The Correspondence Principle in the Statistical Interpretation of Quantum Mechanics

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These results suggest that the natural width of a spectral line may be less than the value to be expected from the classical theory of damping by radiation.

The subject of width of spectral line is one of considerable importance. If the lines are really much narrower than 0.12 X-Unit the radiation cannot come from a damped oscillating electron. The mechanism must be such as to maintain a pure harmonic oscillation of constant amplitude until the quantum of energy is completely emitted. Such a train of waves would need to have a great number of elements and so have considerable length. An alternate hypothesis would be that a quantum is an entity (the word "pulse" is avoided) that may be resolved into a train of waves by the crystal grating. In this case the width of a spectral line would depend on the degree of perfection of the crystal. The quantum theory of crystal grating action advanced by Duane[8] might also give a narrow spectral line, whose width would be a property of the crystal grating and not of the radiation.

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THE CORRESPONDENCE PRINCIPLE IN THE STATISTICAL INTERPRETATION OF QUANTUM MECHANICS

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In studying the very significant statistical interpretation put on the quantum mechanics by the "transformation theory" of Dirac[1] and Jordan,[2] the writer at first experienced considerable difficulty in understanding how the quantum formulas for averages and probabilities merge into the analogous classical expressions in the region of large quantum numbers and also, of course, in the limit $h = 0$. In the present note we shall aim to trace through the asymptotic connection between the formulas of the two theories, which does not seem to have been quite adequately elucidated in existing papers.

In the transformation theory a diagonal element of a matrix which

represents a function f and whose rows and columns are indexed with respect to a set of variables α_k ($k = 1, \ldots s$), is interpreted as the average value of the function f when the α's are specified, and all values of the variables β_k canonically conjugate to the α's are supposed equally probable. The latter supposition is necessary if accurate values are assigned to the α's; it is, in fact, a basic axiom of transformation theory that it is impossible to give accurate values simultaneously to both coördinates and their conjugate momenta. If $f(p,q)$ be a matrix function of the dynamical variables $p_1, \ldots, p_s, q_1, \ldots, q_s$, its diagonal elements in the α-scheme of indexing are given by the formula

$$f(p,q)(\alpha\alpha) = \int \ldots \int (\alpha/q)f(ih\partial/\partial q;q)(q/\alpha)dq_1 \ldots dq_s \quad (1)$$

where $f(ih\partial/\partial q; q)$ is the operator function obtained by replacing p_k by the operator $ih\partial/\partial q_k$ ($k = 1, \ldots s$) in the function $f(p, q)$. Here (q/α) is the "probability amplitude" or "transformation function" associated with the passage from the p,q to α,β system of variables. We use Dirac's notation in the main, except that we do not designate numerical quantities by primes, and the reader is referred to his paper if desirous of more background for the present article. In particular we use Dirac's h, which is $1/2\pi$ times the usual Planck's constant 6.55×10^{-27}. According to Pauli, Dirac and Jordan the expression

$$|(q/\alpha)|^2 dq_1 \ldots dq_s \quad (2)$$

is the probability of a given configuration in the q's when the α's are specified. Dirac shows that (2) is deducible as a special case of (1) by, in particular, taking f to be the product of certain of his δ-functions. We assume throughout the paper that the factors in non-commutative products are so ordered that the matrices representing all real variables are Hermitian. This permits us to use the relation $(\alpha/q) = (q/\alpha)^*$, where, as is customary in quantum literature, the asterisk * means the conjugate imaginary. Otherwise it would be necessary to replace $|(q/\alpha)|^2$ by $(q/\alpha)(\alpha/q)$ in (2).

Let us now suppose that the α's and β's are a set of variables which make the energy a diagonal matrix. We do this merely for concreteness, as in most problems the transformation of interest is one governed by the Hamiltonian function, and some further remarks on the general case, to which the proof still applies, are given at the end of the paper. Under the present supposition (q/α) is identical with a Schroedinger wave function $\psi(q_1, \ldots, q_s; \alpha_1, \ldots, \alpha_s, t)$ and satisfies his equation

$$[H(ih\partial/\partial q; q) + ih\partial/\partial t](q/\alpha) = 0 \quad (3)$$

where $H(p; q)$ is the Hamiltonian function. In dealing with problems connected with a definite stationary state it is often convenient to choose the α's and β's to be a set of action and angle variables; the arguments

α_k entering in (q/α) are then, except for a factor h, merely a set of quantum numbers. This restriction, however, is unnecessary and in many problems, especially the aperiodic affairs involved in the collisions of electrons with atoms, it is impossible to use exclusively true action and angle variables. In an isolated atom with one electron in a definite state, equation (3) can be regarded as determining a statistical charge density representing the average over all phases, and the interpretation of $|(q/\alpha)|^2$ as an electron density (though not necessarily in a purely statistical sense) was proposed by Schroedinger and others even before the advent of transformation theory.

Let us now seek the classical analogs of the preceding formulas. It has been abundantly emphasized in the literature[3,4,5] that the analog of the wave equation (3) is the Hamilton-Jacobi equation

$$H(\partial S/\partial q;\ q) + \partial S/\partial t = 0. \tag{4}$$

Let $S(q_1, \ldots . q_s;\ \alpha_1, \ldots \alpha_s, t) + C$ be a "complete integral" of (4) involving s independent arbitrary constants $\alpha_1, \ldots, \alpha_s$ besides the trivial additive constant C. Then the equations

$$p_k = \partial S/\partial q_k,\ \beta_k = \partial S/\partial \alpha_k,\ (k = 1, \ldots, s) \tag{5}$$

define a canonical transformation from the p,q system to a set of new variables $\alpha_1, \ldots, \alpha_s;\ \beta_1, \ldots, \beta_s$. By this transformation a function $f(p;\ q)$ of the original variables is converted into a function $F(\alpha;\ \beta)$ of the new ones. Let us suppose that for given α's all values of the β's are equally probable so that probability is proportional to the volume in the β-space. Like the usual assumptions concerning "weight" in statistical mechanics, this is a statistical hypothesis not included in the classical analytical dynamics by itself. The average value of $f(p;\ q) = F(\alpha;\ \beta)$ for given α's is then

$$A\int \ldots \int F(\alpha;\ \beta) d\beta_1 \ldots d\beta_s. \tag{6}$$

Let us change the variables of integration from the β's to the q's. The integrand then is expressed in terms of the q's and α's and by (5) the expression (6) thus becomes

$$A\int \ldots \int f(\partial S/\partial q;\ q)\, \Delta dq_1 \ldots dq_s \tag{7}$$

where Δ is the functional determinant

$$\Delta = \frac{\partial(\beta_1, \ldots, \beta_s)}{\partial(q_1, \ldots, q_s)} = \left| \frac{\partial^2 S}{\partial q_k \partial \alpha_j} \right| \tag{8}$$

of the transformation from the β's to the q's with the α's kept fast. With our statistical assumption the probability that the system will be in a given configuration $dq_1 \ldots . dq_s$ is clearly

$$A d\beta_1 \ldots d\beta_s = A \Delta dq_1 \ldots dq_s. \tag{9}$$

The constant A is determined by the requirement that the total probability be unity so that

$$1/A = \int \ldots \int \Delta dq_1 \ldots dq_s. \tag{10}$$

For the correspondence principle to be valid, equation (1) must pass into (7) and (2) into (9) in the limiting case of very large quantum numbers (or, more generally, large values of the variables α_k). This is equivalent to letting h approach zero, as in either case the ratios h/α_k vanish in the limit. It is well known that for small values of the h/α_k, a first approximation to the wave or transformation function (q/α) is $Ce^{S/ih}$, where C is a constant and S is the classical action function satisfying (4). This approximation is, however, not adequate to yield the correspondence principle, for it is easily shown that with only this approximation equations (1) and (2) approach expressions analogous to (7) and (9) except for the important difference that the functional determinant Δ is wanting. It is, however, proved below that a second approximation is

$$(q/\alpha) = A^{\frac{1}{2}} \Delta^{\frac{1}{2}} e^{S/ih} \tag{11}$$

where the constant A has the value (10). From this it follows immediately that (1) and (2) do indeed merge asymptotically into (7) and (9) for it is readily seen (cf. Eq. (13) below) that

$$f(ih\partial/\partial q;\ q)(\Delta^{\frac{1}{2}} e^{S/ih}) = \Delta^{\frac{1}{2}} e^{S/ih} f(\partial S/\partial q;\ q) + \ldots$$

where the dots denote terms which vanish in the limit $h = 0$, and where $f(\partial S/\partial q;\ q)$ means the function obtained by replacing the operators $ih\partial/\partial q_k$ by the expressions $\partial S/\partial q_k$.

The essential contribution of the present paper is the proposition that (q/α) when calculated to the second approximation, always contains the factor $\Delta^{\frac{1}{2}}$ involving the functional determinant (8). This degree of approximation is, in fact, a very necessary one, for it is needed to get the terms in (q/α) which do not vanish in the limit $h = 0$. Solutions through the second approximation have already been given by Wentzel, Brillouin and Kramers for the special case of a particle vibrating in one dimension, and careful examination of these solutions show that they do contain a factor which is proportional to $\Delta^{\frac{1}{2}}$ though not expressed directly in this form (e.g., Kramers' factor is written $y^{-1/4}$ where $y = 2m(W - V)$). The significance and proof of the universal presence of this factor do not appear to have previously been given.

Before passing to the proof of (11) it is well to mention that (3) and (4) are the general wave and Hamilton-Jacobi equations, in which the time is included in the solutions rather that the perhaps commoner equations from which the time is eliminated by substituting the energy constant

W for the operator $-ih\partial/\partial t$ in (3) and for $-\partial S/\partial t$ in (4). We use the equations inclusive of t for two reasons. In the first place they are more general, as Dirac[1] shows probability amplitudes can be calculated from (3) even in non-conservative systems, where the energy no longer has a constant value W. Thus our proof includes the case that t enters explicitly in H, even though our notation does not list t among its arguments. When t does so enter, H ceases to be a diagonal matrix, but meaning is still given to the transformation defined by (3) or (4) by interpreting the α's and β's as constants of integration, especially in Dirac's sense of values of variables at a specified time. In the second place by working with the equations inclusive of t, the manipulation of some of the functional determinants involved in the proof (especially near Eq. 18) is somewhat simplified even in the conservative case, as the formulas exhibit a certain amount of symmetry between t and the other variables. In conservative systems the solutions of (3) and (4) take the form

$$(q/\alpha) = \varphi(q_1, \ldots q_s;\alpha_1, \ldots \alpha_s)e^{iWt/h}, S = S(q_1, \ldots, q_s; \alpha_1, \ldots, \alpha_s) - Wt, \quad (12)$$

where φ and S do not involve t. In such systems we are interested in a definite energy, and so it is necessary to use one solution (12) rather than the general solution of (3) which is a linear combination of solutions each corresponding to a different W. (Developments in such combinations, however, are often useful in the non-conservative case.) With a definite W the distinction between (q/α) and φ is trivial, for then $|(q/\alpha)|^2$ equals $|\varphi|^2$ and hence it is immaterial if we replaced (q/α) by φ and (α/q) by φ^* in (1) or (2).

Proof of Equation (11).—The first step is to establish the following lemma: If $f(ih\partial/\partial q;\ q)$ be any function of the operators $ih\partial/\partial q_k$ and variables q_k $(k = 1, \ldots, s)$ which has Hermitian symmetry, and if S and G be any two functions of the q's not involving the constant h, then

$$f\left(ih\frac{\partial}{\partial q};\ q\right)(Ge^{S/ih}) = e^{S/ih}\left\{Gf\left(\frac{\partial S}{\partial q};\ q\right) + ih\sum_{k=1}^{s}\left(f_k'\frac{\partial G}{\partial q_k} + \frac{1}{2}G\frac{\partial f_k'}{\partial q_k}\right)\right\} + \ldots \quad (13)$$

where the neglected terms are proportional to second and higher powers of h, and where f_k' is an abbreviation for the derivative $\partial f/\partial(\partial S/\partial q_k)$ of $f(\partial S/\partial q;\ q)$ with respect to the argument $\partial S/\partial q_k$.

We shall prove (13) in an inductive fashion somewhat similar to Born, Heisenberg and Jordan's[6] way of proving their formulas for the differentiation of matrices. We shall show that if (13) holds for two functions $f = u$ and $f = v$, it holds likewise for $f = u + v$ and $f = uv + vu$. Since (13) obviously holds for $f = q_j$ and $f = ih\partial/\partial q_j$ this suffices to establish that

(13) holds for the most general function f constructed by repeated additions and Hermitian-ordered multiplications. For $f = u + v$ the proof is trivial. To give the proof for $f = uv + vu$ we first take $f = v$ in (13) and have

$$\mathbf{v}(Ge^{S/ih}) = e^{S/ih}\left[Gv + ih\sum\left(v_k'\partial G/\partial q_k + \frac{1}{2}G\partial v_k'/\partial q_k\right)\right] \quad (14)$$

where we have for brevity omitted to write in the arguments of v, but use bold-face type to distinguish operator from algebraic functions. Next again apply (13), this time taking f equal to u and G equal to the entire part of (14) in square brackets. This gives, discarding terms in h^2,

$$\mathbf{uv}(Ge^{S/ih}) = e^{S/ih}\Big[Guv + ih\sum\Big(uv_k'\partial G/\partial q_k + \frac{1}{2}uG\partial v_k'/\partial q_k + u_k'v\partial G/\partial q_k + u_k'G\partial v/\partial q_k + \frac{1}{2}vG\partial u_k'/\partial q_k\Big)\Big].$$

The right-hand side is not quite symmetric in u and v, but if we add to this the analogous expression for $\mathbf{vu}(Ge^{S/ih})$, we find on collecting terms and noting that $(uv)_k' = u_k'v + uv_k'$, the desired result

$$(\mathbf{uv} + \mathbf{vu})(Ge^{S/ih}) = e^{S/ih}\left\{2Guv + 2ih\sum\left[(uv)_k'\partial G/\partial q_k + \frac{1}{2}G\partial(uv)_k'/\partial q_k\right]\right\}.$$

Let us now, following Brillouin,[4] Wentzel[3] and Eckart,[5] seek to build up a solution of the wave equation (3) as a power series in h, taking $(q/\alpha) = e^{(S+S_1h+S_2h^2+\ldots)/ih}$. We denote the first exponent by S rather than S_0 for reasons that appear later. To the approximation which we desire, S_2 and higher terms may be disregarded. If we denote e^{-iS_1} by G, the assumed solution takes the form

$$(q/\alpha) = Ge^{S/ih}. \quad (15)$$

We can now substitute (15) in (3) and apply (13), taking $f = H$. If we then equate to zero the part of the resulting equation which is independent of h, we see that S satisfies an equation identical with (4) and thus agrees with the classical action function. As a second approximation we equate to zero the part of the resulting equation which is proportional to h, and this yields a partial differential equation

$$\sum_k\left(H_k'\partial G/\partial q_k + \frac{1}{2}G\partial H_k'/\partial q_k\right) + \partial G/\partial t = 0 \quad (16)$$

for determining G. Now, by definition, a complete integral $S(q, \alpha, t)$ of (4) reduces the left-hand side of (4) identically to zero and, consequently,

the derivatives of the left side of (4) with respect to each of the α's must vanish. Therefore, we have the identities

$$\sum_k H'_k \partial^2 S/\partial\alpha_j \partial q_k = -\partial^2 S/\partial\alpha_j \partial t \quad (j = 1, \ldots, s) \tag{17}$$

since H involves the α's only through the arguments $\partial S/\partial q_k$ and since H'_k means the derivative of H with respect to $\partial S/\partial q_k$.

We shall first solve (16) for the special case of a conservative system with one degree of freedom, as here the algebra is somewhat simpler than in the general case. In a conservative system (q/α) involves t only in the exponential fashion (12) and we may without essential loss of generality suppose that in (15) the time factor is incorporated entirely in $e^{S/ih}$ rather than in G, so that $\partial S/\partial t = -W$, $\partial G/\partial t = 0$. This supposition means only that we use in our first approximation the same energy as that appropriate to the final solution. Furthermore, with only one degree of freedom the subscripts are unnecessary and the functional determinant (8) reduces to the single term $\partial^2 S/\partial q \partial\alpha$. Thus (17) becomes $H'\Delta = dW/d\alpha$ and substitution of this in (16) gives

$$\Delta \partial G/\partial q - \frac{1}{2} G \partial\Delta/\partial q = 0$$

as $dW/d\alpha$ is independent of q. This equation integrates immediately into $G = A^{\frac{1}{2}}\Delta^{\frac{1}{2}}$ which is just the desired result, as we later show that the constant of integration has the value given by (10).

Passing now to the general case, the solution of the s simultaneous linear equations (17) for the H'_k is

$$H'_k = -\frac{1}{\Delta} \frac{\partial(\beta_1, \ldots, \beta_s)}{\partial(q_1, \ldots, q_{k-1}, t, q_{k+1}, \ldots, q_s)}. \tag{18}$$

Here we have used the definitions of the β's and Δ given in (5) and (8). Thus the H'_k are expressible as quotients of functional determinants. If we now change the dependent variable in (16) from G to θ by the substitution

$$G = \Delta^{\frac{1}{2}}\theta \tag{19}$$

we find that (16) reduces to

$$\sum H'_k \partial\theta/\partial q_k = -\partial\theta/\partial t \tag{20}$$

as many terms cancel in virtue of (18) and the mathematical identity[7]

$$\sum_{k=1}^{s} \frac{\partial}{\partial q_k} \frac{\partial(\beta_1, \ldots, \beta_s)}{\partial(q_1, \ldots, q_{k-1}, t, q_{k+1}, \ldots, q_s)} = \frac{\partial\Delta}{\partial t}$$

for any s functions $\beta_1, \ldots, \beta_s$ of $s + 1$ variables $q_1, \ldots, q_s, t$.

Equation (20) is clearly satisfied by taking θ equal to a constant C. This, however, is not the most general solution. Instead a "complete integral" of (20) is

$$\theta = C + \sum_j c_j \partial S/\partial \alpha_j \tag{21}$$

where the c_j are constants, for substitution of a typical term of (21) reduces (20) to an expression differing from (17) only by a constant factor. Now by (19) the term proportional to c_j in (21) simply adds to (15) very approximately the change $c_j ih\partial(\Delta^{\frac{1}{2}} e^{S/ih})/\partial\alpha_j$ which results in (15) if α_j is altered by a small amount ihc_j/C in a solution originally of the form $(q/\alpha) = C\Delta^{\frac{1}{2}} e^{S/ih}$. Such terms can clearly be made to disappear by assigning proper original values to the α's, for the effect of these terms is equivalent to changing slightly the constants of integration in the solution of (4). Thus we may always take $c_j = 0$, provided we start with a classical solution having the proper values of the α's. The discussion of what values of the α's should be used is beyond the scope of the present paper, as this problem has been considered by others. If the quantum dynamical system is one which has a "discrete" rather than entirely continuous matrix spectrum; i.e., if the classical motion is recurrent, the exponent in the factor $e^{S/ih}$ will be multiple-valued, as the action S does not revert to its original value after a cycle. Consequently, the wave function will meet the necessary requirement of single-valuedness only if the α's are given certain particular values or "Eigenwerte." Wentzel[3] and Brillouin[4] have shown that in multiply periodic systems in which the variables can be separated in (4) the constants in S should as a first approximation be so chosen that the Sommerfeld phase integrals taken over a cycle are integral multiples n_k of Planck's constant. Kramers,[8] however, has shown that because of the singularities at the classical libration limits it is a much better approximation and more rigorous procedure to start with a classical motion in which these integrals equal $n_k + \frac{1}{2}$ times Planck's constant, thus giving "half quanta." A classical solution with half quantum numbers is entirely adequate for our purpose of asymptotic comparison of the two theories.

The preceding paragraph shows that we may, without essential loss of generality, discard all the terms except $\theta = C$ in (21) so that by (15) and (19) the wave function has the desired form (11), if now we denote the constant factor by $A^{\frac{1}{2}}$ rather than C. To complete the proof of the correspondence principle it only remains to show that A has the value (10). Now, the wave function is normalized by the relation

$$\int \cdots \int |(q/\alpha)|^2 dq_1 \ldots dq_s = 1. \tag{22}$$

The integral in (22) is to be taken over the entire q-space, whereas the

classical integral (10) is to be taken only between the limits of libration of the classical motion, which is precisely the region in which S is real. Let us, however, divide the domain of integration in (22) into two parts, I and II, in which S is respectively real and imaginary. The integral over I is then identical with (10), as in I we have by (11) $|(q/\alpha)|^2 = A\Delta$. It can be shown that beyond the classical limits it is always possible to take the exponent S/ih as a negative real number; Kramers[8] shows this is true even though the ordinary series developments fail at the libration limits. With a positive exponent, of course, the integral would not converge and we should not have an "Eigenfunktion." With the negative exponent, however, the integrand has a factor $e^{-2|S|/h}$ and so drops off rapidly beyond the classical limits. The sharpness of dropping off increases as h approaches zero, and in the limit $h = 0$ the entire contribution to (22) comes from the region I, thus giving the desired result (10).

In many dynamical problems the Hamiltonian function involves only even powers of the p's. An approximate solution of (3-12) more general than (11) is then clearly $\varphi = \Delta^{\frac{1}{2}}(A^{\frac{1}{2}}e^{S/ih} + B^{\frac{1}{2}}e^{-S/ih})$ where A and B are constants. Equation (13) shows that then (1) goes over asymptotically into an expression identical with (7) except that now A is replaced by $A+B$, for "cross product" terms involving the product $A^{\frac{1}{2}}B^{\frac{1}{2}}$ have a rapidly fluctuating factor $e^{\pm 2S/ih}$ and so vanish to a high approximation when integrated over the entire q-space. It is these "cross product" terms which give so many problems their "wave nature" in quantum mechanics. As h approaches zero the wave-length becomes shorter and shorter, so that even a small element (2) will contain so many waves that the fluctuating part may be omitted. It is easily seen that the normalization now requires that $1/(A + B)$ equal the integral in (10), so that the correspondence principle still applies.

In equation (22) we have assumed that the α's assume discrete values in the quantum theory. The extension to the case where they assume continuous values occasions no difficulty if one uses the Dirac δ-functions to specify a distribution of the α's about some point.

Non-Diagonal Elements of Heisenberg Matrices.—In the transformation theory the primary emphasis is on the diagonal elements, but it is only a small extension of the previous work to prove the asymptotic identity of the non-diagonal elements of the original Heisenberg matrices with classical Fourier coefficients in multiply periodic systems. We have only to take (α/q) and (q/α) in (1) to refer to different wave functions or stationary states, which we may designate by primes and double primes, and must further specialize the α's and β's to be respectively action and angle variables, as the Heisenberg matrices are indexed with respect to quantum numbers. Equations (11) and (13) show that with this modification formula (1) approaches asymptotically an expression identical with (7) except for

insertion of a factor $e^{(S''-S')/ih}$ in the integrand, for to our degree of approximation we may take $\Delta' = \Delta''$ and hence by (10) $A' = A''$ if the ratios $(\alpha_k'' - \alpha_k')/\alpha_k'$ are small. Now this exponential factor is precisely the factor $e^{-i\Sigma \mathrm{T}_k\beta_k}$ which must be inserted to give the classical formulas, obtained by term-by-term integration, for the Fourier coefficients of an expansion in the β's. This follows inasmuch as for large quantum numbers $S'' - S'$ is approximately $\sum(\partial S/\partial\alpha_k)(\alpha_k''-\alpha_k')$ and by (5) $\beta_k = \partial S/\partial\alpha_k$; further $\alpha_k'' - \alpha_k' = \mathrm{T}_k h$ where the T_k are integers since the action variables α_k are integral (or more accurately half-integral) multiples of h.

We must by all means mention that the asymptotic connection of the Heisenberg matrices with Fourier components has also been proved very elegantly by Eckart[5] with another method. He considered the particular case of one degree of freedom and $f = q$, but his method is readily generalized. The functional determinant factor did not come to light in his work as his method involves only the ratios of the wave functions, making determination of the second approximation unnecessary.

Extension to the Case Where H Does Not Denote the Hamiltonian Function.—In the previous work we have interpreted H in (3) to be the Hamiltonian function, but our proof of the correspondence principle is clearly applicable to the still more general type of transformation determined by (3) in which H is any function of the p's and q's and in which we wish to transform to a set of variables α,β which make the function H (no longer necessarily the energy) a diagonal matrix; i.e., a function W only of the α's. In the general case the variable t need not have the physical significance of the time and can be regarded as simply a mathematical auxiliary useful in throwing the functional determinants into a symmetrical form. Such a formal auxiliary will not enter in H and so starting originally with φ as the probability amplitude we may regard t as simply a parameter introduced in the exponential fashion (12). Of the transformations in which H does not denote the Hamiltonian function, we may note two interesting special cases:

(1) Point transformation. Here Jordan[9] has already derived a rigorous solution which contains the factor $\Delta^{\frac{1}{2}}$.

(2) Case where $\alpha_k = p_k$ $(k = 1, \ldots, s)$. Here H reduces to $\sum ih\partial/\partial q_k$, and the first approximation $\varphi = e^{\Sigma q_k\alpha_k/ih}$ is also a rigorous solution, which incidentally Jordan[2] takes as one of the axioms in his postulational formulation of quantum mechanics. The functional determinant now reduces to unity, as indeed it must since the second and higher approximations are unnecessary. The correspondence principle thus follows directly without even the second approximation, and I am indebted to Dr. J. R. Oppenheimer for calling my attention to this simple result. He further remarks that all problems of calculating statistical averages in quantum mechanics can be reduced to this case (2), for by proper canonical transformations

any matrix f can be expressed as a function of the variables α,β instead of p, q, just as in classical theory we can use (6) as well as (7). The correspondence principle then follows from the asymptotic agreement of classical and quantum canonical transformations. It seems, nevertheless, of interest to have proved the correspondence principle as directly as possible without appeal to iterated transformations.

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[5] C. Eckart, *Proc. Nat. Acad. Sci.*, **12**, 684 (1926).

[6] Born, Heisenberg and Jordan, *Zeits. Physik*, **35**, 563 (1926).

[7] Cf., for instance, Muir, *Theory of Determinants*, vol. II, p. 230. The identity is due originally to Jacobi, *Crelle's Journ.*, **27**, 199 (1846), or *Gesammelte Werke*, iv, 317.

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[9] P. Jordan, *Ibid.*, **38**, 513 (1926); and **41**, 797 (1927).

EXPERIMENTS WITH MODIFIED MUCRONATE ELECTRODES

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1. *Correction.*—In an earlier paper* I found the anode behaving in a way indistinguishable from the cathode. Many repetitions of the experiment since have shown that this is not the case. What probably happened was a spontaneous change of the polarity of the electrical machine for which I was unprepared. Hence the cathode behavior was inadvertently measured twice. The correct graphs are given in the following paragraphs.

2. *Apparatus.*—This is essentially the same as before, consisting of the electrodes E E' (about 2 cm. in diameter) of the spark gap x of a small electrostatic machine. The quill tube from E' leads to the interferometer U-tube beyond U, for measuring the pressure of the electric wind (s, roughly in 10^{-6} atm.). The electrode E is provided with a micrometer screw db carrying the needle n, whose extrusion y beyond the electrode is thus measurable. P and P' are insulated posts and c a constriction of the pipe a, carrying the electrode E and the nut e of the screw. A similar arrangement for the cathode is also provided (not shown).

3. *Moist and Dry Electrodes.*—Observations made in the dark with electrodes moistened, for instance, with glycerine, recorded very marked differences of behavior. There is not room to describe these here further