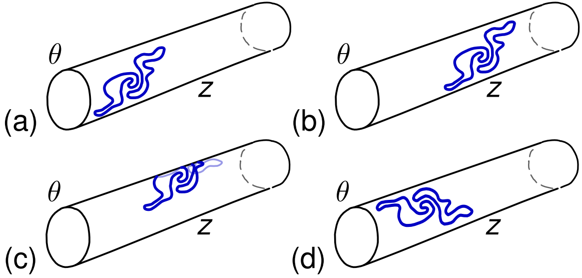
## ChaosBook.org chapter

# relativity for cyclists

10 February 2022, version 17.5.5

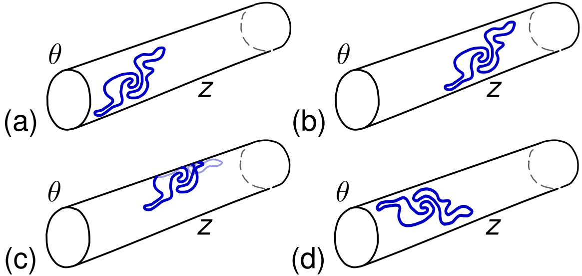
# Das Problem : dancers and drifters




pipe flow

(a) instantaneous global state of a fluid (marked by a ‘swirl’)

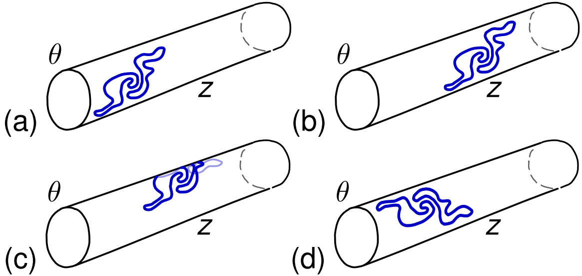
# Das Problem : dancers and drifters




symmetry : a pipe flow solution
translated or rotated or reflected is also a solution

(b)   the state translated by downstream shift $d$ (fluid states are $\mathrm{SO}(2)_z$ equivariant in a stream-wise periodic pipe),

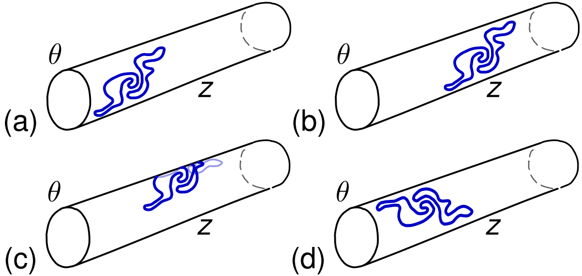
# Das Problem : dancers and drifters




symmetry : a pipe flow solution
translated or rotated or reflected is also a solution

(c) the state translated by *d* and rotated azimuthal by $\phi$ (the two states are $\mathrm{SO}(2)_\theta \times \mathrm{SO}(2)_z$ equivariant)

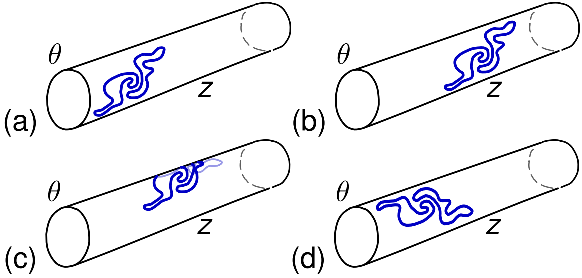
# Das Problem : dancers and drifters




symmetry : a pipe flow solution
translated or rotated or reflected is also a solution

(d) the state reflected and rotated azimuthally by $\phi$ (the two states are $\mathrm{O}(2)_\theta$ equivariant).

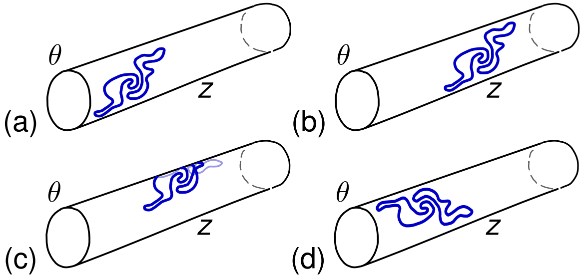
# Das Problem : dancers and drifters




states may also be symmetry-related by time evolution

relative equilibrium : solution that retains its shape while rotating and traveling downstream with constant $c$.

# Das Problem : dancers and drifters




states may also be symmetry-related by time evolution

relative periodic orbit : $\mathcal{M}_p$ a *time dependent*, shape-changing state of the fluid that after a period $T_p$ reemerges as (b), (c), or (d), the initial state translated by $d_p$, rotated by $\phi_p$ and possibly also azimuthally reflected

# Das Problem : don't be stupid

with a continuous symmetry,

there are families of $\infty$-many equivalent states

you do not want to compute the same solution over and over, do you?

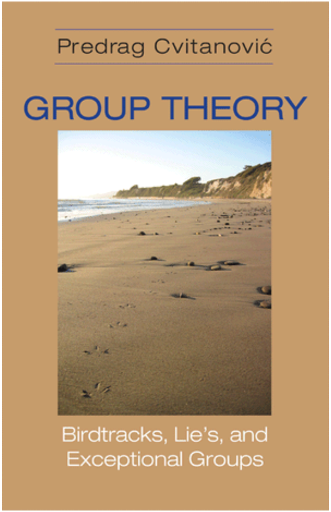
so, you **must** reduce any continuous symmetry

## Happy families are all alike;
## every unhappy family is unhappy in its own way

everybody, her mother,
and Robert MacKay knows how to do this

except the author of

# masters of group theory



# Das Problem : a 5-dimensional drifting attractor

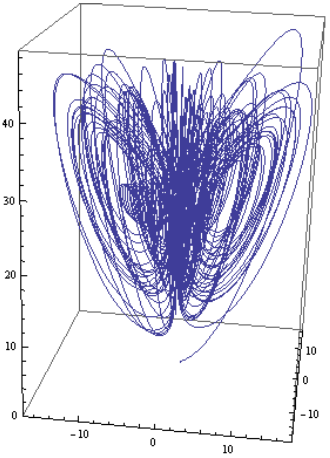
## complex Lorenz equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\sigma x_1 + \sigma y_1 \\ -\sigma x_2 + \sigma y_2 \\ (\rho_1 - z)x_1 - \rho_2 x_2 - y_1 - e y_2 \\ \rho_2 x_1 + (\rho_1 - z)x_2 + e y_1 - y_2 \\ -bz + x_1 y_1 + x_2 y_2 \end{bmatrix}$$

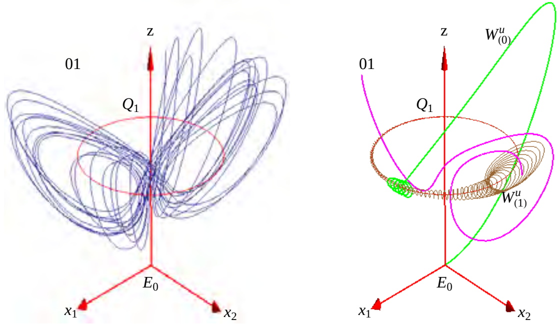
$\rho_1 = 28, \rho_2 = 0, b = 8/3, \sigma = 10, e = 1/10$

- A typical $\{x_1, x_2, z\}$ trajectory
- superimposed: a trajectory whose initial point is close to the relative equilibrium $Q_1$

## attractor



## continuous symmetry induces drifts




- generic chaotic trajectory (blue)
- $E_0$ equilibrium
- $E_0$ unstable manifold - a cone of such (green)
- $Q_1$ relative equilibrium (red)
- $Q_1$ unstable manifold, one for each point on $Q_1$ (brown)
- relative periodic orbit $\overline{01}$ (purple)

# die Lösung

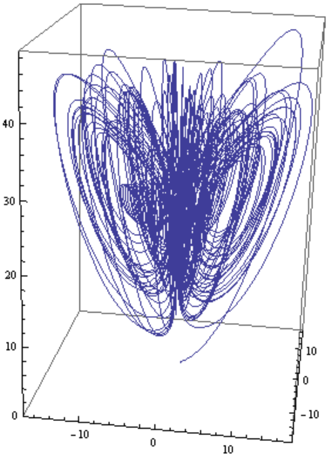
### what to do?

it's a mess

### the goal

reduce this messy strange attractor to something simple

### attractor



# die Lösung

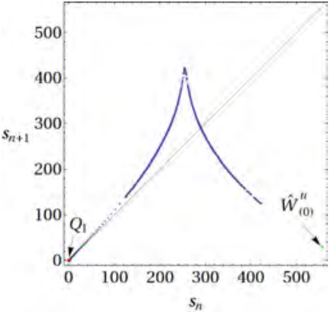
## the goal attained

started in five dimensions : reduced it to one (!)

## but it will cost you

must learn how to reduce (quotient) the $\mathrm{SO}(2)$ symmetry

## 1D return map!



# die Lösung

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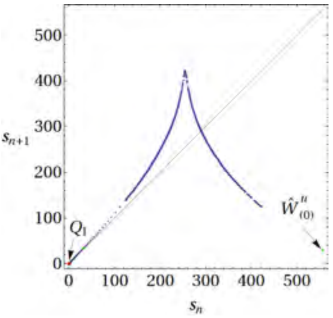



how? hang on, that's what we'll explain here

## symmetries of dynamics

**a flow $\dot{x} = v(x)$ is $G$-equivariant if**

$$v(x) = g^{-1}\, v(g\, x)\,, \qquad \text{for all } g \in G\,.$$

**definition: Lie group**

a topological group $G$ such that
(1) $G$ has the structure of a smooth differential manifold
(2) composition map $G \times G \to G : (g, h) \to gh^{-1}$ is smooth

mystified?

just think "aha, like the rotation group $\mathrm{SO}(3)$..."

# example: $\mathrm{SO}(2)$ invariance

## complex Lorenz equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\sigma x_1 + \sigma y_1 \\ -\sigma x_2 + \sigma y_2 \\ (\rho_1 - z)x_1 - \rho_2 x_2 - y_1 - e y_2 \\ \rho_2 x_1 + (\rho_1 - z)x_2 + e y_1 - y_2 \\ -bz + x_1 y_1 + x_2 y_2 \end{bmatrix}$$

invariant under a $\mathrm{SO}(2)$ rotation by finite angle $\phi$:

$$g(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi & 0 \\ 0 & 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

# example: abelian group $\mathrm{SO}(2)$

$\mathrm{SO}(2)$ : rotations in a plane

reflection $(x, y) \to (-x, y)$ excluded $(\det g = -1)$

if the group $G$ actions consists of two such rotations which commute, the group $G$ is an Abelian group that sweeps out a $T^2$ torus

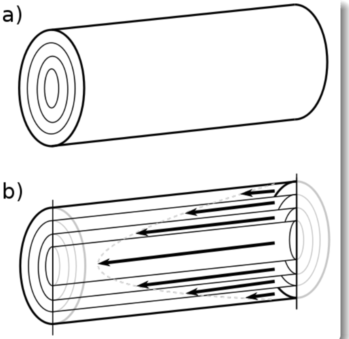
# example: continuous symmetries of pipe flow

**pipe flow**

- periodic streamwise, spanwise
- eqs. under azimuthal flip invariant

**a)** $SO(2)_z \times O(2)_\theta$ symmetry

**b)** laminar sol. is invariant



## group orbits

for any $x \in \mathcal{M}$, the group orbit $\mathcal{M}_x$ of $x$ is the set of all group actions

$$\mathcal{M}_x = \{g\,x \mid g \in G\} \subset \mathcal{M}$$

states in $\mathcal{M}_x$ are physically equivalent

# example: group orbit of a pipe flow relative equilibrium

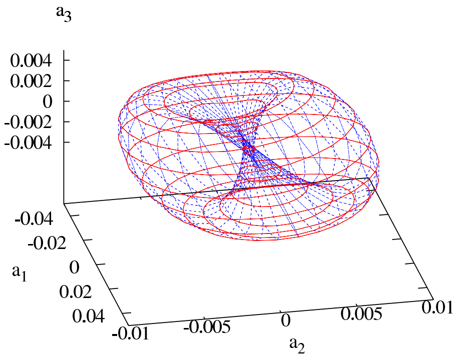
**$\hat{x}'$ = Kerswell *et al* $N2\_M1$ solution, (*Re* $= 2400$, stubby $L = 2.5D$ pipe)**

a very smooth, almost laminar solution

**$\mathrm{SO}(2) \times \mathrm{SO}(2)$ symmetry**
**$\Rightarrow$ group orbit is 2-torus**

projected on

- 2 $\hat{x}'$ group tangents
- 3. axis along the curvature direction




2*d* group orbit (in 100,000 dimension state space) traced out by

- equal increment translations in $\theta$ (dashed blue)
- equal increments in $z$ (solid red)

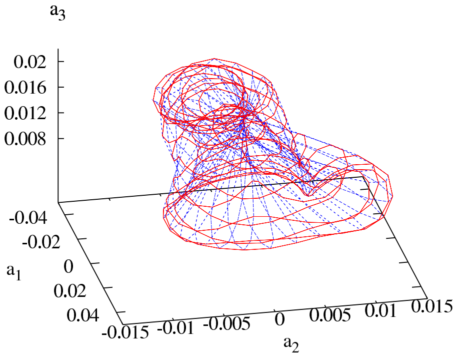
# example: group orbit of a pipe flow turbulent state

$\hat{x}'$ **is Kerswell** ***et al*** $N2\_M1$ **relative equilibrium**

**(** $\mathit{Re} = 2400$**, stubby** $L = 2.5D$ **pipe)**

a turbulent state

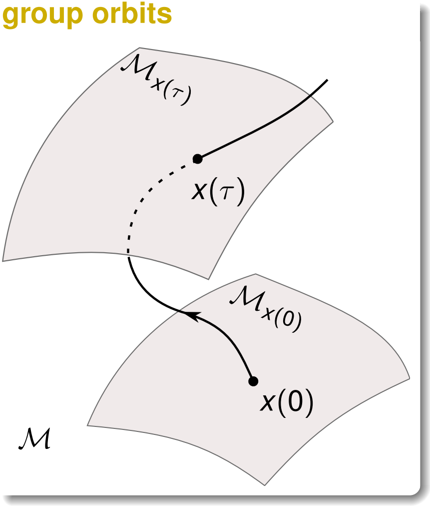
**$\mathrm{SO}(2) \times \mathrm{SO}(2)$ symmetry**
**$\Rightarrow$ group orbit is 2-torus**




group orbits of nonlinear states are highly contorted

## foliation by group orbits

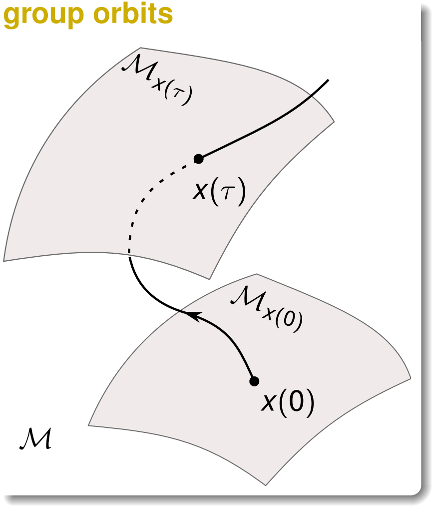
### group orbits




*group orbit* $\mathcal{M}_x$ of $x$ is the set of all group actions

$$\mathcal{M}_x = \{g\, x \mid g \in G\}$$

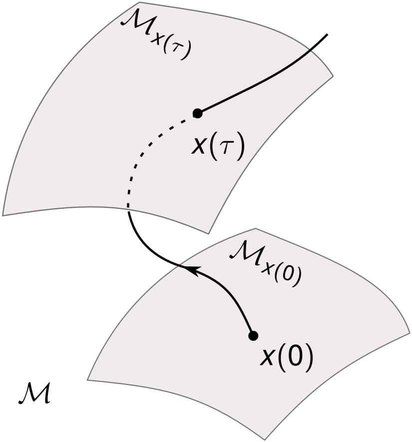
## foliation by group orbits




any point on the manifold $\mathcal{M}_{x(t)}$ is equivalent to any other

# foliation by group orbits

## group orbits




action of a symmetry group foliates the state space into a union of group orbits

each group orbit an equivalence class

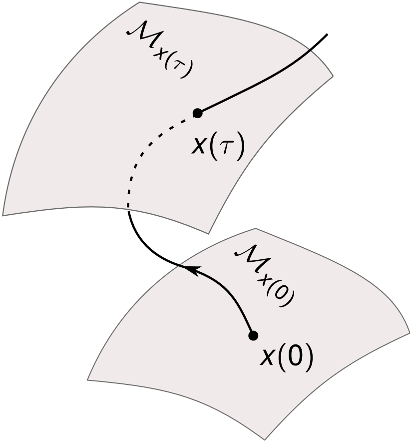
**the goal**

replace each group orbit by a unique point in a lower-dimensional

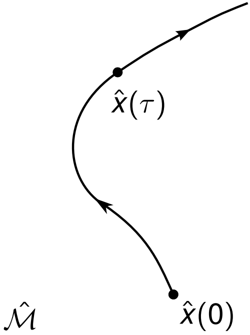
symmetry reduced state space $\mathcal{M}/G$

## symmetry reduction

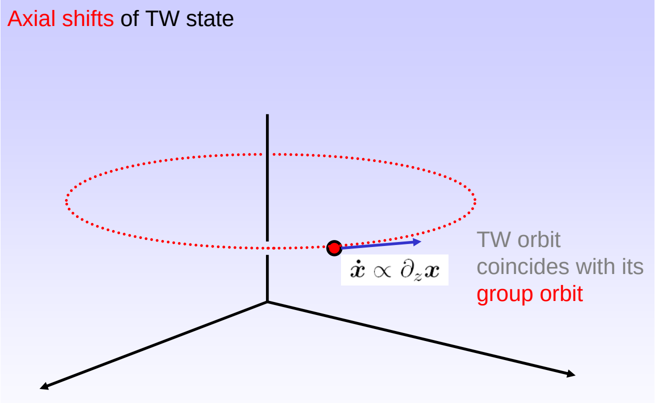
### full state space




### reduced state space



# pedestrian attempt : relative equilibrium or ‘traveling wave’

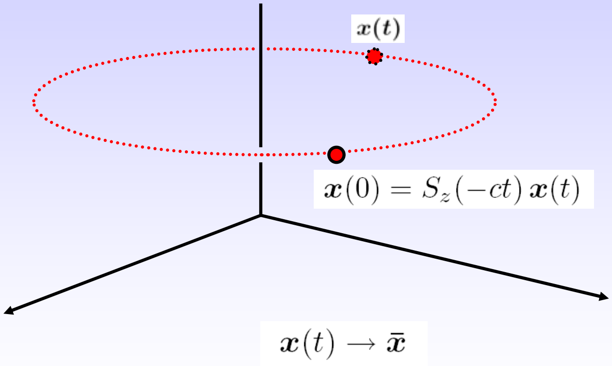



dynamical orbit confined to the group orbit

$$g(\tau)\, x(0) = x(\tau) \in \mathcal{M}_{TW}$$

# pedestrian* attempt : moving frame

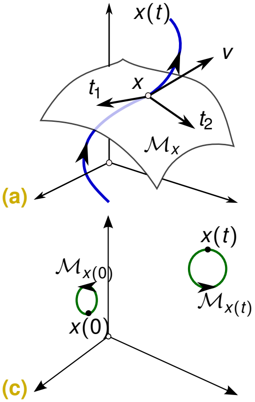
Reduction of TW orbit to point by shifts




relative equilibrium is made stationary by
a counter-rotating ‘frame’

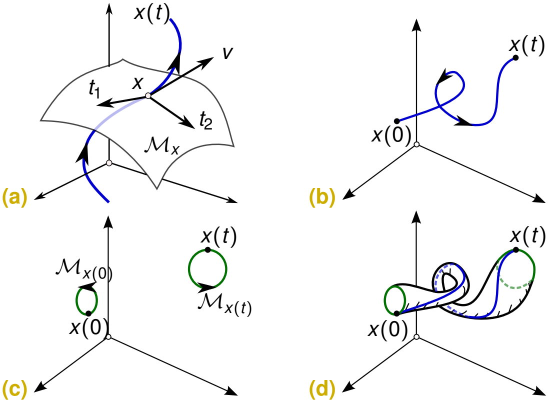
* ‘pedestrian’ = polite word for ‘applied mathematician’

# symmetries of dynamics



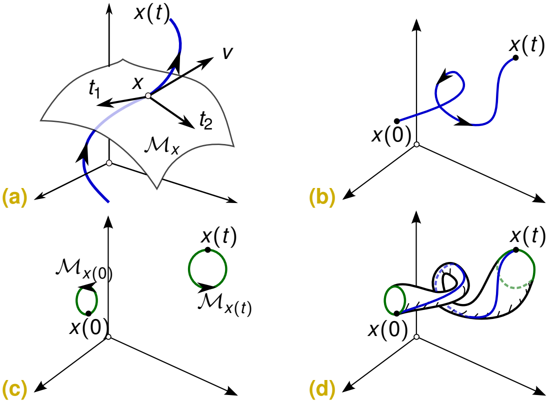

(a) $N$-continuous parameters symmetry : each state space point $x$ owns $(N+1)$ tangent vectors: $v(x)$ along the time flow $x(t)$ and the $N$ group tangents $t_1(x)$, $t_2(x)$, $\cdots$ , $t_N(x)$ along space, tangent to the $N$-dimensional group orbit $\mathcal{M}_x$

# symmetries of dynamics



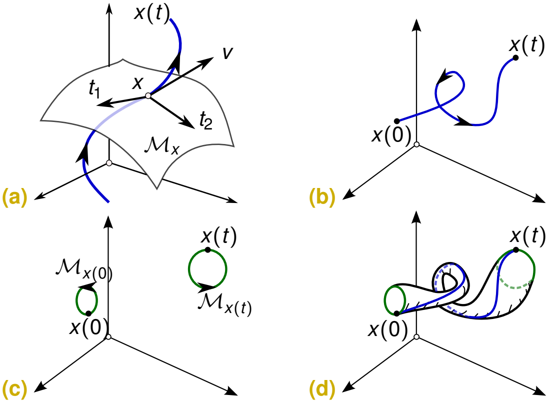

(b) each point has a trajectory (blue) under time evolution

# symmetries of dynamics




(c) each point has a group orbit (green) of symmetry-related states. For $\mathrm{SO}(2)$, this is topologically a circle. Any two points on a group orbit are physically equivalent, but may lie far from each other in state space

## symmetries of dynamics




(d) together, time-evolution and group actions trace out a wurst of physically equivalent solutions

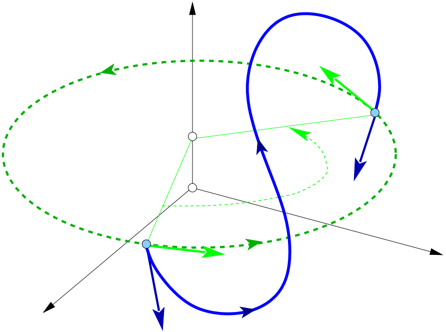
# relative periodic orbit

A relative periodic orbit $p$ is an orbit in state space $\mathcal{M}$ which exactly recurs

$$x_p(t) = g_p x_p(t + T_p)\,, \qquad x_p(t) \in \mathcal{M}_p$$

for a fixed relative period $T_p$ and a fixed group action $g_p \in G$ that “rotates" the endpoint $x_p(T_p)$ back into the initial point $x_p(0)$

# relative periodic orbit : state space visualization



(green dashes) group orbit
(blue) relative periodic orbit orbit

(arrows) velocity, group tangents

each cycle point

$$x_p(0) = g_p x_p(T_p)$$

exactly recurs at a fixed

**relative period**

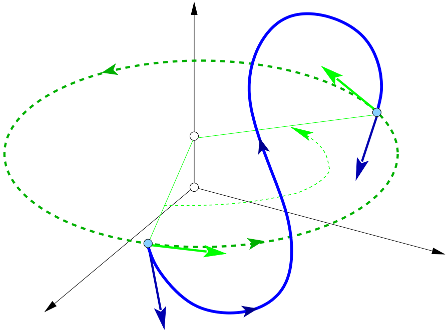
$$T_p$$

but shifted by a fixed

**group action**

$$g_p$$

## relative periodic orbit : state space visualization



group action parameters $\phi = (\phi_1, \phi_2, \cdots \phi_N)$ are irrational:

trajectory sweeps out ergodically the group orbit without ever closing into a periodic orbit

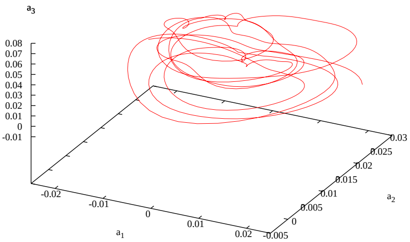
## example : pipe flow relative periodic orbit $\overline{rpo}_{36.72}$

symmetry reduction: full state space trajectory $x(t)$

$$\Rightarrow$$

reduced state space trajectory $\hat{x}(t)$, continuous group induced drifts quotiented out

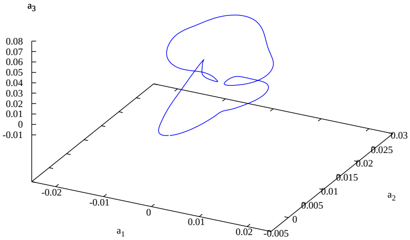
### full state space




traced for two periods:

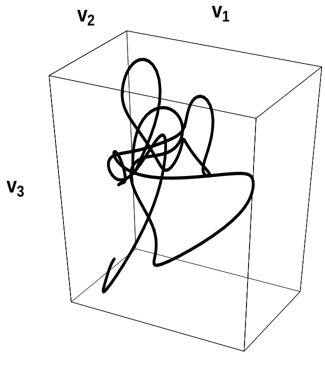
fills quasi-periodically a highly contorted 2-torus

### reduced state space




closes a periodic orbit in one period

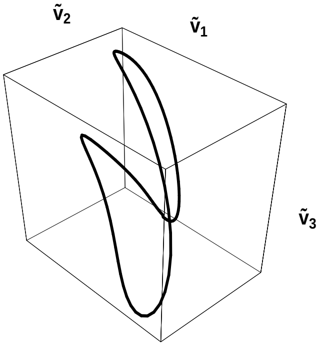
## try a co-moving coordinate frame?




a relative periodic orbit of the Kuramoto-Sivashinsky flow, 128$d$ state space traced for four periods $T_p$, projected on

a stationary state space coordinate frame $\{v_1, v_2, v_3\}$; a mess

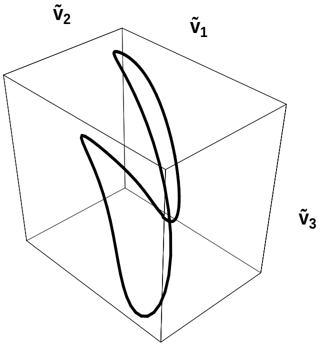
## try a co-moving coordinate frame?




a relative periodic orbit of the Kuramoto-Sivashinsky flow projected on

a co-moving $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ frame

## no good global co-moving frame!




(*b*)

beautiful, but this is no symmetry reduction at all;

all other relative periodic orbits require their own frames, moving at different velocities!

### method of moving frames / slices

cut group orbits by a hypersurface (spatial analogue of time Poincaré section), so

each group orbit of symmetry-equivalent points represented by the single point

cut how?

## inspiration: pattern recognition

you are observing turbulence in a pipe flow, or your defibrillator has a mesh of sensors measuring electrical currents that cross your heart, and

you have a precomputed pattern, and are sifting through the data set of observed patterns for something like it

here you see a pattern, and there you see a pattern that seems much like the first one

how ‘much like the first one?’

take the first pattern

**‘template’ or ‘reference state’**

a point $\hat{x}'$ in the state space $\mathcal{M}$

and use the symmetries of the flow to

**slide and rotate the ‘template’**

act with elements of the symmetry group $G$ on $\hat{x}' \to g(\phi)\,\hat{x}'$

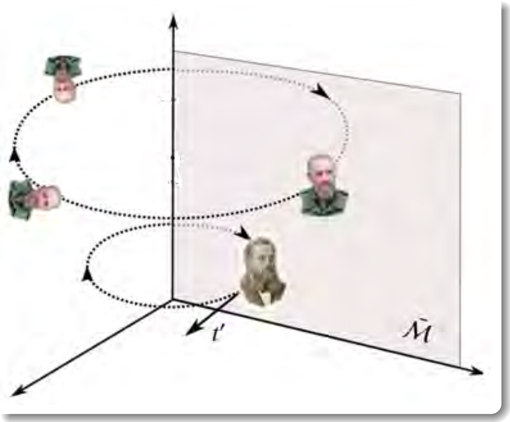
until it overlies the second pattern (a point $x$ in the state space)

**distance between the two patterns**

$$|x - g(\phi)\,\hat{x}'| = |\hat{x} - \hat{x}'|$$

is minimized

# idea: the closest match




template: Sophus Lie

(1) rotate bearded guy $x$ traces out the group orbit $\mathcal{M}_x$

(2) replace the group orbit by the closest match $\hat{x}$ to the template pattern $\hat{x}'$

the closest matches $\hat{x}$ lie in the $(d-N)$ symmetry reduced state space $\hat{\mathcal{M}}$

## distance

assume that $G$ is a subgroup of the group of orthogonal transformations $\mathrm{O}(d)$, and measure distance $|x|^2 = \langle x|x\rangle$ in terms of the Euclidean inner product

numerical fluids: PDE discretization independent L2 distance is

**energy norm**

$$\|\mathbf{u}-\mathbf{v}\|^2 = \langle \mathbf{u}-\mathbf{v}|\mathbf{u}-\mathbf{v}\rangle = \frac{1}{V}\int_\Omega d\mathbf{x}\,(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})$$

experimental fluid:

**image discretization independent distance**

is Hamming distance, or ???

**minimal distance**

is a solution to the extremum conditions

$$\frac{\partial}{\partial \phi_a} |x - g(\phi)\, \hat{x}'|^2$$

but what is

$$\frac{\partial}{\partial \phi_a} g(\phi)\,?$$

## Lie algebras for pedestrians

an element of a compact Lie group:

$$g(\phi) \propto e^{\phi\cdot\mathbf{T}}\,, \qquad \phi\cdot\mathbf{T} = \sum \phi_a \mathbf{T}_a,\ a = 1, 2, \cdots, N$$

$\phi\cdot\mathbf{T}$: *Lie algebra* element
$\phi_a$: parameters of the transformation.

### infinitesimal transformations

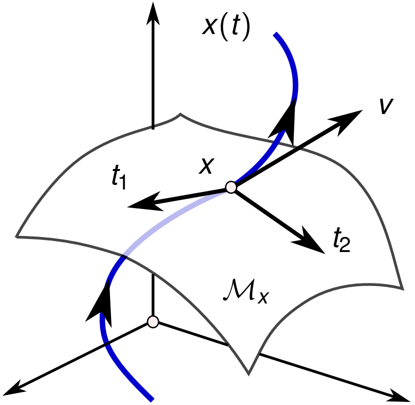
$$g = e^{\delta\phi\cdot\mathbf{T}} \simeq 1 + \phi\cdot\mathbf{T}\,, \qquad |\delta\phi| \ll 1$$

### Lie algebra

- $T_a$ are generators of infinitesimal transformations
- here $T_a$ are $[d \times d]$ antisymmetric matrices
- $T_a$ are elements of the Lie algebra of $G$

## symmetries of dynamics

each state space point $x$ has




time tangent vector $v(x)$ along the time flow $x(t)$

## symmetries of dynamics

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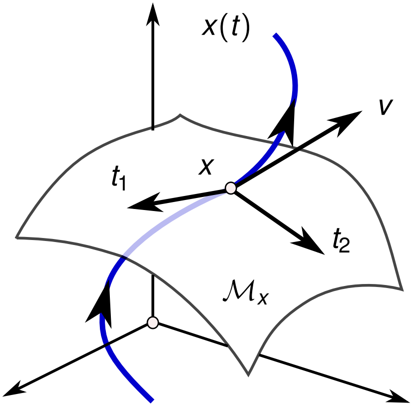



group tangent vectors $t_1(x), t_2(x), \cdots, t_N(x)$ along the $N$-dimensional space group orbit $\mathcal{M}_x$

## example: $\mathrm{SO}(2)$ invariance of complex Lorenz equations

complex Lorenz equations equations are invariant under $\mathrm{SO}(2)$ rotation by finite angle $\phi$:

$$
g(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi & 0 \\ 0 & 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

$\mathrm{SO}(2)$ Lie algebra has one generator of infinitesimal rotations

$$
\mathbf{T} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

## now have the 'slice condition'

### group tangent fields

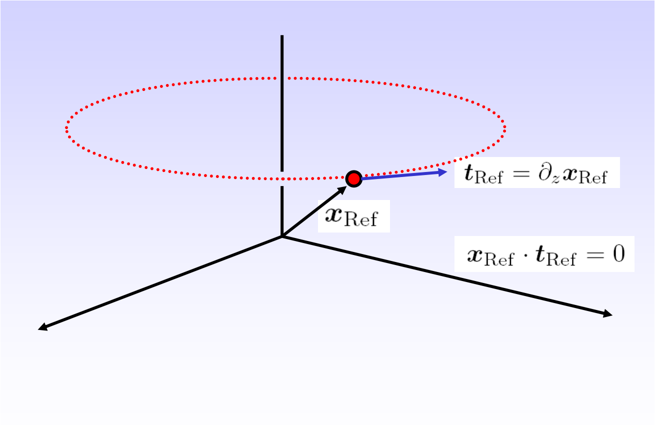
flow field at the state space point $x$ induced by the action of the group is given by the set of $N$ *tangent fields*

$$t_a(x)_i = (\mathbf{T}_a)_{ij} x_j$$

### slice condition

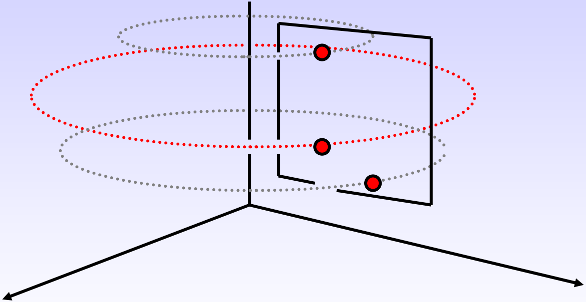
$$\frac{\partial}{\partial \phi_a} |x - g(\phi)\, \hat{x}'|^2 = 2\, \langle \hat{x} - \hat{x}' | t'_a \rangle = 0\,, \qquad t'_a = \mathbf{T}_a \hat{x}'$$

# traveling wave



# traveling wave

Reduce all TWs into a single slice



$$\boldsymbol{x}_i(0) = S_z(-c_i\, t)\, \boldsymbol{x}_i(t)$$

How? - several speeds *c*, possibly unknown

## flow within the slice

slice fixed by $\hat{x}'$

**reduced state space $\hat{\mathcal{M}}$ flow $\hat{v}(\hat{x})$**

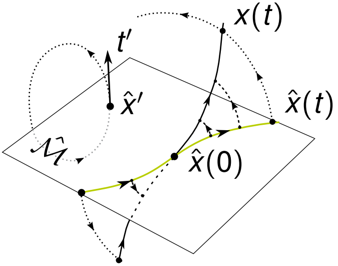
$$\begin{aligned} \hat{v}(\hat{x}) &= v(\hat{x}) - \dot{\phi}(\hat{x}) \cdot t(\hat{x}) \,, \qquad \hat{x} \in \hat{\mathcal{M}} \\ \dot{\phi}_a(\hat{x}) &= (v(\hat{x})^T t'_a)/(t(\hat{x})^T \cdot t') \,. \end{aligned}$$

- $v$ : velocity, full space
- $\hat{v}$ : velocity component in slice
- $\dot{\phi} \cdot t$ : velocity component normal to slice
- $\dot{\phi}$ : reconstruction equation for the group phases

**Cartan derivative**

$$g^{-1}\dot{g}\, x = e^{-\phi\cdot\mathbf{T}} \frac{d}{d\tau} e^{\phi\cdot\mathbf{T}} x = \dot{\phi} \cdot t(x)$$

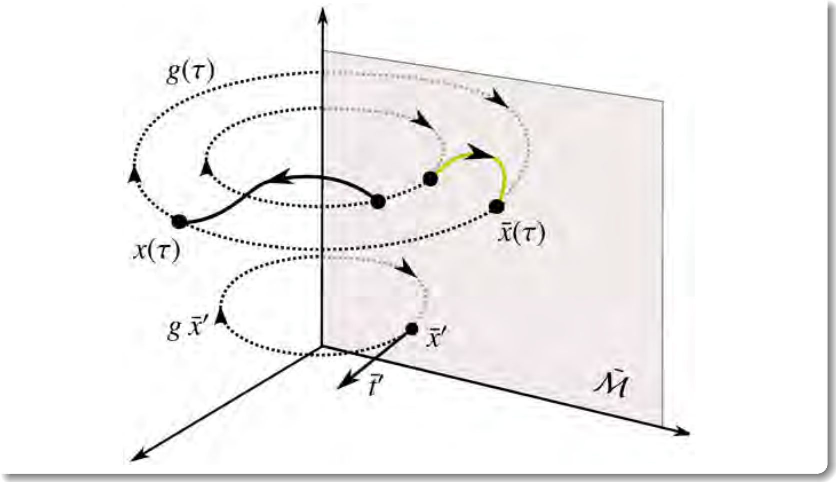
## flow within the slice




slice hyperplane $\hat{\mathcal{M}}$ through the template point $\hat{x}'$, normal to its group tangent $t'$, intersects all group orbits (dotted lines) in a neighborhood of $\hat{x}'$

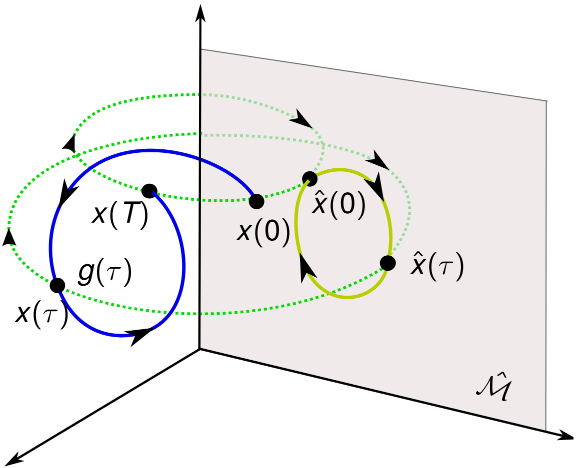
state space trajectory point $x(t)$ (solid black line) and the reduced state space trajectory $\hat{x}(t)$ (solid green line) belong to the same group orbit $\mathcal{M}_{x(t)}$, and are equivalent up to a moving frame group rotation $g(t)$

## flow within the slice



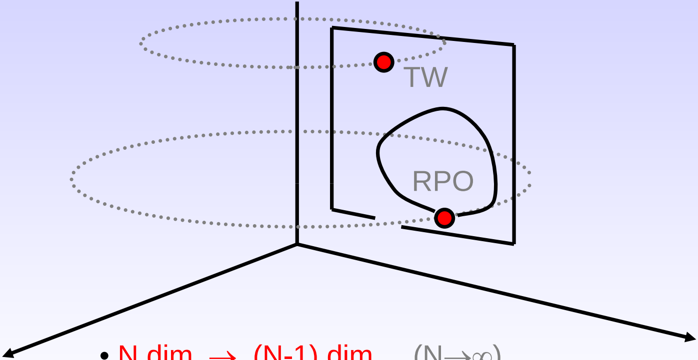

full-space trajectory $x(\tau)$
rotated into the reduced state space $\hat{x}(\tau) = g(\phi)^{-1}x(\tau)$
by appropriate *moving frame* angles $\{\phi(\tau)\}$

## relative periodic orbit→ periodic orbit



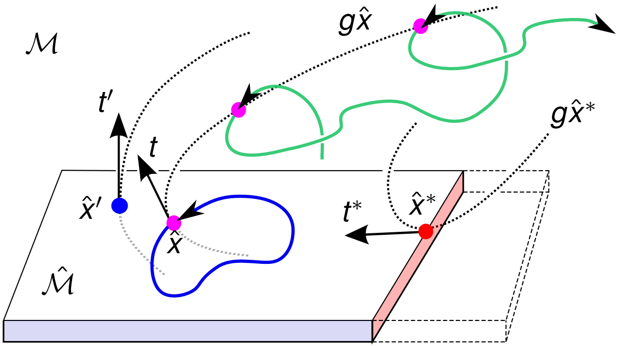

full state space relative periodic orbit $x(\tau)$
is rotated into the reduced state space periodic orbit

# relative equilibria and relative periodic orbits together



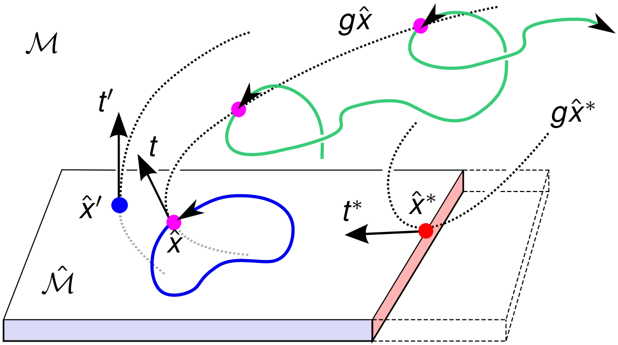

- N dim $\rightarrow$ (N-1) dim    (N$\rightarrow\infty$)
- Automatic removal of strong shift (gives c for TW)
- TW $\rightarrow$ point
- RPO $\rightarrow$ PO

## symmetry reduction by the method of slices




blue point : the template $\hat{x}'$

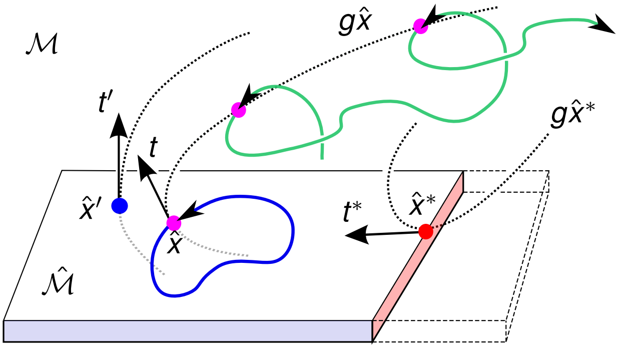
# symmetry reduction by the method of slices




pink points : equivalent to $\hat{x}$ up to a shift, so

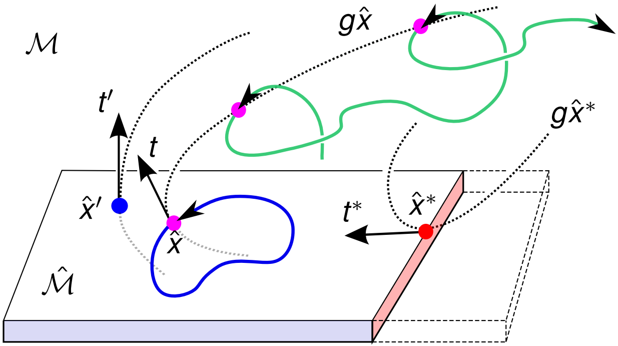
a relative periodic orbit (green) in the $d$-dimensional full state space $\mathcal{M}$ closes into a periodic orbit (blue) in the slice $\hat{\mathcal{M}}$

# symmetry reduction by the method of slices




slice $\hat{\mathcal{M}} = \mathcal{M}/G$ : a $(d-1)$-dimensional slab transverse to the template group tangent $t'$

## symmetry reduction by the method of slices



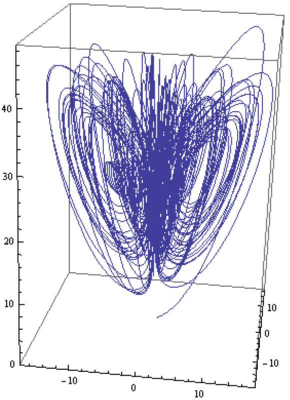

typical group orbit (dotted) crosses the slice hyperplane transversally, with group tangent $t = t(\hat{x})$

## symmetry reduction achieved!

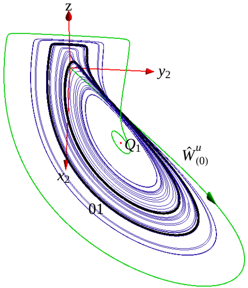
- all points equivalent by symmetries are represented by
  - a single point
- families of solutions are mapped to a single solution
  - relative equilibria become equilibria
  - relative periodic orbits become periodic orbits

# die Lösung : complex Lorenz flow reduced

**full state space**



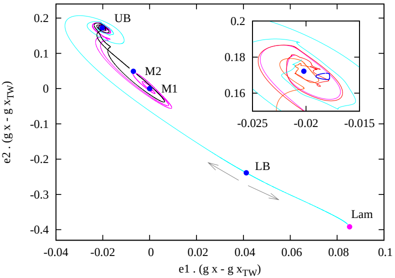

**reduced state space**



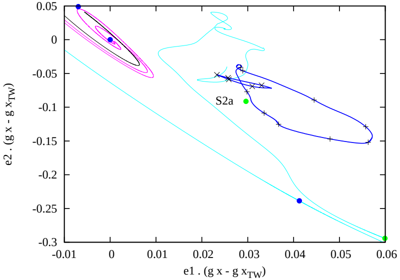

ergodic trajectory was a mess, now the topology is reveled
relative periodic orbit $\overline{01}$ now a periodic orbit

# triumph : all pipe flow solution in one happy family

a typical turbulent state $\hat{x}'$ breaks all symmetries
plot relative equilibria and unstable manifolds



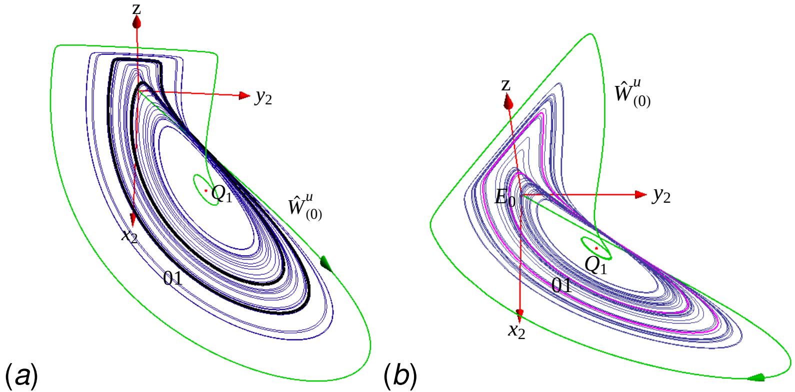

all in the same projection
inset: an expanded view
blue loop: $T = 4.93$
relative periodic orbit




$T = 10.96$ and $T = 36.92$
relative periodic orbits
embedded in turbulence

first ‘turbulent’ relative periodic orbits for pipe flows!

## portrait of complex Lorenz flow in reduced state space




any choices of the slice $\hat{x}'$ exhibit flow discontinuities

## take-home message

rotation into a slice is not an average over 3D pipe azimuthal angle

it is the full snapshot of the flow embedded in the

$\infty$-dimensional state space

NO information is lost by symmetry reduction

- not modeling by a few degrees of freedom
- no dimensional reduction

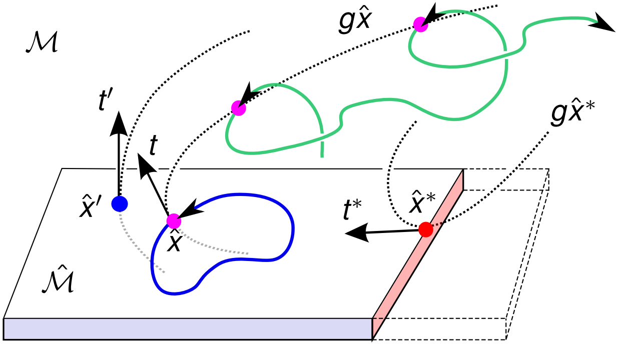
# slice trouble 1

**glitches!**

group tangent of a generic trajectory orthogonal to the slice tangent at a sequence of instants $\tau_k$

$$t(\tau_k)^T \cdot t' = 0$$

# slice trouble 1

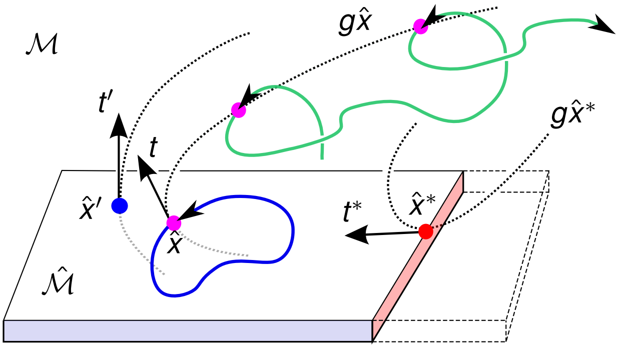



slice hyperplane is almost never a global slice; it is valid up to

slice border, a $(d-2)$-dimensional hypersurface (red) of points $\hat{x}^*$ whose group orbits graze the slice,

i.e. points whose tangents $t^* = t(\hat{x}^*)$ lie in $\hat{\mathcal{M}}$

# slice trouble 1




group orbits beyond the slice border miss the slice hyperplane : the “missing chunk” is here indicated by the dashed lines.

## example: group orbit of a pipe flow turbulent state

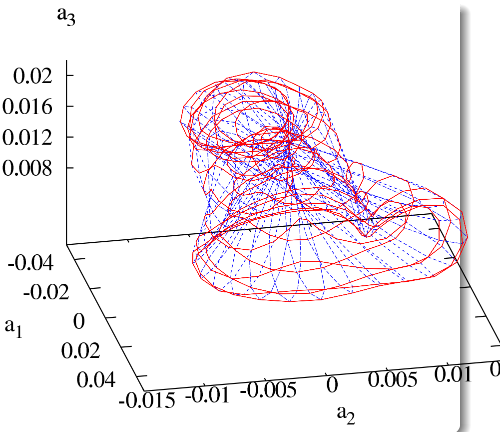
$\hat{x}'$ is Kerswell *et al* $N2\_M1$ relative equilibrium
( $Re = 2400$, stubby $L = 2.5D$ pipe)

**$\mathbf{SO(2) \times SO(2)}$ symmetry**
**$\Rightarrow$ group orbit is 2-torus**

a turbulent state

**distance extremum condition**

$$\frac{\partial}{\partial \phi_a} |x - g(\phi)\, \hat{x}'|^2 = 0$$




group orbits of highly nonlinear states are highly contorted: many extrema, multiple sections by a slice

# How good is your slice?

hyperplane of points $x^*$ defined by being normal to the quadratic Casimir-weighted vector $\mathbf{T}^2\hat{x}'$, such that from the template vantage point their group orbits are not transverse, but locally ‘horizontal,’

$$\langle t(x^*)|t'\rangle = -\langle x^*|\mathbf{T}^2\hat{x}'\rangle = 0$$

(for simplicity, specialize to the $\mathrm{SO}(2)$ case)

## inflection hyperplane

$S$ : set of all points $\hat{x}^*$ which are both

**(a)** in the slice

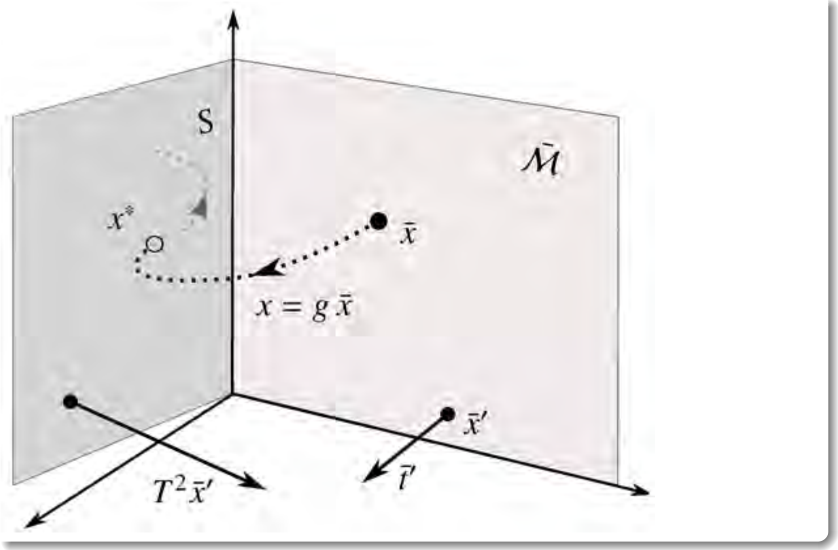
**(b)** whose group tangent $t(\hat{x}^*)$ is also in the slice

$$\begin{aligned} \langle \hat{x}^* | t' \rangle &= 0 \\ \langle t(\hat{x}^*) | t' \rangle &= -\langle \hat{x}^* | \mathbf{T}^2 \hat{x}' \rangle = 0 \end{aligned}$$

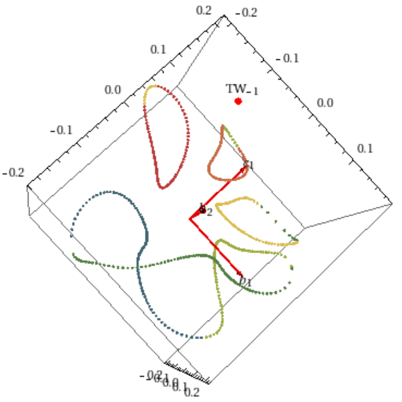
$S$ is the locus of inflection points, a hyperplane through which

- curvature of the distance function changes sign
- local minimum turns into a local maximum

## slice is good up to inflection hyperplane



## slice may cut a relative periodic orbit multiple times




here a single relative periodic orbit is intersected by a slice in 3 separate sections of the relative periodic orbit torus,
and 3 sections that appear to connect to a closed loop

## die Lösung für alle Ihre Probleme : be a cartographer

construct a global atlas by deploying a set of linear Poincaré sections and slices,

each a local chart in the neighborhood of an important equilibrium and/or periodic orbit

# summary

## conclusion

- symmetry reduction by method of slices:
  efficient, allows exploration of high-dimensional flows hitherto unthinkable
- stretching and folding of unstable manifolds in reduced state space organizes the flow

## to be done

- construct Poincaré sections and return maps
- find all (relative) periodic orbits up to a given period
- use the information quantitatively (periodic orbit theory)

## take-home message

if you have a symmetry

use it!

without symmetry reduction,
no understanding of pipe, Couette, ..., flows is possible

## amazing theory! amazing numerics! and still... frustration...

*"Ask your doctor if taking a pill to solve all your problems is right for you."*