that we have to evaluate a determinant, take a logarithm, stuff like that. Would hardly merit still another learned treatise, where it not for the fact that this determinant that we are to compute is fashioned of infinitely many infinitely small pieces. Sounds like statistical mechanics, does it not? Indeed it does, and that is how the problem was solved; in 1960's the pieces were counted, and in 1970's they were weighted and assembled together in a fashion that in beauty and in depth ranks along with thermodynamics, partition functions and path integrals amongst the crown jewels of mathematical physics.

Then something happened that might be without parallel; this is the only area of science where the advent of cheap computation had actually subtracted from our collective understanding. Excitement of "fractal science" of 1980's has popularized methods much inferior to deeper insights of the 1970's, and these computer pictures and numerical plots have now migrated into textbooks.

The goal of this book is to repair the damage, and return the beautiful theory to you. We teach you how to evaluate a determinant, take a logarithm, stuff like that. Should take 20 pages or so. Well, we fail - so far we have not found a way to traverse this material in less than a semester, or 200-300 pages of text. Sorry about that.

1.3 A game of pinball

> Find the quote about making contract to record everything that makes life worth living but is omitted from all the books.
>
> Henry Miller, in *Tropic of Cancer*

That deterministic dynamics leads to chaos is no surprise to anyone who has tried pool, billiards or snooker – that is what the game is about – so we start our story about what is chaos and what to do about it with a game of pinball. This might seem a trifle, but a pinball is to chaotic dynamics what a pendulum is to integrable systems: thinking clearly about what is "chaos" in a pinball will help us tackle more difficult problems, such as computing diffusion constants in deterministic gases, or computing the helium spectrum.

We all have an intuitive feeling for what a pinball does as it bounces between the pinball machine disks, and only high-school level Euclidean geometry is needed to describe the trajectory. Turning this intuition into cold calculation will lead us, in clear physically motivated steps, to almost everything one needs to know about deterministic chaos: from Smale horseshoes, through Cantor sets, Lyapunov exponents, symbolic dynamics, discrete symmetries, bifurcations, pruning, diffusion, all the way to transfer operators, thermodynamic formalism, the classical and quantum zeta functions, and spectral determinants. However, you must realize that the essence of this subject is incommunicable in print; the only way to developed intuition about chaotic dynamics is by computing, and the reader is urged to try to work through the exercises.

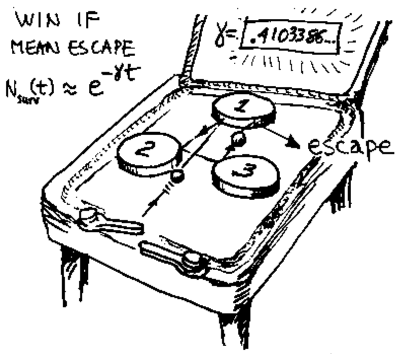


Figure 1.1: Physicist's bare bones pinball.

A physicist's pinball game is a pinball stripped to its bare essentials: three equidistantly placed reflecting disks in a plane, fig. 1.1. Physicist's pinball is free, frictionless, point-like, spinless, perfectly elastic, and noiseless. The only parameter in the system is R/a, the ratio of the center-to-center disk separation R and the disk radius a. Point-like pinballs are shot at the disks from random starting positions and angles; they spend some time bouncing between the disks and then escape. A pinball trajectory is fully determined by specifying (p, q), where q is the position of the collision of the pinball measured as arclength along the reflecting wall, θ is the angle between the outgoing trajectory and the normal to the wall, and $p = \sin\theta$ is the momentum parallel to the wall, fig. 1.5.

At the beginning of 18th century baron Gottfried Wilhelm Leibniz was confident that given the initial conditions one knew what a deterministic system would do far into the future. He wrote in "Von dem Verhängnisse":

> That everything is brought forth through an established destiny is just as certain as that three times three is nine. ...If, for example, one sphere meets another sphere in free space and if their sizes and their paths and directions before collision are known, we can then foretell and calculate how they will rebound and what course they will take after the impact. Very simple laws are followed which also apply, no matter how many spheres are taken or whether objects are taken other than spheres. From this one sees then that everything proceeds mathematically – that is, infallibly – in the whole wide world, so that if someone could have a sufficient insight into the inner parts of things, and in addition had remembrance and intelligence enough to consider all the circumstances and to take them into account, he would be a prophet and would see the future in the present as in a mirror.

Not only that the claim was plain wrong – but Leibniz chose to illustrate his faith in determinism precisely with the type of physical system that we shall use here as a paradigm of "hard" chaos.

1.3.1 What is "chaos"?

Two pinball trajectories that start out very close to each other separate exponentially with time, and in a finite (and in practice, very small) number of bounces their separation attains the size of the whole system, fig. 1.2. This property of *sensitivity to initial conditions* can be quantified as

$$|\delta \mathbf{x}(t)| \approx e^{\lambda t} |\delta \mathbf{x}|$$

where λ, the mean rate of separation of trajectories of the system, is called the *Lyapunov exponent*. For any finite accuracy δx of the initial data, the dynamics is predictable only up to a finite time $T \approx -\ln(\delta x)/\lambda$, despite of the deterministic and for baron Leibniz infallible simple laws that rule the pinball motion.

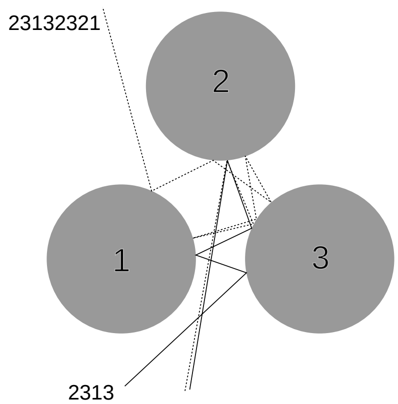


Figure 1.2: Sensitivity to initial conditions: two trajectories that start out very close to each other separate exponentially with time.

A positive Lyapunov exponent in itself does not chaos make. One could try to play 1- or 2-disk pinball, but it would not be much of a game; trajectories would only separate, never to meet again. What is also needed is *mixing*, the coming together again and again of trajectories. While locally the nearby trajectories separate, the interesting dynamics is recurrent confined to a globally finite region of the phase space and thus of necessity the separated trajectories are folded back and can re-approach each other arbitrarily closely, infinitely many times. In the case at hand there are 2^n topologically distinct n bounce trajectories that originate on a given disk. More generally, the number of distinct trajectories with n bounces can be quantified as

$$N(n) \approx e^{hn}$$

where the *topological entropy* h ($h = \ln 2$ in the case at hand) is the growth rate of the number of topologically distinct trajectories. We shall learn how to compute topological entropy in sect. 3.1. Strictly speaking, the correct quantity is the *Kolmogorov entropy*, but we have to postpone this discussion to sect. 12.3.

The appellation "chaos" is a confusing misnomer, as in deterministic dynamics there is no chaos in the everyday sense of the word; everything proceeds mathematically – that is, as baron Leibniz would have it, infallibly. When a physicist says that a certain system exhibits "chaos", he means that the system obeys deterministic laws of evolution, but that the outcome is highly sensitive to small uncertainties in the specification of the initial state. The word "chaos" has in this context taken on a narrow technical meaning. If a deterministic system is locally unstable (positive Lyapunov exponent) and globally mixing (positive Kolmogorov entropy), it is said to be *chaotic*. In a chaotic system any open ball of initial conditions, no matter how small, will in finite time overlap with any other finite region and in this sense spread over the extent of the entire asymptotically accessible phase space. Once this is grasped, the focus of theory shifts from attempting precise prediction (which is impossible) to description of the geometry of the space of possible outcomes, and evaluation of averages over this space. What is to be done with it is what this book is about.

Confronted with a potentially chaotic dynamical system, we analyze it through a sequence of three distinct stages; diagnose, count, measure. First, we determine the intrinsic *dimension* of the system – the minimum number of degrees of freedom necessary to capture its essential dynamics. If the system is very turbulent (description of its long time dynamics requires a space of high dimension) we are, at present, out of luck. We know only how to deal with the transitional regime between regular motions and weak turbulence. In this regime the chaotic dynamics is restricted to a space of low dimension, the number of relevant parameters is small, and we can proceed to the second step; we *count* and *classify* all possible topologically distinct trajectories of the system into a hierarchy whose successive layers require increased precision and patience on part of the observer. If successful, we can proceed with the third step: investigate the *weights* of the different pieces of the system.

Remark **1.2** The chaos word We owe the appellation "chaos" – as well as several other dynamics catchwords – to J. Yorke who entitled a paper [1.17] that he wrote with T. Li in 1973 *"Period 3 implies chaos"*.

1.3.2 Symbolic dynamics

We commence our analysis of the pinball game with the steps I, II: diagnose, count. We shall return to the step III – measure – in sect. 1.3.5 and chapter 10.

With a pinball we are in luck – it is low dimensional system, a free motion in a plane. The motion of a point particle is such that after a collision with one disk it either continues to another disk or it escapes. If we label the three disks by 1, 2 and 3, we can associate to every trajectory an *itinerary*, a sequence of labels which indicates the order in which the disks are visited; for example, the two trajectories in fig. 1.2 have itineraries _2313_, _23132321_ respectively. The itinerary will be finite for a scattering orbit, coming in from infinity and escaping after a finite number of collisions, infinite for a trapped orbit, and infinitely repeating for a periodic orbit. exercise 1.1

Such labeling is the simplest example of *symbolic dynamics* (we will discuss symbolic dynamics at length in chapter 2). There is one obvious restriction to the possible sequences, namely that any two consecutive symbols must differ, since the particle cannot collide two times in succession with the same disk. This is an example of *pruning*, a rule that forbids certain subsequences of symbols. Deriving pruning rules is in general a difficult problem, discussed in chapter 19.

It is important to keep in mind that the choice of symbols (or the associated Markov partitions of sect. 2.1.1) is in no sense unique. For example, as at each bounce we face a choice of proceeding to the next disk or returning to the previous disk, the above 3-letter alphabet can be replaced by a binary $\{0, 1\}$ alphabet, fig. 1.3. A clever choice of an alphabet will reflect important features of the dynamics, such as its symmetries, or the weight of a trajectory corresponding to a given itinerary.

Short periodic trajectories are easily drawn and enumerated, (see fig. 2.6 and fig. 2.5) but it is rather hard to perceive the systematics of the orbits from the shapes of their trajectories. The problem is that we are looking at the projection of a 4-dimensional phase space flow onto a 2-dimensional subspace, the space coordinates. A clearer picture of the dynamics is obtained by constructing a phase space Poincaré section.

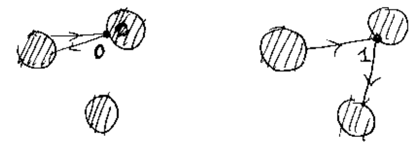


Figure 1.3: Binary labeling of the 3-disk pinball trajectories; a bounce in which the trajectory returns to the preceding disk is labeled 0, and a bounce which results in continuation to the third disk is labeled 1.

Suppose you wanted to play a good game of pinball, that is, get the pinball to bounce as many times as you possibly can – what would be a winning strategy? The simplest thing would be to try to aim the pinball so it bounces many times between a pair of disks – if you managed to shoot it so it starts out as the periodic orbit bouncing along the line connecting two disk centers, it would stay there forever. Your game would be just as good if you managed to get it to keep bouncing between the three disks forever, or place it on any periodic trajectory. So it is pretty clear that if one is interested in playing well, periodic trajectories are important – they form the *skeleton* onto which all trajectories trapped for long times cling.

A trajectory is periodic if it returns to the starting position and momentum in phase space. We shall refer to the set of periodic points that belong to a given periodic orbit as a *cycle*. A bar over a finite block of symbols denotes a periodic itinerary with infinitely repeating basic block; we shall omit the bar whenever it is clear form the context that the trajectory is periodic. A *prime* cycle p of length n_p is a single traversal of the orbit; its label is a non-repeating symbol string of n_p symbols. For example, $\overline{12}$ is prime, but $\overline{2121}$ is not, since it is $\overline{21} = \overline{12}$ repeated.

exercise 1.2

1.3.3 Partitioning with periodic orbits

Physicist's pinball can bounce forever elastically, without losing energy. The position of the ball is described by a pair of numbers (the coordinates on the plane) and its velocity by another pair of numbers (the coordinates of the velocity vector). As far as Baron Leibnitz is concerned, this is a complete description. The motion of the pinball is then the motion of a point in a four-dimensional space.

Suppose that the pinball has just bounced off disk 1. Depending on its position and outgoing angle, it could proceed to either disk 2 or 3. Not much happens in between the bounces – the ball just travels at constant velocity along a straight

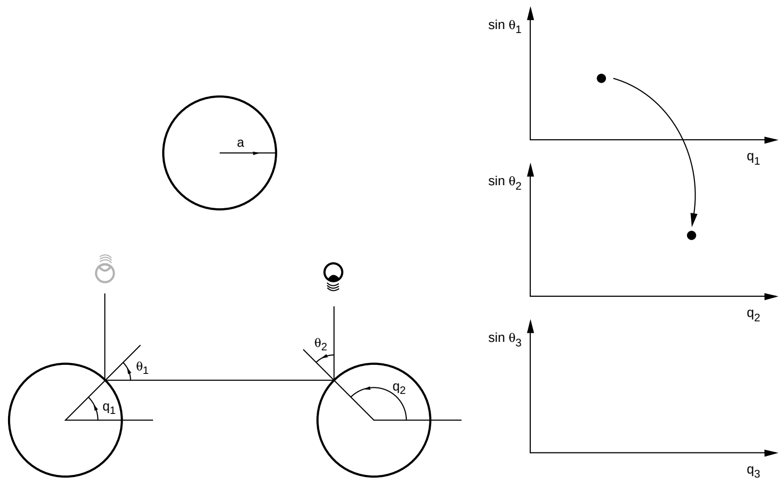
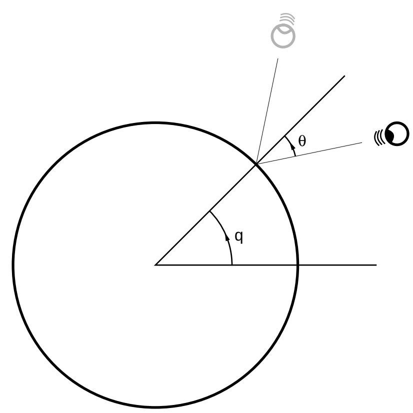


Figure 1.4: The 3-disk pinball coordinates and Poincaré sections.

line – so we can simplify the four-dimensional flow to a two-dimensional map M that takes the coordinates of the pinball from one disk edge to another disk edge.

A trajectory at the moment of impact can defined by marking q_i, the arc-length position of the ith bounce along the billiard wall, and $p_i = \sin\theta_i$, the momentum component parallel to the wall, fig. 1.4. Coordinates $x_i = (p_i, q_i)$ are a convenient choice, because they are phase-space volume preserving and easy to extract from the pinball trajectory. Such section of a flow is called a *Poincaré section* In terms of the Poincaré section, the dynamics is described by the map $M : (p_i, q_i) \mapsto (p_{i+1}, q_{i+1})$ from the boundary of a disk to the boundary of the next disk.



exercise 6.6

Figure 1.5: The coordinate system needed to describe a single bounce of a pinball off a disk. The coordinate q is given by an angle in $[0, 2\pi]$, and the momentum is given by specifying its component $\sin\theta$ tangential to the disk. For convenience, the pinball momentum and the disk radius a are customarily set equal to one.

In this way the phase space flow is reduced to an iterated boundary $\to$ boundary mapping; for billiards this is very natural, as all of the interesting dynamics is determined by the shape of the billiard boundary (details are given in sect. 4.6).

We next mark in the Poincaré section those initial conditions $\Omega_{.s_2}$ which do not escape in one bounce. There are two strips of survivors, as the trajectories originating from one disk can hit either of the other two disks, or escape.

We label the two strips with $s \in \{0, 1\}$. There are four strips $\Omega_{.s_1 s_2}$ that survive four bounces, and so forth, see fig. 1.7. Another way to look at the survivors after two bounces is to plot $\Omega_{s_1.s_2}$, the intersection of $\Omega_{.s_2}$ with the strips $\Omega_{s_1.}$ obtained by time reversal (the velocity changes sign $\sin\theta \to -\sin\theta$). $\Omega_{s_1.s_2}$ is a "rectangle" of nearby trajectories which have arrived from the disk

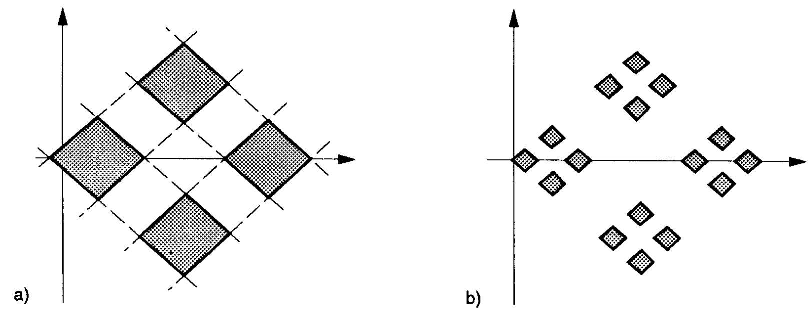


Figure 1.7: The Poincaré section of the phase space for the binary labelled pinball, see also fig. 2.5(b). Indicated are the fixed points $\overline{0}$, $\overline{1}$ and the 2-cycle periodic points $\overline{01}$, $\overline{10}$, together with strips which survive 1, 2, … bounces. Iteration corresponds to the decimal point shift; for example, all points in the rectangle [01.01] map into the rectangle [010.1] in one iteration.
PC: do this figure right!

s_1 and are heading for the disk s_2. Provided that the disks are sufficiently well separated, what emerges is a complete binary Cantor set with the Smale horseshoe foliation (Smale horseshoes are discussed in sect. 2.4.1).

After n bounces the survivors are divided into 2^n distinct strips: the ith strip consists of all points with itinerary $i = s_1 s_2 s_3 \ldots s_n$, $s = \{0, 1\}$. The unstable cycles as a skeleton of chaos are almost visible here: each such patch contains a periodic point $\overline{s_1 s_2 s_3 \ldots s_n}$ with the basic block infinitely repeated. Periodic points are skeletal in the sense that as we look further and further, the strips shrink but the periodic points stay put forever. The unstable periodic orbits are isolated and uniformly sprinkled out over the phase space, as the phase space is tessellated in strips of distinct n-step itineraries, and within each such strip there is a periodic point.

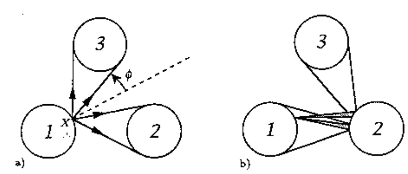


Figure 1.6: A 3-disk pinball consists of three equidistant disks in a plane, with radius $a = 1$ and center-to-center disk separation R. (a) A trajectory starting out from disk 1 can either hit another disk or escape. (b) Hitting two disks in a sequence requires a much sharper aim. The pencils of initial condidtions that hit more and more consecutive disks are nested within each other as in fig. 1.7.

There is a one-to-one relationship between the periodic itineraries and the unstable periodic trajectories: there exists a unique trajectory for every admissible infinite length itinerary, and a unique itinerary labels every trapped trajectory. For example, the only trajectory labeled by $\overline{12}$ is the 2-cycle bouncing along the line connecting the centers of disks 1 and 2; any other trajectory starting out as 12… either eventually escapes or hits the 3rd disk.

So the periodic points are dense on the asymptotic repeller, and their number increases exponentially with the cycle length (in case at hand, as As we shall see, this exponential proliferation of cycles is not as dangerous as it might seem; as a matter of fact, all our computations will be carried out in the $n \to \infty$ limit. Though a careful look at chaotic dynamics might reveal it complex beyond belief, it is still generated by a simple deterministic law, and with some luck and insight, our labeling of possible motions will reflect this simplicity. If the rule that gets us from one level of the classification hierarchy to the next does not depend strongly on the level, the resulting hierarchy is approximately self-similar. We now turn such approximate self-similarity to our advantage, by using it to determine the *weights* of the different pieces of the system.

1.3.4 Escape rate

What is a good physical quantity to compute for a pinball? A repeller *escape rate* is an eminently measurable quantity. An example of such measurement would be an unstable molecular or nuclear state which can be well approximated by a classical potential with possibility of escape in certain directions. The experimental measurement consists in injecting many projectiles into such non-confining potential and measuring their mean escape rate; for a theorist a good game of pinball consists in predicting accurately the asymptotic lifetime (or the escape rate) of a pinball. The thought experiment might consist of injecting the pinball between the disks in some random direction and asking how many times does the ball bounce before it escapes the region between the disks. We now show how the periodic orbit theory accomplishes this for us. exercise 1.3

Consider fig. 1.7 again. In each bounce the initial conditions get thinned out, yielding twice as many thin strips as at the previous bounce. The total area that remains will be the sum of the areas of the strips, so that the fraction of survivors after n bounces is proportional to

$$\hat{\Gamma}_n = \sum_{i=1}^{2^n} \mathcal{M}_i . \qquad (1.1)$$

where i is a binary label of the ith strip, and $\mathcal{M}_i$ is the area of the ith strip. As at each bounce one routinely loses about the same fraction of trajectories, one expects the sum (1.1) to fall off exponentially with n and tend to the limit

$$\hat{\Gamma}_n = e^{-n\gamma_n} \to e^{-n\gamma} . \qquad (1.2)$$

The quantity γ is called the *escape rate* from the repeller. We shall now show that this asymptotic escape rate γ can be extracted from a highly convergent *exact* expansion by reformulating the sum (1.1) in terms of unstable periodic orbits. If, when asked what the 3-disk escape rate is for radius 1, center-center separation 6, velocity 1, you estimate γ to be roughly $0.4103384077693464893384613078192\ldots$, you do not need this book. If you have no clue, hang on.

1.3.5 Size of a partition

Not only do the periodic points keep track of locations and the ordering of the strips, but they also determine their size. The phase space of a generic nonlinear dynamical system is an infinitely interwoven mixture of islands of stability and regions of chaos. As a trajectory evolves, it carries along and distorts its infinitesimal neighborhood. Possible trajectories are qualitatively of three distinct types: they are either asymptotically unstable (positive Lyapunov exponent), asymptotically marginal (vanishing Lyapunov) or asymptotically stable (negative Lyapunov). For an unstbale system such as the pinball, the trajectories that start out in an infinitesimal neighborhood are separated along the *unstable directions*, approach each other along the *stable directions*, and maintain their distance along the *neutral directions*. These directions and the corresponding rates of expansion/contraction are given by the eigenvectors and eigenvalues of the *Jacobian matrix* of the linearized flow around the cycle

$$\mathbf{J}_p := \mathbf{J}^{T_p}\,, \qquad J^t(x(0))_{ij} = \frac{\partial x_i(t)}{\partial x_j(0)}\,.$$

Evaluation of a cycle Jacobian matrix is a longish exercise that we shall go through in sect. 4.6.2 — here we just state the result: after one traversal of the cycle p the beam of neighboring trajectories is defocused in the unstable eigendirection by the factor Λ_p, the expanding eigenvalue of the 2-dimensional Jacobian matrix

$$\Lambda_\pm = \frac{1}{2}\left(\mathrm{tr}\,\mathbf{J}_p \pm \sqrt{(\mathrm{tr}\,\mathbf{J}_p)^2 - 4}\right)\,. \tag{1.3}$$

As the heights of the strips are effectively constant, we can concentrate on their thickness. So if their height is L, then the area of the ith strip is $\mathcal{M}_i = L l_i$ for a strip of width l_i. Each strip i in fig. 1.7 contains a periodic point x_i. The finer the intervals, the smaller is the variation in flow across them, and the strip width l_i is well approximated by the contraction around the periodic point,

$$l_i := a_i/|\Lambda_i|\,, \tag{1.4}$$

where Λ_i is the i-th periodic point expanding eigenvalue (1.3), and a_i is a prefactor which depends on the distribution of initial points (we shall put this estimate on a firm basis in chapter 7). To proceed with the derivation we need the *hyperbolicity* assumption: for large n the prefactors $a_i \approx O(1)$ are overwhelmed by the exponential growth of Λ_i, so we neglect them (see sect. 7.3.1 for a discussion). The prefactors a_i reflect a particular distribution of starting values of x; the asymptotic trajectories are strongly mixed by bouncing chaotically around the repeller and we expect them to be insensitive to smooth variations in the initial distribution. If the hyperbolicity assumption is justified, we can replace

$\mathcal{M}_i = Ll_i$ in (1.1) by $1/\Lambda_i$ and form a formal sum over all periodic orbits of all lengths:

$$
\begin{aligned}
, (z) &= \sum_{n=1}^{\infty} , _n z^n = \sum_{n=1}^{\infty} z^n \sum_i^{(n)} |\Lambda_i|^{-1} \\
&= \frac{z}{|\Lambda_0|} + \frac{z}{|\Lambda_1|} + \frac{z^2}{|\Lambda_{00}|} + \frac{z^2}{|\Lambda_{01}|} + \frac{z^2}{|\Lambda_{10}|} + \frac{z^2}{|\Lambda_{11}|} \\
&\quad + \frac{z^3}{|\Lambda_{000}|} + \frac{z^3}{|\Lambda_{001}|} + \frac{z^3}{|\Lambda_{010}|} + \frac{z^3}{|\Lambda_{100}|} + \ldots
\end{aligned}
\tag{1.5}
$$

Here we have omitted the overall prefactor L as it does not affect the exponent in (1.2) in the $n \to \infty$ limit. For sufficiently small z this sum is convergent (this is discussed in sect. 7.3.5). As for large n the nth level sum (1.1) tends to the limit $e^{-n\gamma}$, the escape rate γ is determined by the smallest $z = e^{\gamma}$ for which (1.5) diverges:

$$
, (z) \approx \sum_{n=1}^{\infty} \left(ze^{-\gamma}\right)^n = \frac{ze^{-\gamma}}{1 - ze^{-\gamma}}
\tag{1.6}
$$

This observation is what motivated the introduction of the sum (1.5) in the first place. Rather than attempting to extrapolate the escape rate from the finite n sums (1.1), we shall determine γ from the singularities of (1.5).

1.3.6 A dynamical zeta function

We could now proceed to estimate the location of the leading singularity of $, (z)$ by extrapolating finite truncations of (1.5) by methods such as Padé approximants. However, as we shall now show, it pays to first perform a simple resummation that converts this divergence into a *zero* of a related function.

If a trajectory retraces itself r times, its derivative is Λ_p^r, where p is a *prime* cycle. A prime cycle is a single traversal of the orbit; its label is a non-repeating symbol string. There is only one prime cycle for each cyclic permutation class. For example, $p = \overline{0011} = \overline{1001} = \overline{1100} = \overline{0110}$ is prime, but $\overline{0101} = \overline{01}$ is not. The stability of a cycle is by the chain rule for derivatives (see (4.45) below) the same everywhere along the orbit, so each prime cycle of length n_p contributes n_p terms to the sum (1.5). Hence (1.5) can be rewritten as

$$
, (z) = \sum_p n_p \sum_{r=1}^{\infty} \left(\frac{z^{n_p}}{|\Lambda_p|}\right)^r = \sum_p \frac{n_p t_p}{1 - t_p}, \qquad t_p = \frac{z^{n_p}}{|\Lambda_p|}
\tag{1.7}
$$

where the index p runs through all distinct *prime* cycles. Note that we have *resumed* the contribution of the cycle p to all times, so truncating the summation up to given p is *not* a finite time $n \leq n_p$ approximation, but an asymptotic, *infinite* time estimate based by approximating stabilites of all cycles by a finite number of shortest cycles. The $n_p z^{n_p}$ factors in the sum suggest rewriting it as a derivative

$$, (z) = -z \frac{d}{dz} \sum_p \ln(1 - t_p) \,.$$

Hence , (z) is a logarithmic derivative of the infinite product

$$1/\zeta(z) = \prod_p (1 - t_p) \,, \qquad t_p = \frac{z^{n_p}}{|\Lambda_p|} \,. \tag{1.8}$$

This function is called the *dynamical zeta function*, and is a prototype of types of formulas that the periodic orbit theory yields; the problem of estimating the asymptotic escape rates from finite n sums such as (1.1) is now reduced to a study of the zeros of the dynamical zeta function (1.8). The escape rate is related by (1.6) to a divergence of , (z), and , (z) diverges whenever $1/\zeta(z)$ or $\zeta(z)$ has a zero.

The critical step in the derivation of the dynamical zeta function was the hyperbolicity assumption, that is assumption of exponential shrinkage of all strips of the pinball repeller. By dropping the a_i prefactors in (1.4), we have given up on any possibility of recovering the precise distribution of starting x (which should anyhow be impossible due to the exponential growth of errors), but in exchange we gain an effective description of the asymptotic behavior of the system. The pleasant surprise implicit in (1.8) is that the infinite time behavior of an unstable system will be as easy to determine as the short time behavior.

1.3.7 Evolution operators

The above derivation of the dynamical zeta function formula for the escape rate has one shortcoming; it estimates the fraction of survivors as a function of the number of pinball bounces, but the physically interesting quantity is the escape rate measured in units of continuous time. For continuous time flows, the escape rate (1.1) is generalized as follows. Define a finite phase space volume V such that a trajectory that exits the V never reenters. For example, any pinball that falls of the edge of a pinball table is gone forever. The fraction of initial x whose trajectories remain within V at time t is is expected to decay exponentially

$$, (t) = \frac{\int_V dx dy \, \delta(y - f^t(x))}{\int_V dx} \;\to\; e^{-\gamma t} \,.$$

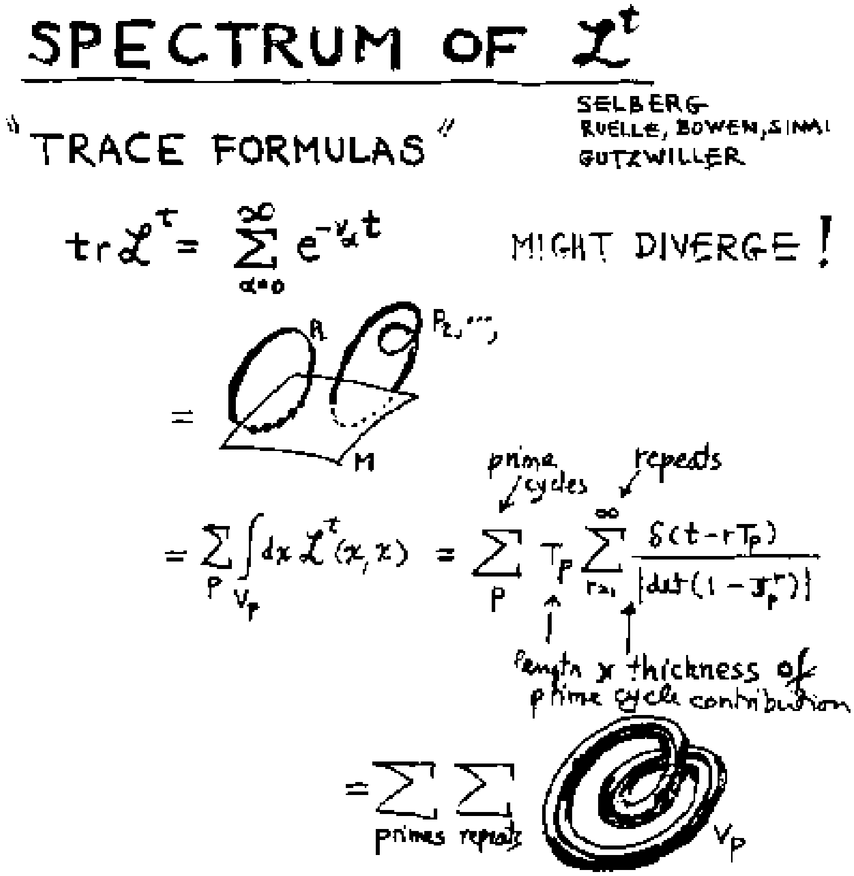


Figure 1.8: The trace of an evolution operator is concentrated in tubes around prime cycles.

The kernel of this integral motivates introduction of the *evolution operator* for a d-dimensional map or a d-dimensional flow

$$\mathcal{L}^t(x,y) = \delta\big(x - f^t(y)\big) \ ,$$

where $\delta(\cdots)$ is the Dirac delta function: for a deterministic flow the initial point y maps into a unique point x at time t. For discrete time, $f^n(x)$ is the n-th iterate of the map f; for continuous flows, $f^t(x)$ is the trajectory of the initial point x.

We shall show in sect. 7.3.4 that integration over the whole phase space yields an expression for $\mathrm{tr}\,\mathcal{L}^t$ as a sum over all prime cycles p and their repetitions

$$\mathrm{tr}\,\mathcal{L}^t = \sum_p T_p \sum_{r=1}^{\infty} \frac{\delta(t - rT_p)}{\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|} \ .$$

A geometrical interpretation of the Jacobian in this formula is that after the r-th return to a Poincaré section, the initial tube V_p has been stretched out along the expanding eigendirections, with the overlap with the initial volume given by $1/|\det(\mathbf{1}-\mathbf{J}_p^r)|$, see fig. 1.8.

A Laplace transform smoothes the above sum over Dirac delta functions in cycle periods and yields the *trace formula* for classical evolution operators:

$$\int_{0_+}^{\infty} dt\, e^{-st} \mathrm{tr}\, \mathcal{L}^t = \sum_p T_p \sum_r^{\infty} \frac{e^{-sT_p r}}{|\det\left(\mathbf{1}-\mathbf{J}_p^r\right)|} \,. \tag{1.9}$$

The beauty of the trace formulas (which we shall discuss in detail in sect. 7.3) lies in the fact that everything on the right-hand-side – prime cycles p, their periods T_p and the stability eigenvalues of $\mathbf{J}_p$ – is an invariant property of the flow, independent of any coordiante choices. A consideration of $\left|\det\left(\mathbf{1}-\mathbf{J}_p\right)\right|$ leads to the conclusion that for continuous time flows the correct weight is obtained by replacing the discrete "topological" time n_p in (1.8) by the cycle period T_p:

$$t_p = e^{-sT_p}/|\Lambda_p|. \tag{1.10}$$

1.3.8 A spectral determinant

exercise 1.5

The eigenvalues of a linear operator are given by the zeros of the appropriate secular determinant. One way to evaluate determinants is to expand them in terms of traces,

$$\log\det(1-\mathcal{L}) = \mathrm{tr}\,\log(1-\mathcal{L}) = \sum_{n=1} \frac{1}{n} \mathrm{tr}\, \mathcal{L}^n \,, \tag{1.11}$$

and in this way the *spectral determinant* of an evolution operator becomes related to its traces, *ie.* periodic orbits (see chapter 7):

$$F(s) = \exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{1}{r} \frac{e^{-sT_p r}}{|\det\left(\mathbf{1}-\mathbf{J}_p^r\right)|} \right) \,. \tag{1.12}$$

The motivation for recasting the eigenvalue problem in this form is sketched in fig. 1.9; exponentiation improves analitycity and promotes a divergence of the trace sum into a zero of a smooth function, the spectral determinant. The heuristically derived dynamical zeta function function (1.8) re-emerges in the process as the leading eigenvalue part of this *exact* spectral determinant. As we shall see in chapter 18, not only is the spectral determinant exact, but it is also preferable in actual calculations, as it has superior convergence properties (this is illustrated by table 10.2).

While various periodic orbit formulas may be formally equivalent, in practice some are vastly preferable to others. Trace formulas, such as the thermodynamic

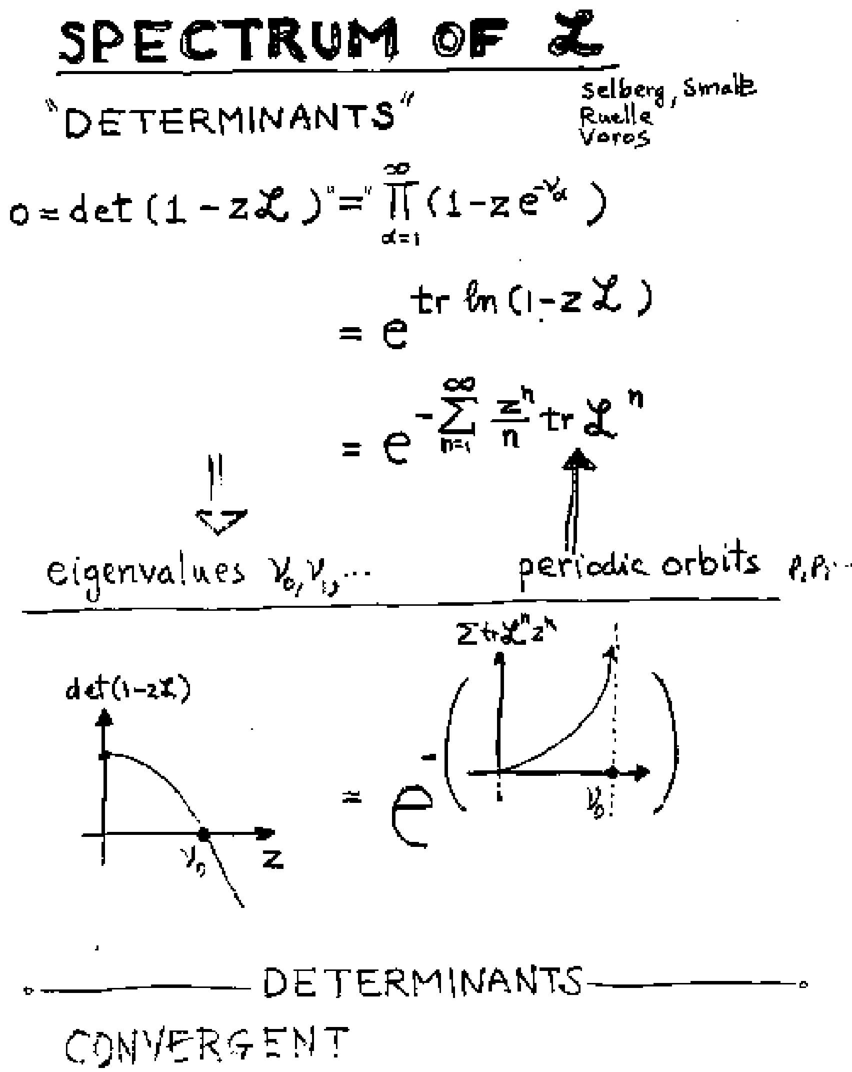


Figure 1.9: Spectral determinant vanishes smoothly at the first eigenvalue while the trace formula diverges.

averages in classical dynamics, and the semi-classical Gutzwiller trace formula in quantum mechanics are difficult to use for anything other than the leading eigenvalue estimates. However, dynamical zeta functions, spectral determinants and the semiclassical zeta function are powerful tools for evaluation of classical and quantum averages in low dimensional chaotic dynamical systems.

1.3.9 Cycle expansions

How are formulas such as (1.8) used? We start by computing the lengths and eigenvalues of the shortest cycles. This usually requires some numerical work, such as the Newton's method searches for periodic solutions; we shall assume that the numerics is under control, and that *all* short cycles up to given length have been found. We shall learn in chapter 5 how to do this for a variety of problems. In our pinball example this can be done by elementary geometrical optics. It is very important not to miss any short cycles, as the calculation is as accurate as the shortest cycle dropped – including cycles longer than the shortest omitted does not improve the accuracy (unless exponentially many mor cycles are included). The result is a list of cycles, their periods and their stabilities, like table 5.1.

Now we formally expand the infinite product (1.8), grouping together the terms of the same total symbol string length

$$\begin{aligned} 1/\zeta &= 1 - t_0 - t_1 - [t_{10} - t_1 t_0] - [(t_{100} - t_{10} t_0) + (t_{101} - t_{10} t_1)] \\ &\quad -[(t_{1000} - t_0 t_{100}) + (t_{1110} - t_1 t_{110}) \\ &\quad +(t_{1001} - t_1 t_{001} - t_{101} t_0 + t_{10} t_0 t_1)] - \ldots \end{aligned} \qquad (1.13)$$

We call the sum of all terms of the same total length n (grouped in brackets above) the nth *curvature correction* c_n, for geometrical reasons we shall explain in the next section and then again in sect. 18.1.2.

The calculation is now straightforward. We substitute the eigenvalues and lengths of prime cycles into the curvature expansion (1.13), and obtain a polynomial approximation to $1/\zeta$. We then vary z in (1.8) or s in (1.10), and determine the escape rate γ by finding the smallest $z = e^\gamma$ or $s = -\gamma$ for which (1.13) vanishes. The rapid convergence is illustrated – as an example – by the significant figures of γ computed from truncations of (1.13) to different maximal cycle lengths, table 10.2.

1.3.10 Shadowing

If you have some experience with numerical estimates of fractal dimensions, you will appreciate how very impressive the convergence such as indicated by table 10.2 is; only three input numbers (the two fixed points $\overline{0}$, $\overline{1}$ and the 2-cycle

$\overline{10}$) already yield the escape rate to 3-4 significant digits! We have omitted an infinity of unstable cycles; so why does approximating the dynamics by a finite number of cycle eigenvalues work so well?

The convergence of cycle expansions of dynamical zeta functions and spectral determinants is a consequence of the smoothness and analyticity of the flows they are constructed from; particularly strong results exist for Axiom A hyperbolic systems, for which the dynamical zeta functions are meromorphic, and the spectral determinants are entire functions. Intuitively, one can understand why these functions should be convergent in terms of the geometrical picture presented in fig. 1.10; the key observation is that the long orbits are *shadowed* by sequences of shorter orbits.

A typical curvature expansion term in (1.13) is a *difference* of a long cycle $\{ab\}$ minus its shadowing approximation by shorter cycles $\{a\}$ and $\{b\}$:

$$t_{ab} - t_a t_b = t_{ab}(1 - t_a t_b / t_{ab}) = t_{ab}\left(1 - \left|\frac{\Lambda_{ab}}{\Lambda_a \Lambda_b}\right| e^{(T_a + T_b - T_{ab})s}\right) \tag{1.14}$$

If all orbits are weighted equally ($t_p = z^{n_p}$), such combinations cancel exactly; if orbits of similar symbolic dynamics have similar weights, the weights in such combinations almost cancel. To understand why such combinations should be small compared to t_{ab}, try to visualize the partition of a chaotic dynamical system's phase space in terms of cycle neighborhoods as a tessellation of the dynamical system, with smooth flow approximated by its periodic orbit skeleton, each "face" centered on a periodic point, and the scale of the "face" determined by the linearization of the flow around the periodic point, fig. 1.10.

The orbits that follow the same symbolic dynamics, such as $\{ab\}$ and a "pseudo orbit" $\{a\}\{b\}$, lie close to each other in the phase space; long shadowing pairs have to start out exponentially close to beat the exponentially growth in separation with time. If the weights associated with the orbits are multiplicative along the flow (for example, the chain-rule products of derivatives) and the flow is smooth, the term in parenthesis in (1.14) falls off *exponentially* with the cycle length, and therefore the curvature expansions are expected to be highly convergent. We shall justify the exponential error estimate in chapter 18. More amazingly, we shall learn that for nice hyperbolic flows the cycle expansion truncation errors can be *superexponentially* small.

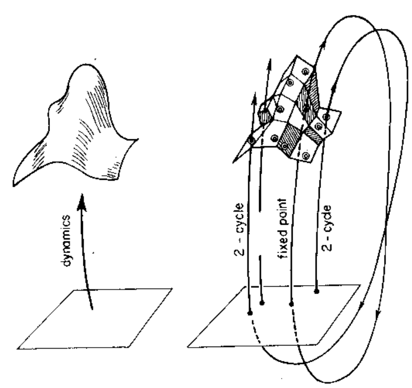


Figure 1.10: Approximation to (a) a smooth dynamics by (b) the skeleton of periodic points, together with their linearized neighborhoods.

1.4 From chaos to statistical mechanics

While the above replacement of dynamics of individual trajectories by evolution operators which propagate densities might feel like just another bit of mathematical physics voodoo, actually something very radical has taken place. Consider a chaotic flow, such as stirring of red and white paint by some deterministic machine. *If* we were able to track individual trajectories, the fluid would forever remain a striated combination of pure white and pure red; there would be no pink. What is more, if we reversed stirring, we would return back to the perfect white/red separation. However, we know that this cannot be true – in a very few turns of the stirring stick the thickness of the layers goes from centimeters to Ångströms, and the result is irreversibly pink.

Understanding the distinction between evolution of individual trajectories and the evolution of the densities of trajectories is key to understanding statistical mechanics – this is the conceptual basis of the second law of thermodynamics, and the origin of irreversibility of the arrow of time for deterministic systems with time-reversible equations of motion: reversibility is attainable for distributions whose measure in the space of density functions goes exponentially to zero with time. While individual trajectories are sensitive to noise, the asymptotic density eigenfunctions are robust.

By going to a description in terms of the asymptotic time evolution operators we give up tracking individual trajectories for long times, but instead gain a very effective description of the asymptotic trajectory densities. This will enable us, for example, to give exact formulas for transport coefficients such as the diffusion constants (see chapter 13) without *any* probabilistic assumptions. The bold claim is that once you understand this, classical ergodicity, wave mechanics and stochastic mechanics will be at your feet, special cases to be worked out at your leisure.

1.5 Quantum pinball

So far, so good – anyone can play a game of classical pinball. But what happens quantum mechanically, that is if we scatter waves rather than pointlike pinballs? Were the pinball a closed system, quantum mechanically one would determine its stationary eigenfunctions and eigenenergies. For open systems one determines instead complex resonances, where the imaginary part of the eigenenergy describes the rate at which the quantum wavefunction leaks out of the central multiple scattering region. One of the pleasant surprises in the development of the theory of chaotic dynamical systems was the discovery that the zeros of dynamical zeta function (1.8) also yield excellent estimates of *quantum* resonances, with the quantum amplitude associated with a given cycle approximated semiclassically by (in an appropriately vague sense) the square root of the classical weight (1.10)

$$t_p = \frac{1}{\sqrt{|\Lambda_p|}} e^{\frac{i}{\hbar}S_p - i\pi m_p/2} , \tag{1.15}$$

with phase given by the Bohr-Sommerfeld action integral S_p, together with an additional geometrical Maslov phase m_p. m_p counts the number of points on the periodic trajectory where the naive semiclassical approximation fails us (see chapter 14).

1.5.1 Quantization of helium

Now we are finally in position to accomplish something altogether remarkable; we shall put together all ingredients that made the pinball unpredictable, and compute a "chaotic" part of the helium spectrum to a shocking accuracy. Poincaré taught us that from the classical dynamics point of view, Helium is an example of the intractable dreaded 3-body problem. Undounted, we forge ahead innocently by considering the *collinear* helium, with zero total angular momentum, and the two electrons on the opposite sides of the nucleus. We set the electron mass to 1, and the nucleus mass to ∞. In these units the helium nucleus has charge 2, the electrons have charge -1, and the Hamiltonian is

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - \frac{2}{r_1} - \frac{2}{r_2} + \frac{1}{r_1 + r_2} . \tag{1.16}$$

The collinear helium has only 3 degrees of freedom and the dynamics can be visualized as a motion in the (r_1, r_2), $r_i \geq 0$ quadrant, fig. 1.11.

The motion in the (r_1, r_2) plane is topologically similar to the pinball motion in a 3-disk system, except that the motion is not free, but in Coulomb potential.

Miraculously, the symbolic dynamics again turns out to be binary, just as in the 3-disk pinball, so we know what cycles need to be computed for the cycle expansion (1.13). This arises because the classical collinear helium is also a repeller; almost all of the classical trajectories escape. A set of shortest cycles up to a given symbol string length then yields an estimate of the helium spectrum; we shall carry this program out in chapter 16. A typical set of the shortes cycles is drawn in fig. 1.11, and a typical comparison of the exact quantum and the cycle expansion eigenenergies is given in table 1.1. What should surprise you is that even though the cycle expansion was based on the *semiclassical approximation* (1.15), which is expected to be good only in the classical large energy limit, the eigenenergies are good to 1% all the way down to the ground state.

Remark **1.3** <u>If this book is not rigorous enough...</u> This text aims to bridge the gap between the physics and mathematics dynamical systems literature.

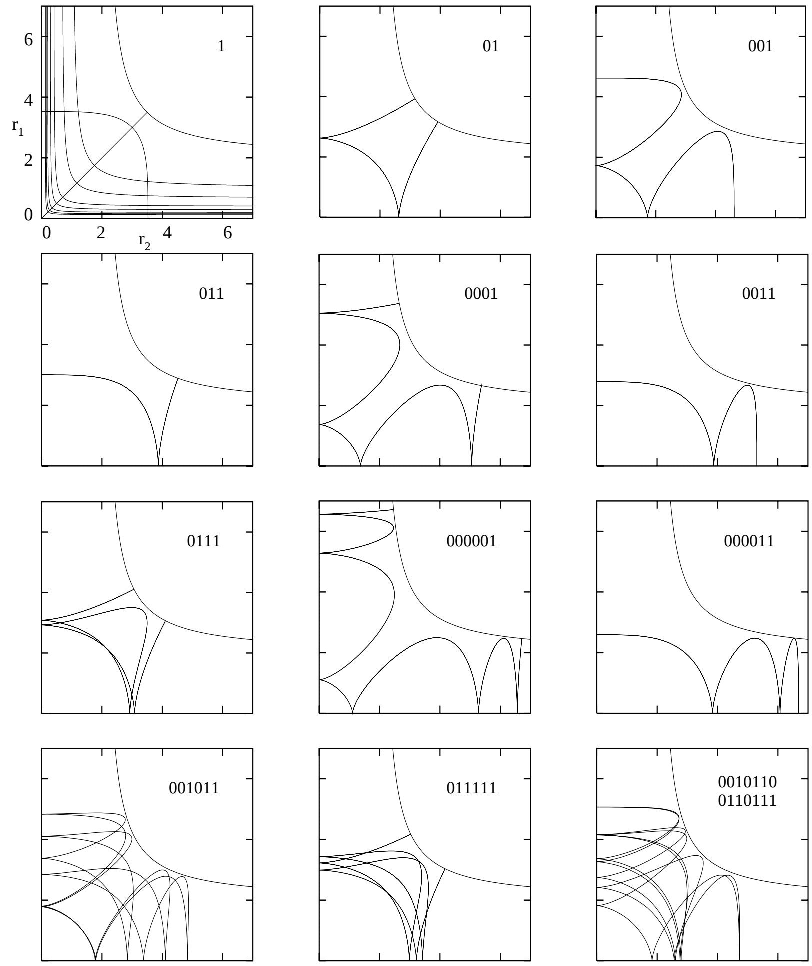


Figure 1.11: Some of the shortest cycles for the collinear helium. The classical collinear electron motion is bounded by the potential barrier $-1 = -2/r_1 - 2/r_2 + 1/(r_1 + r_2)$ and the nucleus $(r_i = 0)$. (Courtesy of Gregor Tanner)

l	QM	cycles
0	2.903721	2.92825
1	2.145974	2.13562
2	2.061272	2.05923
3	2.033587	2.03288
4	2.021177	2.02085
5	2.014563	2.01439
6	2.010626	2.01052
7	2.008094	2.00802
8	2.006370	2.00632
⋮	⋮	⋮
∞	2.0	2.0

Table 1.1: Exact quantum *vs.* cycle expansions energies for the parahelium bound state $(L, S = 0)$ series. See chapter 16 for details. (Courtesy of Gregor Tanner)

The intended audience is the ideal graduate student with a theoretical bent. As an alternative presentation we recommend P. Gaspard's monograph [1.21] which covers much of the same ground in a highly readable and scholarly manner.

This book does not discuss the random matrix theory approach to the "quantum chaology"; no randomness assumptions are made here, rather the goal is to milk the deterministic chaotic dynamics for its full worth. The book concentrates on the periodic orbit theory. The role of unstable periodic orbits was already fully appreciated by Poincaré [1.18], who noted that hidden in the apparent chaos is a rigid skeleton, a tree of *cycles* (periodic orbits) of increasing lengths and self-similar structure, and suggested that the cycles should be the key to chaotic dynamics. Periodic orbits have been at core of much of the mathematical work on the theory of the classical and quantum dynamical systems ever since. We refer the reader to the reprint selection [1.20] for an overview of some of that literature.

The fundamental papers in this field, all still valuable reading, are Smale [1.27], Bowen [1.28] and Sinai [7.25]. Sinai's paper is prescient and offers a vision and a program that ties together dynamical systems and statistical mechanics. It is written for readers versed in statistical mechanics. For a dynamical systems exposition, consult Anosov and Sinai[?]. Markov partitions were introduced by Sinai in ref. [1.10]. The classical text (though certainly not an easy read) on the subject of dynamical zeta functions is Ruelle's 1978 *Statistical Mechanics, Thermodynamic Formalism* [1.22]. In Ruelle's monograph transfer operator technique (or the "Perron-Frobenius theory") and Smale's theory of hyperbolic flows are applied to zeta functions and correlation functions. The hyperbolic case is treated, and the essential spectrum discussed. The Grothendieck theory and Fredholm determinants were introduced in Ruelle's two 1989 papers [1.23, 1.24]. The status of the theory from Ruelle's point of view is compactly summarized in his 1995 Pisa lectures [1.26]. Further excellent mathematical references on thermodynamic formalism are Parry and Pollicott's monograph [9.6] with emphasis on the symbolic dynamics aspects of the formalism, and Baladi's clear and compact review of dynamical zeta functions [1.31].

The most readable introduction to cycle expansions (other than this book) is given in the *Nonlinearity* articles [1.35, 1.36].

The role of "chaos" in statistical mechanics is dissected by Bricmont in his highly readable essay *"Science of Chaos or Chaos in Science?"* [1.37].

Introductions to "quantum chaos" are given in from Gutzwiller's [16.9] and Reichl's textbooks [14.4].

Remark **1.4** <u>Why study pinballs?</u> The 3-disk pinball is to chaotic dynamics what the pendulum is to integrable systems; the simplest physical example that captures the essence of "hard" chaos. Another contender for the title of the "harmonic oscillator of chaos" is the baker's map which is used as the red thread through Ott's introduction to chaotic dynamics [1.38]. Baker's map, is the simplest Hamiltonian dynamical system which is hyperbolic and has positive Kolmogorov entropy. Regrettably, due to its piecewise linearity the baker's map is so nongeneric that it misses all of the curvature corrections structure of cycle expansions of dynamical zeta functions that are central to this treatise (and discussed here in chapter 10).

> A pinball game does miss a number of important aspects of chaotic dynamics: generic bifurcations in smooth flows, the interplay between regions of stability and regions of chaos, intermittency phenomena, and the renormalization theory of the "border of order" between these regions. For this we shall have to turn to dynamics in smooth potentials and smooth dissipative flows.

Resumé

The goal of this text is an exposition of the best of all possible theories of deterministic chaos, and the strategy is: 1) count, 2) weigh, 3) add up.

A motion on a strange attractor can be approximated by shadowing the orbit by a sequence of nearby periodic orbits of finite length. The theory presented here is based on the observation that the motion in dynamical systems of few degrees of freedom is often organized around a few *fundamental* cycles. These short cycles capture the skeletal topology of the motion in the sense that any long orbit can approximately be pieced together from the fundamental cycles. This notion is here made precise by approximating orbits by primitive cycles, and evaluating associated curvatures. A curvature measures the deviation of a longer cycle from its approximation by shorter cycles; the smoothness of the dynamical system implies exponential (or faster) fall-off for (almost) all curvatures. The technical prerequisite for implementing this shadowing is a good understanding of the symbolic dynamics of the classical dynamical system. The resulting cycle expansions offer an efficient method for evaluating classical and quantum periodic orbit sums; accurate estimates can be obtained by using as input the lengths and eigenvalues of a few prime cycles.

To keep exposition simple we have here illustrated the utility of cycles and their curvatures by a pinball game, but the remainder of this book should give the reader some confidence in a general applicability of the periodic orbit theory. The formalism should work for any average over any chaotic set which satisfies two conditions:

1. the weight associated with the observable under consideration is multiplicative along the trajectory

2. the set is organized in such a way that the nearby points in the symbolic dynamics have nearby weights.

The theory is applicable to evaluation of a broad class of averages such as the Lyapunov exponents, transport coefficients, or those used in the extraction of generalized dimensions. One of the surprises is that the quantum mechanics of classically chaotic systems is very much like the classical mechanics of chaotic systems; one needs nearly the same zeta functions and cycle expansions, with the same group theory factorizations and dependence on the topology of the classical flow. Each application requires determination of the physically correct cycle weight t_p – the rest proceeds as outlined above.

Guide to exercises

God can afford to make mistakes. So can Dada!

Dadaist Manifesto

The essence of this subject is incommunicable in print; the only way to develop intuition about chaotic dynamics is by computing, and the reader is urged to try to work through the essential exercises. Not to fragment the text too much, the exercises are indicated by text margin boxes such as the one on this margin, and collected at the end of each chapter. The problems that you should do have **underlined titles**. *The rest (smaller type, in italics) are optional.* Difficult optional problems are marked by any number of *** stars. By the end of the course you should have completed at least three projects: (a) compute everything for a 1-dimensional repeller, (b) compute escape rate for a 3-disk pinball, (c) compute a part of the quantum 3-disk or the helium spectrum. The essential steps are: exercise 10.2

- **Dynamics**

  1. count prime cycles, exercise 1.1, exercise 2.1, exercise 2.4
  2. pinball simulator, exercise 1.4, exercise 5.13
  3. pinball stability, exercise 5.10, exercise 5.13
  4. pinball periodic orbits, exercise 5.14, exercise 5.15
  5. helium integrator, exercise 4.11, exercise 5.16
  6. helium periodic orbits, exercise 16.1, exercise 5.18

- **Averaging, numerical**

  1. pinball escape rate, exercise 9.15
  2. Lyapunov exponent, exercise 12.4, or pressure, exercise 11.4 or exercise 11.7.

- **Averaging, periodic orbits**

  1. cycle expansions, exercise 10.1, exercise 10.2
  2. pinball escape rate, exercise 10.5, exercise 10.6
  3. cycle expansions for averages, exercise 10.1, exercise 11.1
  4. cycle expansions for diffusion, exercise 13.1
  5. pruning, Markov graphs, exercise 3.4, exercise **??**
  6. desymmetrization exercise 17.1

7. intermittency, phase transitions exercise 23.5
8. ortho-, para-helium, lowest eigenenergies ?!

If you happen to generate a nice postscript figure illustrating a problem, let us include the figure into these notes. Nothing seems to be more time consuming than generating figures.

References

[1.1] A. Einstein, *Verh. Deutsch. Phys. Ges.* **19**, 82 (1917).

[1.2] M. Born, *Vorlesungen über Atommechanik* (Springer, Berlin 1925). English translation: *The Mechanics of the Atom*, (F. Ungar Publishing Co., New York 1927).

[1.3] M. Jammer, *The Conceptual Development of Quantum mechanics* (McGraw-Hill, New York 1966).

[1.4] J. G. Leopold and I. Percival, *J. Phys. B*, **13**, 1037 (1980).

[1.5] J. Mehra and H. Rechtenberg, *The Historical Development of the Quantum Theory* (Springer, New York 1982).

[1.6] A. Pais, *Inward Bound: of Matter and Forces in the Physical World* (Oxford Univ. Press, Oxford 1986).

[1.7] A. Pais, *Niels Bohr's Times, in Physics, Philosophy and Polity* (Oxford Univ. Press, Oxford 1991).

[1.8] D. Wintgen, K. Richter and G. Tanner, " The semiclassical helium atom", *CHAOS* **2**, 19 (1992).

[1.9] J.W.S. Rayleigh, *The Theory of Sound* (Macmillan, London 1896; reprinted by Dover, New York 1945)

[1.10] Ya.G. Sinai, *"Construction of Markov partitions"*, *Funkts. Analiz i Ego Pril.* **2**, 70 (1968). English translation: *Functional Anal. Appl.* **2**, 245(1968).

[1.11] R. Bowen, *Markov partitions for Axiom A diffeomorphisms"*, *Amer. J. Math.* **92**, 725 (1970).

[1.12] D. Ruelle, *Transactions of the A.M.S.* **185**, 237 (197?).

[1.13] N. Metropolis, M.L. Stein and P.R. Stein, *On Finite Limit Sets for Transformations on the Unit Interval*, *J. Comb. Theo.* **A15**, 25 (1973).

[1.14] J. Milnor and W. Thurston, *"On iterated maps of the interval"*, in A. Dold and B. Eckmann, eds., *Dynamical Systems, Proceedings, U. of Maryland 1986-87*, *Lec. Notes in Math.* **1342,** 465 (Springer, Berlin, 1988).

[1.15] W. Thurston, *"On the geometry and dynamics of diffeomorphisms of surfaces"*, *Bull. Amer. Math. Soc. (N.S.)* **19**, 417 (1988).

[1.16] P. Cvitanović, G.H. Gunaratne and I. Procaccia, *Phys. Rev.* **A 38**, 1503 (1988).

[1.17] T. Li and J. Yorke, *"Period 3 implies chaos"*, *Amer. Math. Monthly* **82**, 985 (1975).

[1.18] H. Poincaré, *Les méthodes nouvelles de la méchanique céleste* (Guthier-Villars, Paris 1892-99)

[1.19] For a very readable exposition of Poincaré's work and the development of the dynamical systems theory up to 1920's see J. Barrow-Green, *Poincaré and the Tree Body Problem*, (Amer. Math. Soc., Providence R.I., 1997).

[1.20] R.S. MacKay and J.D. Miess, *Hamiltonian Dynamical Systems* (Adam Hilger, Bristol 1987)

[1.21] P. Gaspard, *Chaos, Scattering and Statistical Mechanics* (Cambridge Univ. Press, Cambridge 1997).

[1.22] D. Ruelle, *Statistical Mechanics, Thermodynamic Formalism*, (Addison-Wesley, Reading MA, 1978).

[1.23] D. Ruelle, *"The thermodynamical formalism for expanding maps"*, *Commun. Math. Phys.* **125**, 239(1989).

[1.24] D. Ruelle, *"An extension of the theory of Fredholm determinants"*, *Inst. Hautes Études Sci. Publ. Math.* **72**, 175-193 (1990).

[1.25] D. Ruelle, *Dynamical Zeta Functions for Piecewise Monotone Maps of the Interval*, (Amer. Math. Soc., Providence, NJ 1994)

[1.26] D. Ruelle, *"Functional determinants related to dynamical systems and the thermodynamic formalism*, preprint IHES/P/95/30 (March 1995).

[1.27] S. Smale, *Differentiable Dynamical Systems*, *Bull. Am. Math. Soc.* **73**, 747 (1967).

[1.28] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Springer Lecture Notes in Math. **470** (1975).

[1.29] P. Cvitanović, *"Invariant measurements of strange sets in terms of cycles"*, *Phys. Rev. Lett.* **61**, 2729 (1988).

[1.30] W. Parry and M. Pollicott, *Zeta Functions and the periodic Structure of Hyperbolic Dynamics*, *Astérisque* **187–188** (Société Mathématique de France, Paris 1990).

[1.31] V. Baladi, *"Dynamical zeta functions"*, in *Real and Complex Dynamical Systems*, B. Branner and P. Hjorth, ed. (Kluwer, Dordrecht, 1995).

[1.32] *Universality in Chaos, 2. edition*, P. Cvitanović, ed., (Adam Hilger, Bristol 1989).

[1.33] M.J. Feigenbaum, *J. Stat. Phys.* **52**, 527 (1988), reprinted in ref. [1.32]; lectures in ref. [1.34].

[1.34] P. Zweifel, G. Gallavotti and M. Anile, eds., *Non-linear Evolution and Chaotic Phenomena* (Plenum, New York 1987).

[1.35] R. Artuso, E. Aurell and P. Cvitanović, "Recycling of strange sets I: Cycle expansions", *Nonlinearity* **3**, 325 (1990).

[1.36] R. Artuso, E. Aurell and P. Cvitanović, "Recycling of strange sets II: Applications", *Nonlinearity* **3**, 361 (1990).

[1.37] J. Bricmont, *"Science of Chaos or Chaos in Science?"*, available on `mp_arc@fireant.ma.utexas.edu` preprint server, #96-116.

[1.38] E. Ott, *Chaos in Dynamical Systems* (Cambridge, 1993)